Critical Points of Functions
Skoltech Lecture course Fall 2020
Problem Set I

Discriminants of families of functions

The *evolute* Σ of a plane smooth curve $C \subset \mathbb{R}^2$ is defined as an envelope of the family of its normal lines, that is, it is defined by the condition that all normal lines are tangent to Σ. Equivalently, it can be defined as the curve of *curvature centers*, that is the curve of centers of osculating circles (circles having tangency of order more that one with the curves).

Recall that the discriminant of a family of functions is the locus of parameter values for which the corresponding function of the family has a degenerate critical point.

Define $S_q(x) = \|x-q\|^2$, $x \in C$, $q \in \mathbb{R}^2$. We consider S as a family of functions on C depending on the point $q \in \mathbb{R}^2$.

1. Prove that Σ is the *discriminant* of the family S. More explicitly, we have

   - $S_q'(x) = 0$ iff q belongs to the normal line to C at the point x;
   - $S_q'(x) = S_q''(x) = 0$ iff q is the curvature center;
   - $S_q'(x) = S_q''(x) = S_q'''(x) = 0$ iff, in addition, x is a local extremum of curvature. The evolute has a (generically, semicubical) cusp at q in this case.

2. Compute in draw the evolute of
   (a) parabola $y = x^2$;
   (b) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   (Hint: please use the result of the previous problem.)

3. Compute and draw the discriminant of the following 2-parameter family of polynomials:

$$F_{a,b}(x) = x^4 + a\,x^2 + b\,x.$$

4. Compute and draw the discriminant of the following 3-parameter family of polynomials:

$$F_{a,b,c}(x) = x^5 + a\,x^3 + b\,x^2 + c\,x.$$

   For each stratum of the discriminant, identify the qualitative behaviour of the corresponding polynomial.

The surface of the previous problem has the name *swallowtail*.

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Problem Set II

Local algebra and Milnor number

The Milnor number of the singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is the dimension of its local algebra

$$\mu = \dim \frac{\mathbb{C}[[x_1, \ldots, x_n]]}{(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})}.$$

The Milnor number is finite iff the origin $0 \in \mathbb{C}^n$ is an isolated critical point of f.

1. Compute Milnor numbers of the following singularities:
   (a) A_k: $f(x) = x^{k+1}$;
   (b) D_k: $f(x, y) = y^{k-1} + x^2\, y$;
   (c) E_6: $f(x, y) = x^3 + y^4$;
   (d) F_7: $f(x, y) = x^3 + xy^3$;
   (e) E_8: $f(x, y) = x^3 + y^5$;
   (f) J_{10}: $f(x, y) = x^3 + y^6 + a\, x^2y^2$, $4a^3 + 27 \neq 0$;
   (g) X_9: $f(x, y) = x^4 + y^4 + a\, x^2y^2$, $a^2 \neq 4$;
   (e) P_8: $f(x, y, z) = x^3 + y^3 + z^3 + a\, x\, y\, z$, $a^3 + 27 \neq 0$;
   (f) $f(x, y) = x^4 + x^2y^2 + y^5$.

Consider a tuple of positive rational numbers $\alpha_1, \ldots, \alpha_n, N$. A function $f(x_1, \ldots, x_n)$ is said to be *quasihomogeneous* of degree N with respect to the weights $\alpha_1, \ldots, \alpha_n$ of the variables $x_1, \ldots, x_n$, respectively, if for any $t > 0$ we have $f(t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x^n) = t^N f(x_1, \ldots, x_n)$. Equivalently, the Taylor expansion of f at the origin involves those monomials $x_1^{k_1} \ldots x_n^{k_n}$ only whose vector of exponents $(k_1, \ldots, k_n)$ belongs to the hyperplane $\alpha_1 k_1 + \cdots + \alpha_n k_n = N$. There are finitely many such monomials, therefore, a quasihimogeneous function is a polynomial.

Theorem. *The Milnor number of an isolated quasihomogeneous singularity is uniquely determined by the homogeneity weights $\alpha_1, \ldots, \alpha_n, N$.*

2. Assuming the statement of Theorem above is true, find an explicit formula for the Milnor number of an isolated quasihomogeneous singularity. Check this formula for the singularities of Problem 1.

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Problem Set III

Versal deformations

1. Find a basis of local algebras and write down versal deformations of simple complex critical point function singularities.

