Integrable systems of particles and nonlinear equations. Lecture 9

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The KP hierarchy

The KP equation is the first member of a whole infinite hierarchy of so-called higher KP equations that are compatible with it. One of the ways to introduce the hierarchy is the technique of pseudo-differential operators.

Pseudo-differential operators. A pseudo-differential operator is a series of the form $\sum_{k=0}^{\infty} v_k \partial_x^{N-k}$, where v_k are functions, and the operator ∂_x has the standard commutation relation with any function: $\partial_x f = f' + f\partial_x$. Multiplying both sides of this equality from the right and from the left by ∂_x^{-1}, one can understand it as a rule of moving the operator ∂_x^{-1} through a function: $\partial_x^{-1} f = f\partial_x^{-1} - \partial_x f' \partial_x^{-1}$. Repeating this procedure, we arrive at the commutation rule

$$\partial_x^{-1} f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} + \dots .$$

Pseudo-differential operators are multiplied as Laurent series, taking into account that the symbol ∂_x does not commute with the coefficient functions. For brevity we right $\partial_x f$, meaning the composition of the operator of multiplication by the function f and the differential operator ∂_x. We hope that this will not lead to a misunderstanding. In the more detailed notation the composition is written as $\partial_x \circ f$, but, in our opinion, a systematic use of this notation makes it more difficult to read formulas.

Problem. For any functions f, g prove the following identities in the algebra of pseudo-differential operators:

$$\text{a)} \qquad (\partial_x - g)^{-1} f = \sum_{n=0}^{\infty} (-1)^n f^{(n)} (\partial_x - g)^{-n-1},$$

$$\text{b)} \qquad e^{-f} \partial_x^{-1} e^{f} = (\partial_x + f')^{-1},$$

$$\text{c)} \qquad \partial_x^n f = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(k)} \partial_x^{n-k}, \quad n \geq 0,$$

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d) $\quad \partial_x^{-n} f = \sum_{k\geq 0} (-1)^k \begin{pmatrix} k+n-1 \\ k \end{pmatrix} f^{(k)} \partial_x^{-n-k}, \quad n > 0.$

Here $\begin{pmatrix} n \\ k \end{pmatrix} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient. Note that identities c) and d) can be unified in one identity by extending the definition of binomial coefficients to arbitrary complex numbers n with the help of the formula

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)(n-2)\dots(n-k+1)}{1\cdot 2\cdot 3\cdot \ldots \cdot k},$$

then at $n < 0$ $\begin{pmatrix} n \\ k \end{pmatrix} = (-1)^k \begin{pmatrix} k-n-1 \\ k \end{pmatrix}$. To prove identities c) and d), one can use induction in n.

Given a pseudo-differential operator $P = \sum_{k=0}^{\infty} v_k \partial_x^{N-k}$, we call the number N the order of the operator. Let P_+ be its differential part (i.e. sum of the terms with non-negative powers of ∂_x: $P_+ = \sum_{k=0}^{N} v_k \partial_x^{N-k}$), then $P_- = P - P_+$ is sum of the terms with negative powers. The operation of conjugation defined as $\partial_x^\dagger = -\partial_x$, $f^\dagger = f$, $(AB)^\dagger = B^\dagger A^\dagger$ can be extended to the algebra of pseudo-differential operators:

$$\left(\sum_{k=0}^{\infty} v_k \partial_x^{N-k}\right)^\dagger = \sum_{k=0}^{\infty} (-\partial_x)^{N-k} v_k.$$

The KP hierarchy: Lax equations and zero curvature representation. Consider a pseudo-differential operator $\mathcal{L}$ of the form

$$\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \ldots,$$

in which the coefficients u_i are functions of x. It is called the Lax operator of the KP hierarchy. Introduce evolution in the space of functions of x by the Lax equations

$$\partial_{t_m} \mathcal{L} = [(\mathcal{L}^m)_+ , \, \mathcal{L}], \qquad m = 1, 2, 3, \ldots.$$

Here t_m are evolution parameters ("times"). Each of these equations define an infinite system of evolution equations for the infinite set of functions u_i of the form $\partial_{t_j} u_i = \mathcal{P}_{ij}(\{u_l\})$, where $\mathcal{P}_{ij}(\{u_l\})$ are some differential polynomials of u_l. The generators of the t_m-flows,

$$A_m = (\mathcal{L}^m)_+,$$

are differential operators of order m. For example, $A_1 = \partial_x$, $A_2 = \partial_x^2 + 2u_1$, $A_3 = \partial_x^3 + 3u_1 \partial_x + u_2$.

