Integrable systems of particles and nonlinear equations. Lecture 8

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The Kadomtsev-Petviashvili equation and its singular solutions

Starting from this section, we will deal with integrable nonlinear equations and will show that their finite-dimensional reductions give systems of the CM and RS type. More precisely, let us consider solutions of a nonlinear integrable equation such that they have poles in one of the variables (usually, the space variable). Then the dynamics of the poles in another (time) variable turns out to be given by equations of motion of a system of the CM or RS type, i.e., the poles move in time as particles of an integrable many-body system. The proof of this statement suggested by Krichever simultaneously provides one with a systematic method of finding the Lax pairs for many-body systems. We will start with the Kadomtsev-Petviashvili (KP) equation introduced in 1970. It is related to the CM system.

The KP equation

The KP equation for a function $u = u(x, y, t)$ of 3 variables has the form

$$3u_{yy} = \Big(4u_t - 6uu_x - u_{xxx}\Big)_x.$$

In Physics, this equation is called KP2; the equation KP1 is obtained from it by making the variable y purely imaginary: $y \to iy$. Properties of solutions and their physical applications in these two cases are very different, but we will not discuss them because we are mainly interested in alrebraic matters. Solutions of the KP equation that do not depend on y satisfy the well known Korteveg-de-Vries (KdV) equation $4u_t = 6uu_x + u_{xxx}$.

The KP equation has a commutation representation as the condition of commutativity of certain differential operators. Consider the differential operators

$$A_2 = \partial_x^2 + u, \quad A_3 = \partial_x^3 + \tfrac{3}{2}\, u\partial_x + w.$$

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Problem. Show that the KP equation is equivalent to the condition $[\partial_y - A_2, \partial_t - A_3] = 0$, or

$$\partial_t A_2 - \partial_y A_3 + [A_2, A_3] = 0.$$

This condition is called the Zakharov-Shabat equation, or the zero curvature representation.

The Zakharov-Shabat equation is the compatibility condition for the overdetermined system of two linear problems

$$\partial_y \psi = A_2 \psi, \quad \partial_t \psi = A_3 \psi$$

for a function ψ, which is called the wave function, since the first equation has the form of the non-stationary Schredinger equation in imaginary time with the potential u. The function ψ plays a very important role in constructing exact solutions to the KP equation: the method is to solve the linear problems first. It is very important for what follows that the compatibility of the linear problems implies existence of a whole family of solutions $\psi = \psi(z)$ depending on a complex parameter z, which does not enter the equations. It is called the spectral parameter. For example, if $u = w = 0$, then the solution for the wave function depending on the spectral parameter is of the form $\psi = e^{xz+yz^2+tz^3}$.

The KP equation can be represented in the bilinear form.

Problem. Show that the substitution $u = 2\partial_x^2 \log \tau$ brings the KP equation to a bilinear equation for the function τ.

The function τ is called the tau-function. It plays a very important role in the theory of the KP and other integrable equations.

The KP equation has a lot of exact solutions of a very different nature. Among them, mostly well known are soliton solutions which can be found explicitly in a closed form. For example, the one-soliton solution has the form

$$u(x, y, t) = \frac{(p-q)^2}{2\cosh^2\left(\frac{1}{2}(p-q)x + \frac{1}{2}(p^2-q^2)y + \frac{1}{2}(p^3-q^3)t\right)},$$

where p, q are parameters. Note that this solution, extended to the complex plane, has an imaginary period and one second order pole in x in the fundamental domain. Position of the pole depends on y и t. There exist also solutions with a rational dependence on x, as well as periodic solutions with one or two periods having N poles x_i in the fundamental domain. We will show that the poles x_i as functions of y move as particles of the CM system (rational, trigonometric or elliptic).

Rational solutions of the KP equation and dynamics of their poles

We begin with rational solutions. It is easy to see that if a solution of the KP equation has a pole in x at some point x_i, then this pole has to be of the second order with zero residue and with the coefficient -2 in front of it, so any rational solution decreasing to zero at infinity can be represented in the form

$$u(x, y, t) = -2\sum_{j=1}^{N} \frac{1}{(x - x_j(y, t))^2}.$$

The tau-function is then a polynomial in x:

$$\tau(x,y,t)=\prod_{j=1}^{N}(x-x_j(y,t)).$$

The roots (which are poles of u) depend on y,t. We will assume that all of them are distinct, i.e., all zeros of the tau-function are of the first order.

