# Master test 2020

Skoltech, Center for Advanced Studies

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## Rules:

- Test duration is 3 hours.
- There are 9 problems on 2 pages.
- Please, provide the key points of solutions for all problems. Correct answers are important, but not enough, we want to see that they are not consequences of the wrong solutions. You may also attach a scan or photo of your solution, please send a separate pdf file for every problem.
- There is no requirement to write down complete and rigorous proofs, but all statements should be correct.

## Problems:

1. **Residues.**
   Find residues of the form $\frac{dx}{y}$ on the compactification of the affine curve defined by the equation $y^2 = (x-1)(x-2)(x-3)$.

2. **Roots of unity.**
   Find product of all $n$-th complex roots of unity.

3. **Homology.**
   Find homology groups $H_n(X, \mathbb{Z})$ of the topological space $X \subset \mathbb{R}^3$,
   $$X = \left\{(x,y,z) \,\middle|\, \left((\sqrt{x^2+y^2}-1)^2 + z^2 - 1/4\right)\left((\sqrt{x^2+y^2}-2)^2 + z^2 - 1/4\right) = 0\right\}.$$

4. **Fermions.**
   Consider the system of $n$ identical non-interacting non-relativistic fermions with mass $m$, spin-$\frac{1}{2}$, and without other internal quantum numbers, moving in the potential
   $$V(x) = \frac{m\omega^2 x^2}{2}.$$
   Find energy of the ground state of this system.

5. **Integral.**
   Evaluate the integral
   $$\int_0^{+\infty} \frac{\log \frac{x}{r}}{R^2 + x^2} dx,$$
   where $R$ and $r$ are positive constants, and log — natural logarithm function.

6. **Symplectic forms.**
Which of the forms below are symplectic inside the real 4-dimensional unit ball $\sum_{i=1}^{4} x_i^2 < 1$ ?

   (a) $\omega_e = x_1 dx_2 + x_3 dx_4$
   (b) $\omega_d = dx_1 \wedge dx_2 + dx_1 \wedge dx_4$
   (c) $\omega_b = (2 - x_3) dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3 + dx_3 \wedge dx_4$
   (d) $\omega_a = (2 + x_3) dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3 + dx_3 \wedge dx_4$
   (e) $\omega_c = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4$

   Show, why the others are not symplectic.

7. **Flyby time.**
Consider a particle of mass $m$ with energy $E > U_0 > 0$ moving in the potential

$$U(x) = \frac{U_0}{\cosh^2(\alpha(x - L))} - \frac{U_0}{\cosh^2(\alpha(x + L))},$$

   where $\alpha > 0$. At time $t = 0$ it's coordinate is $x = -2L$, and it moves in the positive direction. It takes time $\tau = \tau(E, U_0, L)$ to get to the point with coordinate $x = 2L$. Compute the limit

$$\delta\tau(E, U_0) = \lim_{L\to\infty} \left(\tau(E, U_0, L) - \tau(E, 0, L)\right).$$

8. **Hilbert–Poincare.**
Find Hilbert–Poincare series of the graded algebra $\mathbb{C}[x, y]/(x - y)$. Express it as an expansion of a rational function.

9. **Representations.**
Let $W_q$ be an associative algebra over $\mathbb{C}$ defined by generators and relations

$$W_q = \langle x, x^{-1}, y, y^{-1} \mid xy = qyx, xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\rangle.$$

   Find all $q \in \mathbb{C}$, $q \neq 0$, such that algebra $W_q$ has any finite-dimensional representation.