The Lax eqiation for $m = 1$ states that $\partial_{t_1} \mathcal{L} = [\partial_x, \mathcal{L}]$, or $\partial_{t_1} u_i = \partial_x u_i$, which allows one to identify t_1 with x. More precisely, the evolution in t_1 simply shifts the argument x of all functions: $u_i(x) \to u_i(x + t_1)$. The evolution in higher times can not be expressed in such a simple form.

There is another representation of the KP hierarchy which is equivalent to the one given above. In this alternative representation the Lax operator does not participate in an explicit form, only the differential operators A_m take part. We will show that the Lax equations imply the equations

$$\partial_{t_m} A_n - \partial_{t_n} A_m - [A_m, A_n] = 0$$

for all $m, n \geq 1$, which are called the Zakharov-Shabat equations, or zero curvature equations. For the proof we note that by virtue of the Lax equations it holds $\partial_{t_m} \mathcal{L}^n = [A_m, \mathcal{L}^n]$ for all n, and then

$$\begin{aligned}
& \partial_{t_m} (\mathcal{L}^n)_+ - \partial_{t_n} (\mathcal{L}^m)_+ - [A_m, A_n] \\
= \;& \Big(\partial_{t_m} \mathcal{L}^n - \partial_{t_n} \mathcal{L}^m - [A_m, A_n]\Big)_+ \\
= \;& \Big([A_m, \mathcal{L}^n] - [A_n, \mathcal{L}^m] - [A_m, A_n]\Big)_+ \\
= \;& \Big([A_m, \, \mathcal{L}^n - A_n] - [A_n, \mathcal{L}^m]\Big)_+ \\
= \;& \Big([(\mathcal{L}^m)_+, (\mathcal{L}^n)_-] - [(\mathcal{L}^n)_+, \mathcal{L}^m]\Big)_+ \\
= \;& \Big([\mathcal{L}^m, (\mathcal{L}^n)_-] + [\mathcal{L}^m, (\mathcal{L}^n)_+]\Big)_+ = \Big([\mathcal{L}^m, \mathcal{L}^n]\Big)_+ = 0.
\end{aligned}$$

The inverse statement is also true: the full collection of the Zakharov-Shabat equations imply the Lax equations. Clearly, the Zakharov-Shabat equations are equivalent to the commutation relation $[\partial_{t_m} - A_m, \, \partial_{t_n} - A_n] = 0$.

Each of the Zakharov-Shabat equations provides a closed system of a finite number of differential equations for a finite number of unknown functions. However, this system in general can not be represented in an evolution form; it contains derivatives with respect to three times $x = t_1$, t_m, t_n. For $n > m$, the system contains $n - 1$ equations for the functions $u_1, u_2, \ldots, u_{n-1}$. These systems are usually referred to as equations of the KP hierarchy. The simplest non-trivial case is $m = 2$, $n = 3$. Denoting $t_1 = x$, $t_2 = y$, $t_3 = t$, $u = 2u_1$, $w = u_2$, we arrive at a system of two equations for u and w, from which w can be excluded. The resulting equation for u is the KP equation. In general case the system can not be reduced to a single equation.

Linear problems and tau-function. The Zakharov-Shabat equations are compatibility conditions for a system of linear problems for a wave function ψ:

$$\partial_{t_m} \psi = A_m \psi.$$

Compatibility means existence of a large set of common solutions. They can be found as a series in a spectral parameter z. The spectral parameter plays a very important role, although it does not enter the linear equations explicitly. Let $\mathbf{t}$ denote the set of times t_m, $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$. The following standard notation is useful:

$$\xi(\mathbf{t}, z) = \sum_{k \geq 1} t_k z^k.$$

The wave function can be found in the form

$$\psi(x, \mathbf{t}; z) = e^{xz+\xi(\mathbf{t},z)}\Big(1 + \xi_1(x, \mathbf{t})z^{-1} + \xi_2(x, \mathbf{t})z^{-2} + \ldots\Big).$$

Introduce also the conjugated (dual) wave function ψ^* (hereafter the star does not mean the complex congugation). It satisfies the conjugate linear equations

$$-\partial_{t_m}\psi^* = A_m^\dagger \psi^*$$

and can be represented as a series of the form

$$\psi^*(x, \mathbf{t}; z) = e^{-xz-\xi(\mathbf{t},z)}\Big(1 + \xi_1^*(x, \mathbf{t})z^{-1} + \xi_2^*(x, \mathbf{t})z^{-2} + \ldots\Big).$$