   The classification of simple real critical point singularities is obtained from the complex one by insertion of signs $\pm$ in the normal forms in certain cases. For example, the complex singularity D_4 has two nonequivalent real forms $D_4^{\pm}$: $y^3 \pm x^2 y$. The corresponding reduced (i.e. not involving constant terms) versal deformations are

$$y^3 \pm x^2 y + a\,y^2 + b\,y + c\,x.$$

2. Compute and draw the discriminants of the above tree-parameter family of functions.

   The singular surfaces of the previous problem for the singularities D_4^+ and D_4^- are caller *purse* and *pyramid*, respectively. These singularities (along with the swallowtail) are typical singularities of discriminants of tree-parameter families of functions (in any number of variables).

   Consider a smooth surface $M \subset \mathbb{R}^3$. Its *focal set* is the discriminant of the following family of functions: $S_q(x) = \|x - q\|^2$, $x \in M$, $g \in \mathbb{R}^3$. We consider S as a family of functions on M depending on the point $q \in \mathbb{R}^3$. Similarly to the evolute of a plain curve, the focal set of a surface reflects its differential geometry.

3. Prove:

   - x is a critical point of $S_q(x) = 0$ iff q belongs to the normal line to M at the point x;
   - x is a degenerate critical point of S_q iff, in addition to the above property, the distance between q and x is inverse to one of the two principal curvatures of M at x;
   - the function S_q has singularity $D_4^{\pm}$ at x iff, in addition to the above property, x is an *umbilical* point of M, that is, the two principle curvatures are equal at this point.

4. Study (and draw) the focal set of an ellipsoid. How many umbilical points does ellipsoid have? What are their types (D_4^+ or D_4^-)?

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Problem Set IV

Simple singularities and finite subgroups of $SU(2)$

Du Val singularities are isolated surface singularities of the form $f(x,y,z)=0$ where f is a simple critical point function singularity stabilized to the case of functions in 3 variables. Here is the list:

$$\begin{aligned}
A_k:&\quad x^{k+1}+y\,z=0,\quad k\geq 1\\
D_k:&\quad y^{k-1}+x^2\,y+z^2=0,\quad k\geq 4;\\
E_6:&\quad x^3+y^4+z^2=0;\\
E_7:&\quad x^3+xy^3+z^2=0;\\
E_8:&\quad x^3+y^5+z^2=0.
\end{aligned}$$

The ADE classification of simple singularities is related mysteriously to other classifications in mathematics. One of those classifications are *finite subgroups of* $SO(3)$. They include the cyclic group (as the group of symmetries of an oriented regular polygon), the dihedral group (as the group of symmetries of the two-sided regular polygon), and three exceptional groups (as the groups of symmetries of the tetrahedron, octahedron, and icosahedron, respectively). In order to relate these groups to the singularities of functions, consider the two-sheeted covering $SU(2)\to SO(3)$ and denote by $\Gamma\subset SU(2)$ the full preimage of the corresponding group. The group Γ being the subgroup of $SU(2)$ acts on the complex plane and the quotient surface $\mathbb{C}^2/\Gamma$ is singular with the corresponding Du Val singularity at the origin.

More explicitly, let u,v be the coordinates in $\mathbb{C}^2$. Then, in all cases, the ring of Γ-invariant polynomials in u and v has three generators. Denote these polynomials by $x(u,v)$, $y(u,v)$, and $z(u,v)$. These polynomials are not algebraically independent. They obey a polynomial relation of the form $f(x,y,z)=0$ from the list of Du Val singularities.

1. Let $\Gamma\subset SU(2)$ be generated by the transformation $(u,v)\mapsto(\xi u,\xi^{-1}v)$ where $\xi=e^{\frac{2\pi i}{n}}$ is the nth primitive root of unity. Find the generators of the ring of Γ-invariant polynomials and the relation between these generators.

2. Let $\Gamma\subset SU(2)$ be generated by two transformation $(u,v)\mapsto(v,-u)$ and $(u,v)\mapsto(\xi u,\xi^{-1}v)$ where $\xi=e^{\frac{2\pi i}{2n}}$ is the primitive root of unity of degree $2n$ (the degree is necessarily even since Γ contains $-\mathrm{Id}$ which is the square of the first transformation). Find the generators of the ring of Γ-invariant polynomials and the relation between these generators.

