July 2020, Skoltech, Center for Advanced Studies

Rules:

- Test duration is 3 hours.
- There are 9 problems on 2 pages.
- Please, provide the key points of solutions for all problems. Correct answers are important, but not enough, we want to see that they are not consequences of the wrong solutions.
- You may use LaTeXnotations to type your solutions: we can either read it, or compile. Scans or photos of solutions are also fine, if the proctor allows it. In this case please send a separate file for every problem.

Problems:

1. **Integral.**
   Calculate the integral (the integration contour is oriented counter-clockwise)
   $$\frac{1}{2\pi i}\oint\limits_{|z|=1}\frac{zdz}{\cos\frac{1}{z}+\cos\frac{2}{z}}.$$

2. **Quantum mechanics.**
   Find eigenvalues and their degeneracies for the following Hamiltonian:
   $$\hat{H}=\sum_{i=0}^{N-1}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x$$
   acting on $\left(\mathbb{C}^2\right)^{\otimes(N+1)}$, where
   $$\hat{\sigma}_i^x=\begin{pmatrix}1&0\\0&1\end{pmatrix}^{\otimes i}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}^{\otimes(N-i)}.$$

3. **Matrices over $\mathbb{F}_p$.**
   How many 2×2 matrices over the field $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$, p is a prime number, have trace 1 and determinant 0?

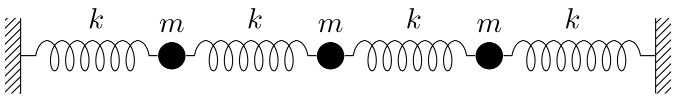
4. **Zeroes in the unit circle.**
   How many zeroes does the function $f(z)=z^8+10z^3+1$ have in the unit circle $|z|<1$?

5. **Abelian groups.**
   Find all possible non-isomorphic abelian groups A such that there exists the short exact sequence
   $$0\to\mathbb{Z}/10\mathbb{Z}\to A\to\mathbb{Z}/15\mathbb{Z}\to 0.$$
   Present *only one* group from each isomorphism class.

6. **Product of matrices.**
Find

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -1 & \lambda \end{pmatrix}^{100} .$$

7. **Classical mechanics.**
Find frequencies of all normal modes of the system in the figure below:



Its Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k}{2}\left(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2\right).$$

8. **Representation.**
Consider a map φ from the Lie algebra $\mathfrak{sl}_2$ to the algebra of differential operators

$$\varphi(e) = \alpha\frac{d^2}{dz^2}, \quad \varphi(f) = \beta z^2.$$

   (a) Find for which α, β this map can be lifted to a representation of $\mathfrak{sl}_2$ on the space of polynomials $\mathbb{C}[z]$.

   (b) Express quadratic Casimir C (i.e. quadratic central element of the universal enveloping algebra $U(\mathfrak{sl}_2)$) in terms of e, f, h. Calculate its action in this representation.

9. **Morse functions.**
Consider a three-dimensional torus with coordinates $x, y, z \in [0, 2\pi]$. For the following functions

   (a) $f_a = \sin x \sin y \sin z$,

   (b) $f_b = \sin x + \sin y + \sin z$,

figure out if they are Morse functions. For Morse functions find critical points and their indices.