It can be shown that the set of all linear problems is equivalent to the following integral relation:

$$\oint_{C_\infty} \psi(x, \mathbf{t}; z)\psi^*(x, \mathbf{t}'; z)dz = 0,$$

which holds for all $\mathbf{t}, \mathbf{t}'$, and where the contour C_∞ is a big circle around ∞ of radius $R \to \infty$. In other words, the coefficient at $1/z$ in the expansion of the expression under the integral in a Laurent series is equal to 0. In its turn, this integral relation implies existence of a function $\tau(x, \mathbf{t})$ such that

$$\psi(x, \mathbf{t}; z) = e^{xz+\xi(\mathbf{t},z)}\, \frac{\tau(x, \mathbf{t} - [z^{-1}])}{\tau(x, \mathbf{t})},$$

$$\psi^*(x, \mathbf{t}; z) = e^{-xz-\xi(\mathbf{t},z)}\, \frac{\tau(x, \mathbf{t} + [z^{-1}])}{\tau(x, \mathbf{t})},$$

where we use the notation

$$\mathbf{t} \pm [z^{-1}] = \Big\{t_1 \pm z^{-1}, t_2 \pm \tfrac{1}{2}z^{-2}, t_3 \pm \tfrac{1}{3}z^{-3}, \ldots\Big\}.$$

The function $\tau(x, \mathbf{t})$ is the tau-function of the KP hierarchy. The integral relation for the wave functions is then rewritten as the following bilinear integral functional relation for the tau-function:

$$\oint_{C_\infty} e^{\xi(\mathbf{t}-\mathbf{t}',z)}\tau(x, \mathbf{t} - [z^{-1}])\tau(x, \mathbf{t}' + [z^{-1}])dz = 0.$$

It serves as a generating relation for all equations of the KP hierarchy. Differential equations of the hierarchy are obtained by expansion of the integral bilinear relation in powers of $\mathbf{t} - \mathbf{t}'$. The coefficient functions u_i in the Lax operator are expressed as combinations of derivatives of the tau-function with respect to the times. For example,

$$u_1 = \partial_x^2 \log\tau, \quad u_2 = \frac{3}{2}(\partial_x^3 \log\tau + \partial_{t_2}\partial_x \log\tau).$$

(Recall that $\partial_x = \partial_{t_1}$.)

Later we will need a corollary of the integral bilinear relation, which is obtained from it by differentiating with respect to t_m and setting $\mathbf{t}' = \mathbf{t}$ after that. As a result, we obtain the relation

$$\frac{1}{2\pi i}\oint_{C_\infty} z^m \psi(x, \mathbf{t}; z)\psi^*(x, \mathbf{t}; z)dz = \partial_{t_m}\partial_x \log\tau(x, \mathbf{t}).$$

Correspondence with the CM system on the level of hierarchies

We have seen that the dynamics of poles of rational with respect to $x = t_1$ solutions of the KP equation in the time t_2 coincides with the Hamiltonian flow of the rational CM system with the Hamiltonian $H_2 = \mathrm{tr} L^2$. It turns out that this correspondence can be extended to the whole hierarchy: the dynamics of poles of rational with respect to $x = t_1$ solutions in any of the higher times t_m coincides with the Hamiltonian flow of the rational CM system with the Hamiltonian $H_m = \mathrm{tr} L^m$. This result was obtained by Shiota [13] in 1994. Here we present the proof in a modified form.

The tau-function for rational solutions with poles at x_j is a polynomial with the roots x_j:

$$\tau(x, \mathbf{t}) = \prod_{j=1}^{N} (x - x_j(\mathbf{t})),$$

so

$$u_1 = -\sum_{j=1}^{N} \frac{1}{(x - x_j(\mathbf{t}))^2}.$$

The wave functions ψ, ψ^* have simple poles at x_j. They can be represented in the form

$$\psi = e^{xz+\xi(\mathbf{t},z)} \Big(1 + \sum_{j=1}^{N} \frac{c_j}{x - x_j(\mathbf{t})}\Big),$$

$$\psi^* = e^{-xz-\xi(\mathbf{t},z)} \Big(1 + \sum_{j=1}^{N} \frac{c_j^*}{x - x_j(\mathbf{t})}\Big).$$

Plugging them in the relation

$$\frac{1}{2\pi i} \oint_{C_\infty} z^m \psi(x, \mathbf{t}; z) \psi^*(x, \mathbf{t}; z) dz = \partial_{t_m} \partial_x \log \tau(x, \mathbf{t}),$$

obtained above, we get:

$$\frac{1}{2\pi i} \oint_{C_\infty} dz\, z^m \Big(1 + \sum_j \frac{c_j^*}{x - x_j}\Big)\Big(1 + \sum_j \frac{c_j}{x - x_j}\Big) = \sum_j \frac{\partial_{t_m} x_j}{(x - x_j)^2}.$$

Equating the coefficients at the highest poles, we obtain:

$$\partial_{t_m} x_j = \frac{1}{2\pi i} \oint_{C_\infty} z^m c_j^* c_j dz.$$

The coefficients c_i, c_i^* can be regarded as components of the vectors

$$\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}, \qquad \mathbf{c}^* = (c_1^*, \ldots, c_N^*).$$