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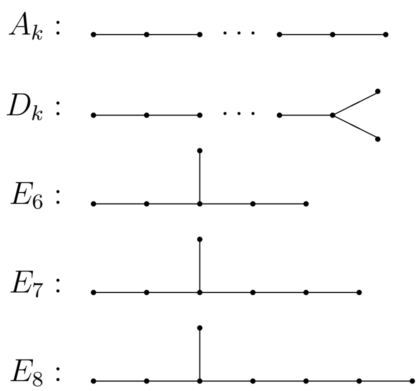
Problem Set V

Minimal resolutions and Dynkin diagrams

1. Describe explicitly the minimal sequence of blowups resolving each of the DuVal singularity. Show (case by case) that all components of the exceptional divisor of this resolution are rational (i.e. holomorphically equivalent to $\mathbb{C}P^1$) and their number is equal to the corresponding Milnor number. Find the incidence graphs of these components (the vertices of this graph correspond to components of the exceptional divisor and the vertices are connected by an edge iff the corresponding components do intersect).

The graphs of the previous problem are known as *Dynkin diagrams*. They appear also in the classification of simple Lie algebras, root systems, and the corresponding reflection groups:



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Problem Set VI

Adjacencies

We say that the singularity f is *adjacent* to the singularity g (notation: $g \leftarrow f$) is there exists a one-parameter family of functions F_t such that the germ of F_0 at the origin is right equivalent to f and the germ of F_t at the origin is right equivalent to g for each value of $t \neq 0$.

1. Consider the family of functions $F_t(x,y) = x^3 + (y^2 + t\,x)^2$. For $t = 0$ the singularity type of this function at the origin is obviously E_6. Find the singularity type of F_t at the origin for $t \neq 0$. Which adjacency is realized by this family?

   Adjacencyies of simple singularities are shown on the following diagram.

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] & A_8 \arrow[l] & \cdots \arrow[l] \\
 & & & D_4 \arrow[ul] & D_5 \arrow[l] \arrow[ul] & D_6 \arrow[l] \arrow[ul] & D_7 \arrow[l] \arrow[ul] & D_8 \arrow[l] \arrow[ul] & \cdots \arrow[l] \\
 & & & & & E_6 \arrow[ul] \arrow[uul] & E_7 \arrow[l] \arrow[ul] \arrow[uul] & E_8 \arrow[l] \arrow[ul] \arrow[uul] &
\end{tikzcd}$$

2. Prove (case by case) the existence of adjacencies shown on the diagram by presenting explicitly the corresponding deformations of adjacent singularities. (The most complicated and rather nontrivial cases are $A_{k-1} \leftarrow D_k$ and $A_{k-1} \leftarrow E_k$.)

3. Prove (also case by case) that there is no adjacency of the form $D_k \leftarrow A_m$, $E_k \leftarrow A_m$, $E_k \leftarrow D_m$.

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Problem Set VII

Milnor fiber

1. Prove that the complex hypersurface in $\mathbb{C}^n$ given by the equation $z_1^2 + \cdots + z_n^2 = \epsilon$, $\epsilon \neq 0$, is diffeomorphic to the total space of the tangent bundle of the sphere S^{n-1}.

2. Prove that the intersection of the complex hypersurface of the previous problem with the open unit ball is also diffeomorphic to the total space of the tangent bundle of the sphere S^{n-1}, provided that $|\epsilon|$ is small enough. Give the precise bound for ϵ such that the assertion of this problem holds true.

3. Compute the monodromy transformation of the Milnor fiber of the Morse singularity $z_1^2+\cdots+z_n^2$ and the action of the monodromy transformation in its homology groups.

4. Consider the complex curve V in $\mathbb{C}^2$ given by the equation $x^n + y^n = 1$. This complex curve considered as a real manifold is a surface of some genus g with some number of punctures. Find the genus and the number of punctures.

   Any surface with punctures is homotopy equivalent to a wedge product of circles.

5. Find the homotopy type of the surface of the previous problem. Compute its homology group $H_1(V)$. Compare the rank of this group with the Milnor number of the critical point singularity $f(x, y) = x^n + y^n$.