Substituting the function u in this form into the KP equation, one arrives at rather complicated expressions from which it is difficult to see that the variables y,t actually separate. The problem of the pole dynamics of rational solutions to the KP equation was solved by Krichever in 1978. His method consists in the substitution of the pole ansats for u not into the nonlinear equation but into the corresponding linear equation for the wave function. This allows one to separate the variables from the very beginning and to obtain the Lax representation for the CM system. Of course, applying this method, one should somehow derive or assume the corresponding pole ansats for the ψ-function. It is easy to see that for the rational solutions of the type discussed above the wave function has to have simple poles at the points x_i.

Let us present some details of the derivation of the equations of motion for x_j's. Consider their dependence on y. This variable will play the role of time, and derivatives with respect to it will be denoted by dot. The linear equation for ψ has the form

$$\dot{\psi}=\partial_x^2\psi+u\psi,$$

where u is the sum of terms with second order poles. The function ψ will be found in the form

$$\psi=e^{xz+yz^2}\left(c_0(z)+\sum_{i=1}^{N}\frac{c_i(z,y)}{x-x_i(y)}\right),$$

where z is the spectral parameter and c_i are some x-independent coefficients. Plugging this in the linear problem, we get:

$$e^{-xz-tz^2}(\partial_t-\partial_x^2)\left[e^{xz+tz^2}\Big(c_0+\sum_{i=1}^{N}\frac{c_i}{x-x_i}\Big)\right]+2\Big(\sum_{i=1}^{N}\frac{1}{(x-x_i)^2}\Big)\Big(c_0+\sum_{i=1}^{N}\frac{c_i}{x-x_i}\Big)=0.$$

The left-hand side is a rational function of x with poles of the first and second orders at $x=x_i$ (possible poles of the third order cancel identically) such that it is equal to 0 at infinity. Therefore, it is enough to choose the coefficients c_i so that all the poles cancel. Equating the coefficients in front of each pole to 0, we obtain the following system of $2N$ linear equations for the coefficients $c_1,\dots,c_N$:

$$\left\{\begin{array}{l}(\dot{x}_i+2z)c_i+2\displaystyle\sum_{k\neq i}\frac{c_k}{x_i-x_k}=-2c_0 \quad \text{(сокращение полюсов 2-го порядка)},\\[3ex] \dot{c}_i+2c_i\displaystyle\sum_{k\neq i}\frac{1}{(x_i-x_k)^2}-2\sum_{k\neq i}\frac{c_k}{(x_i-x_k)^2}=0 \quad \text{(сокращение полюсов 1-го порядка)}.\end{array}\right.$$

The coefficient c_0 can be put equal to 1 because it only changes the common multiplier of the ψ-function. These equations van be compactly written in the matrix form:

$$\left\{\begin{array}{l}(L-zI)\mathbf{c}=c_0\mathbf{e},\\[2ex] \dot{\mathbf{c}}=M\mathbf{c},\end{array}\right.$$

where I is the unity matrix, $\mathbf{c} = (c_1, \dots, c_N)^{\mathrm{T}}$, $\mathbf{e} = (1, \dots, 1)^{\mathrm{T}}$ are N-component column vectors, and $N \times N$ matrices L, M are of the form

$$L_{ik} = -\frac{1}{2}\,\dot{x}_i \delta_{ik} - \frac{1-\delta_{ik}}{x_i - x_k}\,,$$

$$M_{ik} = -\delta_{ik} \sum_{j \neq i} \frac{2}{(x_i - x_j)^2} + \frac{2(1-\delta_{ik})}{(x_i - x_k)^2}.$$

We recognize the Lax pair for the rational CM system with $g = 1$. Compatibility of the overdetermined system for c_i's implies the Lax equation

$$\dot{L} + [L, M] = 0,$$

which is equivalent to the equations of motion, as we saw before. The function ψ is then found as

$$\psi = e^{xz + yz^2}\Big(1 - \mathbf{e}^{\mathrm{T}}(xI - X(y))^{-1}(zI - L(y))^{-1}\mathbf{e}\Big),$$

where $X = \mathrm{diag}(x_1, \dots, x_N)$.