From the analysis of the pole dynamics in $y = t_2$ with other times fixed, we have:

$$\mathbf{c} = -(zI - L)^{-1} \mathbf{e}, \qquad \mathbf{c}^* = \mathbf{e}^{\mathrm{T}} (zI - L)^{-1},$$

where L is the Lax matrix of the rational CM system (with $g = 1$). Then

$$\partial_{t_m} x_j = \operatorname*{res}_{\infty} \sum_{k,k'} z^m \Big(\frac{1}{zI - L}\Big)_{kj} \Big(\frac{1}{zI - L}\Big)_{jk'}$$

$$= \operatorname*{res}_{\infty} \operatorname{tr}\left(z^m E \frac{1}{zI-L} E_j \frac{1}{zI-L} \right),$$

where E is the matrix whose all entries are equal to 1, and E_j is the diagonal matrix with 1 at the place jj and zeros otherwise. Recall that $E_j = -\partial L/\partial p_j$, as well as the commutation relation $[L, X] = E - I$, from which we have

$$E = LX - XL + I.$$

Now we find:

$$\operatorname{tr}\left(E \frac{1}{zI-L} E_j \frac{1}{zI-L} \right) = -\operatorname{tr}\left((LX - XL + I) \frac{1}{zI-L} \frac{\partial L}{\partial p_j} \frac{1}{zI-L} \right)$$

$$= -\operatorname{tr}\left(X \frac{1}{zI-L} \frac{\partial L}{\partial p_j} \frac{L}{zI-L} \right) + \operatorname{tr}\left(X \frac{L}{zI-L} \frac{\partial L}{\partial p_j} \frac{1}{zI-L} \right) - \operatorname{tr}\left(\frac{1}{zI-L} \frac{\partial L}{\partial p_j} \frac{L}{zI-L} \right)$$

$$= \operatorname{tr}\left(X \frac{1}{zI-L} \frac{\partial L}{\partial p_j} \right) - \operatorname{tr}\left(X \frac{\partial L}{\partial p_j} \frac{1}{zI-L} \right) + \frac{\partial}{\partial p_j} \operatorname{tr} \frac{1}{zI-L}.$$

Since X and $\partial L/\partial p_j$ are diagonal matrices, the first two terms cancel each other due to the cyclic property of the trace, and we are left with

$$\partial_{t_m} x_j = -\frac{\partial}{\partial p_j} \operatorname*{res}_{\infty} \left(z^m \operatorname{tr} \frac{1}{zI-L} \right) = \frac{\partial}{\partial p_j} \operatorname{tr} L^m = \frac{\partial H_m}{\partial p_j}.$$

We have obtained the first half of the Hamiltonian equations for the t_m-flow.

To derive the other half, we differentiate the relation

$$\partial_{t_m} x_j = -\operatorname*{res}_{\infty} \Big(z^m c_j^* c_j \Big)$$

with respect to t_2:

$$\partial_{t_m} \dot{x}_j = -\operatorname*{res}_{\infty} \Big(z^m (c_j^* \dot{c}_j + \dot{c}_j^* c_j) \Big)$$

(the t_2-derivative is denoted by the dot).

From the analysis of the pole dynamics in t_2 it follows that $\dot{\mathbf{c}} = M\mathbf{c}$, $\dot{\mathbf{c}}^* = -\mathbf{c}^* M$, where M is the M-matrix of the Lax pair fot the rational CM system (with $g = 1$). Hence

$$\partial_{t_m} p_j = \frac{1}{2} \partial_{t_m} \dot{x}_j = -\frac{1}{2} \sum_k \operatorname*{res}_{\infty} \Big(z^m (c_i^* M_{jk} c_k - c_k^* M_{kj} c_j) \Big)$$

$$= \frac{1}{2} \operatorname*{res}_{\infty} \left[z^m \operatorname{tr}\left(E \frac{1}{zI-L} [E_j, M] \frac{1}{zI-L} \right) \right]$$

$$= \operatorname*{res}_{\infty} \left[z^m \operatorname{tr}\left(LX - XL + I) \frac{1}{zI-L} \frac{\partial L}{\partial x_j} \frac{1}{zI-L} \right) \right].$$

Note that the matrix $[E_j, M] = 2\partial L/\partial x_j$ has matrix elements

$$[E_j, M]_{ik} = 2 \frac{\delta_{ij} - \delta_{jk}}{(x_i - x_k)^2}.$$

This expression has the same form as the one for $\partial_{t_m} x_i$, but instead of derivative with respect to p_j we now have the derivative with respect to x_j. Repeating the chain of equalities given above, we finally obtain:

$$\partial_{t_m} p_j = -\frac{\partial}{\partial x_j}\,\mathrm{tr} L^m = -\frac{\partial H_m}{\partial x_j}.$$

This is the second half of the Hamiltonian equations. Therefore, we have shown that the t_m-flows of the KP hierarchy correspond to the higher Hamiltonian flows of the CM system generated by the Hamiltonians $H_m = \mathrm{tr} L^m$.