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Problem Set VIII

Real morsifications and computation of the intersection form

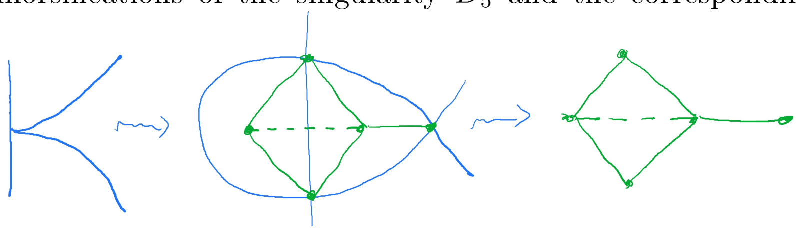
To each singularity type with Milnor number μ one can associate two bilinear forms on the integer lattice $\mathbb{Z}^\mu$: one is symmetric, and another is skewsymmetric. These are intersection forms in the middle homology of the Milnor fiber for the stabilization of the corresponding singularity with odd and even number of variables, respectively. By Picard-Lefshetz theorem, the intersection form determines uniquely the monodromy group. We will concentrate mostly in the symmetric case, where the monodromy group is generated by the reflections in the hyperplanes orthogonal to vanishing cycles,

$$a \mapsto a - 2\frac{(a,\Delta)}{(\Delta,\Delta)}\Delta$$

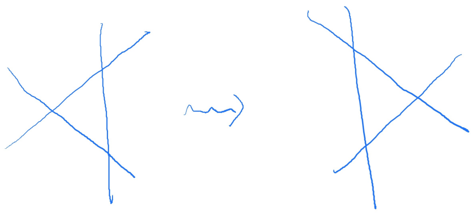
where Δ is one of the vanishing cycles. Remark that $(\Delta,\Delta) = -2 \neq 0$ (here we assume that $n \equiv 3 \pmod 4$; if $n \equiv 1 \pmod 4$ then the sign is opposite).

Let the critical point function singularity be represented by a polynomial $f(x,y)$ with real coefficients. Its real deformation $\tilde{f}(x,y)$ is called *morsification* if it has μ distinct nondegenerate critical points near the origin, and moreover, (a) all these points are real (b) the critical value at every saddle point is 0, the critical value at every local maximum is positive, and the critical value at every local minimum is negative. The method of A'Campo–Gusein-Zade computes the Dynkin graph encoding the matrix of the symmetric intersection form in a suitable basis of vanishing cycles from the geometry of the real curve $\tilde{f}(x,y) = 0$. Let us emphasize that we derive information about the Milnor fiber of the stabilized functin with odd number of variables from the properties of the original function in two variables.

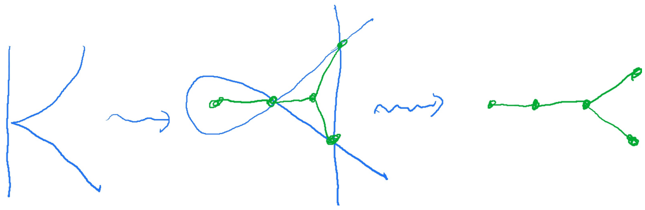
Namely, we put one vertex of the Dynkin graph to every double point of the curve $\tilde{f}(x,y) = 0$ and one vertex at the middle of each compact domain of its complement. We connect by an edge a vertex corresponding to each domain to each its corner, and we also draw a broken edge for each arc of the curve separating bounded domains. For example, here is one of the possible real morsifications of the singularity D_5 and the corresponding Dynkin graph.



The real morsification and the Dynkin graph is not unique. For example, it is allowed to apply to the original curve $\tilde{f}(x,y) = 0$ a sequence of the following kind of moves called *admissible*.



Every such move leads to just a linear invertible change of basis of vanishing cycles. For example, here is an alternative real morsification and a different Dynkin graph of the same singularity D_5:



1. The Chebyshev polynomials $T_n(x)$ are determined by the relation

$$T_n(\cos(t)) = \cos(n\,t).$$

   Prove that all its $n-1$ critical points are real and belong to the segment $[-1, 1]$. Moreover, all local maxima have equal critical value 1 and all local minima have equal critical value -1.

2. Show that $T_{n+1}(x) + y^2 - 1$ is a real morsification of the singularity A_k. Compute the Dynkin diagram associated with this real morsification.

3. Show that $T_n(x) + T_m(y)$ is a real morsification of the singularity $x^n + y^m$. Compute the Dynkin diagram associated with this real morsification.

4. The computation of the previous problem can be applied, in particular, to the singularities E_6 and E_8. However, the Dynkin diagrams in these cases do not agree with the standard Dynkin diagrams E_6 and E_8. Show that after a suitable sequence of admissible moves of the curve $\tilde{f}(x, y) = 0$ it can be brought to the form that gives the standard Dynkin diagrams labelled by the same symbols.

5. Compute real morsifications and Dynkin diagrams for the remaining simple singularities D_k and E_7. Show that if the real morsifications for these singularities are chosen in a suitable way (or after applying a suitable sequence of admissible moves), the Dynkin diagram takes the standard form labelled by the same symbols.