April 2021, Skoltech, Center for Advanced Studies

Rules

- Test duration is 3 hours.
- There are 9 problems on 2 pages.
- Please, provide the key points of solutions for all problems. Correct answers are important, but not sufficient, we want to see that they are not consequences of the wrong solutions.
- You may use LaTeX notation to type your solutions: we can read it, or compile. Scans or photos of solutions are also fine, if the proctor allows it. In this case please send a separate file for each problem.

Problems

1. **Integral.**
   Compute the integral
   $$\int_0^{2\pi} d\phi \left(\frac{1}{5 + e^{-i\phi}} - \frac{1}{5 - 3e^{-i\phi}} \right).$$

2. **Exponential map.**
   Describe all $SL(2,\mathbb{C})$ matrices that are not in the image of the exponential map
   $$\exp : \mathfrak{sl}(2,\mathbb{C}) \to SL(2,\mathbb{C}).$$

3. **Inverse cubic potential.**
   A classical non-relativistic particle of mass m moves in three dimensions along a closed trajectory in the potential
   $$U(r) = -\frac{\alpha}{|\vec{r}|^3}, \quad \alpha > 0,$$
   with angular momentum L. Find the minimal and the maximal value of $|\vec{r}|$ on this trajectory.

4. **Matrix polynomial.**
   Find a non-zero polynomial $P(z) \in \mathbb{Q}[z]$ of minimal possible degree with rational coefficients such that $P(A) = 0$, where
   $$A = \begin{pmatrix} \sqrt{3} & 1 \\ 2 & \sqrt{3} \end{pmatrix}.$$

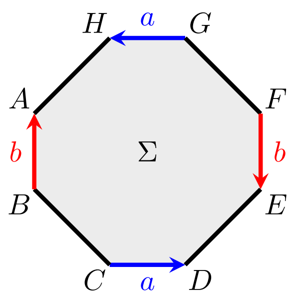
5. **Quantum mechanics.**
   Find spectrum of the Hamiltonian
   $$H = a^\dagger a + b^\dagger b + \frac{\lambda}{2}(a + a^\dagger)(b + b^\dagger), \quad 0 < \lambda < 1,$$
   where $a, b, a^\dagger, b^\dagger$ are annihilation and creation operators with the commutation relations
   $$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0.$$

6. **Space of polynomials.**
   Find the dimension of the vector space of homogeneous polynomials of degree k in n variables.

7. **Singular points.**
Let $F(z_1, z_2, z_3) = z_1^5 + \sum_{i,j=1}^{3} a_{ij} z_i z_j$ be a polynomial with complex coefficients. It defines a map $F : \mathbb{C}^3 \to \mathbb{C}$. Assume that any fiber of this map has only isolated singularities. Under this assumption find minimal and maximal possible number of singular fibers.

8. **Limit.**
Find the limit
$$\lim_{N\to\infty} \sqrt{N} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^N}.$$

9. **Homology.**
Find homology groups with integer coefficients for the following surface obtained by gluing sides of the octagon Σ:



One should glue (identify) two blue sides, $\overrightarrow{CD}$ and $\overrightarrow{GH}$, labelled by a, according to the direction of the arrows, and also glue two red sides, $\overrightarrow{BA}$ and $\overrightarrow{FE}$, labelled by b, according to the direction of their arrows. The other four sides are not glued.