The correspondence between the KP flows and Hamiltonian flows of the CM system can be extended to the trigonometric and elliptic cases. However, in these cases its form is not so simple, and the proofs become technically more difficult. In the trigonometric case, the t_m-flows of the KP hierarchy correspond to CM flows with the Hamiltonians

$$\mathcal{H}_m = \frac{1}{2(m+1)\gamma}\mathrm{tr}\Big((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\Big),$$

which are linear combinations of $H_m = \mathrm{tr} L^m$. The detailed proof can be found in [14]. The elliptic case was addressed in [15]. The result is as follows. Introduce the function $\lambda(z)$ which is defined from the equation of the spectral curve

$$\det\Big((z+\zeta(\lambda))I - L(\lambda)\Big) = 0$$

with the Lax matrix of the elliptic CM system $L(\lambda)$ depending on the spectral parameter λ. As $z \to \infty$, this function has the expansion

$$\lambda(z) = -Nz^{-1} + \sum_{m\geq 1} \mathcal{H}_m z^{-m-1}.$$

It is the generating function of the Hamiltonians

$$\mathcal{H}_m = -\operatorname*{res}_{\infty}(z^m \lambda(z))$$

of the elliptic CM system, which generate the dynamics of poles of elliptic solutions in the times t_m.

Matrix KP hierarchy and elliptic solutions of the matrix KP equation

The KP equation (and the whole hierarchy) has a matrix generalization, when the coefficient functions in the pseudo-differential Lax operator are matrices of size $n \times n$. We will show that elliptic solutions of the matrix KP equation lead to the spin CM system as the dynamics of the poles and matrix coefficients in front of the poles.

Multi-component KP hierarchy. We begin with a more general multi-component KP hierarchy; the matrix hierarchy is its subhierarchy. The independent variables are n infinite sets of continuous times

$$\mathbf{t} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n\}, \qquad \mathbf{t}_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \qquad \alpha = 1, \ldots, n.$$

In addition, it is convenient to introduce a variable x such that

$$\partial_x = \sum_{\alpha=1}^{n} \partial_{t_{\alpha,1}}.$$

The hierarchy is an infinite set of compatible evolution equations in the times $\mathbf{t}$ for matrix functions of the variable x.

In the Lax-Sato formalism, the main object is the pseudo-differential operator with matrix coefficients of the form

$$\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \dots,$$

where the coefficients $u_i = u_i(x, \mathbf{t})$ are $n \times n$ matrices. They depend on x and on all the times:

$$u_k(x, \mathbf{t}) = u_k(x + t_{1,1}, x + t_{2,1}, \dots, x + t_{n,1}; t_{1,2}, \dots, t_{n,2}; \dots).$$

One should also introduce n matrix pseudo-differential operators матричных псевдодифференциальных операторов $\mathcal{R}_1, \dots, \mathcal{R}_n$ of the form

$$\mathcal{R}_\alpha = E_\alpha + u_{\alpha,1} \partial_x^{-1} + u_{\alpha,2} \partial_x^{-2} + \dots,$$

where E_α is the diagonal $n \times n$ matrix in which the element (α, α) is equal to 1, and all other elements are 0. The operators $\mathcal{L}$, $\mathcal{R}_1, \dots, \mathcal{R}_n$ are constrained by the conditions

$$\mathcal{L}\mathcal{R}_\alpha = \mathcal{R}_\alpha \mathcal{L}, \quad \mathcal{R}_\alpha \mathcal{R}_\beta = \delta_{\alpha\beta} \mathcal{R}_\alpha, \quad \sum_{\alpha=1}^{n} \mathcal{R}_\alpha = I.$$

The Lax equations are as follows:

$$\partial_{t_{\alpha,k}} \mathcal{L} = [A_{\alpha,k},\, \mathcal{L}], \quad \partial_{t_{\alpha,k}} \mathcal{R}_\beta = [A_{\alpha,k},\, \mathcal{R}_\beta], \quad A_{\alpha,k} = \left(\mathcal{L}^k \mathcal{R}_\alpha\right)_+, \quad k = 1, 2, 3, \dots.$$

It is instructive to introduce the matrix pseudo-differential "wave operator" (or dressing operator) $\mathcal{W}$ with matrix elements

$$\mathcal{W}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k \geq 1} \xi_{k,\alpha\beta}(x, \mathbf{t}) \partial_x^{-k},$$

where $\xi_{k,\alpha\beta}(x, \mathbf{t})$ are some matrix functions. The operators $\mathcal{L}$ and $\mathcal{R}_\alpha$ are obtained from the "bare" operators $I\partial_x$ and E_α by the "dressing" with the help of the wave operator:

$$\mathcal{L} = \mathcal{W} \partial_x \mathcal{W}^{-1}, \qquad \mathcal{R}_\alpha = \mathcal{W} E_\alpha \mathcal{W}^{-1}.$$

It is clear that there is a freedom in the definition of the wave operator: it can be multiplied from the right by an arbitrary pseudo-differential operator with constant coefficients, i.e., by a series in inverse powers of ∂_x starting from I such that it commutes with E_α for all α.

An important role in the theory is played by the wave function Ψ and its dual Ψ^*. The wave function is defined as the result of action of the wave operator to the exponential function:

$$\Psi(x, \mathbf{t}; z) = \mathcal{W} \exp\Big(xzI + \sum_{\alpha=1}^{n} E_\alpha \xi(\mathbf{t}_\alpha, z)\Big).$$

By definition, the operators ∂_x^{-k} act as $\partial_x^{-k} e^{xz} = z^{-k} e^{xz}$. Clearly, the expansion of the wave function as $z \to \infty$ has the following form:

$$\Psi_{\alpha\beta}(x, \mathbf{t}; z) = e^{xz+\xi(\mathbf{t}_\beta, z)}\Big(\delta_{\alpha\beta} + \xi_{1,\alpha\beta} z^{-1} + \xi_{2,\alpha\beta} z^{-2} + \ldots\Big).$$

The dual function is introduced by the formula

$$\Psi^*(x, \mathbf{t}; z) = \exp\Big(-xzI - \sum_{\alpha=1}^{n} E_\alpha \xi(\mathbf{t}_\alpha, z)\Big)\mathcal{W}^{-1}.$$

Here we adopt the convention that the operators ∂_x that are contained in $\mathcal{W}^{-1}$ act not to the right but to the left. The left action is defined as $f \overleftarrow{\partial_x} \equiv -\partial_x f$.

It can be proved that the wave function satisfy the linear equations

$$\partial_{t_{\alpha,m}} \Psi(x, \mathbf{t}; z) = A_{\alpha,m} \Psi(x, \mathbf{t}; z),$$

where $A_{\alpha,m}$ is the differential operator оператор $A_{\alpha,m} = \Big(\mathcal{W} E_\alpha \partial_x^m \mathcal{W}^{-1}\Big)_+$, and the dual function satisfy the conjugated equation

$$-\partial_{t_{\alpha,m}} \Psi^*(x, \mathbf{t}; z) = \Psi^*(x, \mathbf{t}; z) A_{\alpha,m}.$$

Here the operator $A_{\alpha,m}$ acts to the left.

Matrix KP hierarchy. The matrix KP hierarchy is a subhierarchy of the multi-component KP which is obtained by the following restriction of the independent variables: $t_{\alpha,m} = t_m$ for all α and m, so that the vector field ∂_{t_m} coincides with $\sum\limits_{\alpha=1}^{n} \partial_{t_{\alpha,m}}$. The wave function of the matrix KP hierarchy has the expansion

$$\Psi_{\alpha\beta}(x, \mathbf{t}; z) = e^{xz+\xi(\mathbf{t}, z)} \left(\delta_{\alpha\beta} + \xi_{1,\alpha\beta}(\mathbf{t}) z^{-1} + O(z^{-2})\right),$$

where $\xi(\mathbf{t}, z) = \sum\limits_{k\geq 1} t_k z^k$. The wave function and its dual satisfy the linear equations

$$\partial_{t_m} \Psi(\mathbf{t}; z) = A_m \Psi(\mathbf{t}; z), \qquad -\partial_{t_m} \Psi^\dagger(\mathbf{t}; z) = \Psi^\dagger(\mathbf{t}; z) B_m, \qquad m \geq 1,$$

where A_m is the differential operator $A_m = \Big(\mathcal{W} \partial_x^m \mathcal{W}^{-1}\Big)_+$. At $m = 1$ we have $\partial_{t_1} \Psi = \partial_x \Psi$, so we can identify $\partial_x = \partial_{t_1} = \sum\limits_{\alpha=1}^{N} \partial_{t_{\alpha,1}}$. This means that the evolution in t_1 is simply a shift of x: $\xi_k(x, t_1, t_2, \ldots) = \xi_k(x + t_1, t_2, \ldots)$. At $m = 2$ we have the linear equations

$$\partial_{t_2} \Psi = \partial_x^2 \Psi + 2V(x, \mathbf{t})\Psi,$$

$$-\partial_{t_2} \Psi^* = \partial_x^2 \Psi^* + 2\Psi^* V(x, \mathbf{t})$$

(note that Ψ, Ψ^* and V are matrices, and the order is important), which have the form of matrix non-stationary Schrodinger equations with the potential

$$V(x, \mathbf{t}) = -\partial_x \xi_1(x, \mathbf{t}).$$

Elliptic solutions. Consider solutions that are elliptic functions of x, and find their evolution in the time t_2. Unlike in the scalar case, where the coefficient in front of each pole was equal to a fixed constant, in the matrix case these coefficients are dynamical variables, and their dynamics should be found together with the dynamics of poles. This problem was solved in the paper [17], where it was shown that this dynamics coincides with that of the spin CM system.

Like in the scalar case, we address the linear problems. Suppose that the wave functions Ψ, Ψ^* (and the coefficient ξ_1), as functions of x, have simple poles at N points x_i, $i = 1, \dots, N$. One can show (here we omit the arguments) that the residues at the poles are matrices of rank 1. Then it is natural to parametrize them by column vectors $\mathbf{a}_i = (a_i^1, a_i^2, \dots, a_i^n)^{\mathrm{T}}$, $\mathbf{b}_i = (b_i^1, b_i^2, \dots, b_i^n)^{\mathrm{T}}$:

$$\xi_{1,\alpha\beta} = S_{\alpha\beta} - \sum_i a_i^\alpha b_i^\beta \zeta(x - x_i),$$

where $S_{\alpha\beta}$ does not depend on x. Therefore,

$$V_{\alpha\beta}(x, \mathbf{t}) = -\sum_i a_i^\alpha b_i^\beta \wp(x - x_i).$$

Components of the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ are going to be the spin variables in the spin CM system.

Like in the scalar case, the wave functions can be represented as linear combinations of the elementary double-Bloch functions:

$$\Psi_{\alpha\beta} = e^{zx+\xi(\mathbf{t},z)} \sum_i a_i^\alpha c_i^\beta \Phi(x - x_i, \lambda),$$

$$\Psi^*_{\alpha\beta} = e^{-zx-\xi(\mathbf{t},z)} \sum_i c_i^{*\alpha} b_i^\beta \Phi(x - x_i, -\lambda),$$

where c_i^α, $c_i^{*\alpha}$ are components of some x-independent vectors $\mathbf{c}_i = (c_i^1, \dots, c_i^n)^{\mathrm{T}}$, $\mathbf{c}_i^* = (c_i^{*1}, \dots, c_i^{*n})^{\mathrm{T}}$.

Consider first the linear equation for Ψ. Substituting the explicit form of Ψ and V, we see that the both sides have poles at $x = x_i$ up to the third order. Equating the coefficients in front of the poles of different orders, we arrive at the conditions

- at $\frac{1}{(x-x_i)^3}$: $\quad b_i^\nu a_i^\nu = 1;$
- at $\frac{1}{(x-x_i)^2}$: $\quad -\dfrac{1}{2}\,\dot{x}_i c_i^\beta - \sum\limits_{j\neq i} b_i^\nu a_j^\nu c_j^\beta \Phi(x_i - x_j, \lambda) = z c_i^\beta;$
- at $\frac{1}{x-x_i}$:
$$\partial_{t_2}(a_i^\alpha c_i^\beta) = \wp(\lambda) a_i^\alpha c_i^\beta - 2\sum_{j\neq i} a_i^\alpha b_i^\nu a_j^\nu c_j^\beta \Phi'(x_i - x_j, \lambda) - 2c_i^\beta \sum_{j\neq i} a_i^\nu b_j^\nu a_j^\alpha \wp(x_i - x_j),$$

where dot means the t_2-derivative. Here and below summation over repeated Greek indices is assumed. In a similar way, cancellation of poles in the linear problem for the dual wave function Ψ^* leads to the conditions

- at $\frac{1}{(x-x_i)^3}$: $\quad b_i^\nu a_i^\nu = 1$ (as above);

- at $\frac{1}{(x-x_i)^2}$: $\quad -\dfrac{1}{2}\,\dot x_i c_i^{*\alpha} - \sum\limits_{j\neq i} c_j^{*\alpha} b_j^{\nu} a_i^{\nu}\Phi(x_j-x_i,\lambda) = z c_i^{*\alpha};$

- at $\frac{1}{x-x_i}$: $\quad \partial_{t_2}(c_i^{*\alpha} b_i^{\beta}) = -\wp(\lambda) c_i^{*\alpha} b_i^{\beta}$

$$+\,2\sum_{j\neq i} c_j^{*\alpha} b_j^{\nu} a_i^{\nu} b_i^{\beta}\Phi'(x_j-x_i,\lambda) + 2c_i^{*\alpha}\sum_{j\neq i} b_i^{\nu} a_j^{\nu} b_j^{\beta}\wp(x_i-x_j).$$

We have used the obvious property $\Phi(x,-\lambda) = -\Phi(-x,\lambda)$. The conditions coming from cancellation of the third order poles are constraints on the vectors $\mathbf{a}_i$, $\mathbf{b}_i$. The other conditions in matrix form can be written as follows:

$$\left\{\begin{array}{l} (zI-L(\lambda))\mathsf{c}^{\beta}=0,\\[2ex] \dot{\mathsf{c}}^{\beta}=M(\lambda)\mathsf{c}^{\beta},\end{array}\right. \qquad \left\{\begin{array}{l} \mathsf{c}^{*\alpha}(zI-L(\lambda))=0,\\[2ex] \dot{\mathsf{c}}^{*\alpha}=\mathsf{c}^{*\alpha}M^{*}(\lambda),\end{array}\right.$$

where $\mathsf{c}^{\beta} = (c_1^{\beta},\dots,c_N^{\beta})^{\mathrm{T}}$, $\mathsf{c}^{*\alpha} = (c_1^{*\alpha},\dots,c_N^{*\alpha})$ are N-component vectors, and $L(\lambda)$, $M(\lambda)$, $M^*(\lambda)$ are $N\times N$ matrices

$$L_{ij}(\lambda) = -\frac{1}{2}\,\dot x_i\delta_{ij} - (1-\delta_{ij}) b_i^{\nu} a_j^{\nu}\Phi(x_i-x_j,\lambda),$$

$$M_{ij}(\lambda) = (\wp(\lambda)-\Lambda_i)\delta_{ij} - 2(1-\delta_{ij}) b_i^{\nu} a_j^{\nu}\Phi'(x_i-x_j,\lambda),$$
$$M^*_{ij}(\lambda) = -(\wp(\lambda)-\Lambda^*_i)\delta_{ij} + 2(1-\delta_{ij}) b_i^{\nu} a_j^{\nu}\Phi'(x_i-x_j,\lambda).$$

Here

$$\Lambda_i = \frac{\dot a_i^{\alpha}}{a_i^{\alpha}} + 2\sum_{j\neq i}\frac{a_j^{\alpha} b_j^{\nu} a_i^{\nu}}{a_i^{\alpha}}\,\wp(x_i-x_j), \quad -\Lambda_i^* = \frac{\dot b_i^{\alpha}}{b_i^{\alpha}} - 2\sum_{j\neq i}\frac{b_i^{\nu} a_j^{\nu} b_j^{\alpha}}{b_i^{\alpha}}\,\wp(x_i-x_j)$$

does not depend on the index α (in these formulas there is summation over ν but no summation over α). In fact $\Lambda_i = \Lambda_i^*$, so $M^*(\lambda) = -M(\lambda)$. Indeed, multiplying the formulas for Λ_i, Λ_i^* by $a_i^{\alpha} b_i^{\alpha}$ (no summation!), summing over α and then summing the two equations, we get $\Lambda_i - \Lambda_i^* = \partial_{t_2}(a_i^{\alpha} b_i^{\alpha}) = 0$ by virtue of the constraint $a_i^{\alpha} b_i^{\alpha} = 1$.

The condition of compatibility of the linear system states that

$$(\dot L + [L,M])\mathsf{c}^{\beta} = 0.$$

Write the equations for Λ_i, Λ_i^* in the form

$$\dot a_i^{\alpha} = \Lambda_i a_i^{\alpha} - 2\sum_{j\neq i} a_j^{\alpha} b_j^{\nu} a_i^{\nu}\wp(x_i-x_j),$$

$$\dot b_i^{\alpha} = -\Lambda_i b_i^{\alpha} + 2\sum_{j\neq i} b_i^{\nu} a_j^{\nu} b_j^{\alpha}\wp(x_i-x_j)$$

(in this form they look like equations of motion for the spin variables). They guarantee vanishing of the off-diagonal matrix elements of the matrix $\dot L + [L,M]$. Vanishing of the diagonal elements provides equations of motion for the poles x_i:

$$\ddot x_i = 4\sum_{j\neq i} b_i^{\mu} a_j^{\mu} b_j^{\nu} a_i^{\nu}\wp'(x_i-x_j).$$

The gauge transformation $a_i^\alpha \to a_i^\alpha q_i$, $b_i^\alpha \to b_i^\alpha q_i^{-1}$ with $q_i = \exp\Bigl(\int^{t_2} \Lambda_i dt\Bigr)$ gets rid of Λ_i, so we can put $\Lambda_i = 0$. This gives the equations of motion:

$$\dot a_i^\alpha = -2\sum_{j\neq i} a_j^\alpha b_j^\nu a_i^\nu \wp(x_i - x_j), \quad \dot b_i^\alpha = 2\sum_{j\neq i} b_i^\nu a_j^\nu b_j^\alpha \wp(x_i - x_j)$$

obtained from the Lax equation before.

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