Elliptic solutions to the KP equation

Let us now consider elliptic (i.e., double-periodic) in x solutions to the KP equation. Such solutions must have poles. As was already argued, these poles must be of the second order with zero residue, with the coefficient -2 in front of each pole. Consider a solution with N such poles in the fundamental domain. The general form of a double-periodic function u satisfying these conditions is

$$u(x, y, t) = -2\sum_{i=1}^{N} \wp(x - x_i(y, t)) + c,$$

where c is a constant. The corresponding tau-function has the form

$$\tau = e^{Q(x,y,t)} \prod_{i=1}^{N} \sigma(x - x_i(y, t)),$$

where $Q(x, y, t)$ is a quadratic form in the variables x, y, t.

As before, we consider the dynamics of the poles in y which in this context plays the role of time. We apply the same method, addressing the linear problems. In the linear equation for ψ,

$$\dot{\psi} = \partial_x^2 \psi + u\psi,$$

the coefficient function u is double-periodic, so it is natural to look for solutions among double-Bloch functions, i.e., the functions that satisfy the conditions $\psi(x + 2\omega_\alpha) = B_\alpha \psi(x)$ with some Bloch multipliers B_α. The simplest non-constant double-Bloch function is the exponential function e^{ax} with arbitrary constant a; all other double-Bloch functions must have poles. The transformation $\psi(x) \mapsto \tilde{\psi}(x) = \psi(x)e^{ax}$ does not change position of the poles of any function and preserves the double-Bloch property. Let B_α be the Bloch multipliers for ψ, then the ones for $\tilde{\psi}$ are $\tilde{B}_\alpha = B_\alpha e^{2a\omega_\alpha}$. We call two pairs of Bloch multipliers equivalent if they are connected by such a transformation with some a. Note that for all pairs of equivalent Bloch multipliers the quantity $B_1^{\omega_2} B_2^{-\omega_1}$ is the same and depends on the equivalence class only.

The simplest double-Bloch function that is not equivalent to the exponential function is the Lamé-Hermite function

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)} e^{-\zeta(\lambda)x}.$$

As a function of x, it has a single simple pole in the fundamental domain (at $x=0$), and the Bloch multipliers are $e^{2(\zeta(\omega_\alpha)\lambda - \zeta(\lambda)\omega_\alpha)}$. A double-Bloch solution to the linear problem can be found as a linear combination of the Lamé-Hermite functions with poles at the points x_i multiplied by an exponential function:

$$\psi = e^{xz+yz^2} \sum_{i=1}^{N} c_i \Phi(x - x_i, \lambda).$$

The coefficients c_i may depend on y and z but not on x. The Bloch multipliers for this function are

$$B_\alpha = e^{2(\omega_\alpha z + \zeta(\omega_\alpha)\lambda - \zeta(\lambda)\omega_\alpha)}.$$

This representation is analogous to the representation of a rational function in the form of a sum of pole terms.

The function $\tilde{\psi} = -\partial_y \psi + \partial_x^2 \psi + u\psi$ is a double-Bloch function, too, with the same Bloch multipliers. If one chooses the coefficients c_i in such a way that it has no poles, then it can be only equal to an exponential function of the form Ce^{ax}, which, however, has a pair of Bloch multipliers not equivalent to B_1, B_2. Therefore, $C=0$ and the function $\tilde{\psi}$ is identically zero. Plugging u and ψ in the linear problem in the form written above, we have:

$$-\sum_i \dot{c}_i \Phi(x-x_i) + \sum_i c_i \dot{x}_i \Phi'(x-x_i) + 2z \sum_i c_i \Phi'(x-x_i) + \sum_i c_i \Phi''(x-x_i)$$
$$-2\left(\sum_i \wp(x-x_i)\right)\left(\sum_k c_k \Phi(x-x_k)\right) + c\sum_i c_i \Phi(x-x_i) = 0,$$

where we have omitted the second argument of the function Φ for brevity. Different terms of this expression have poles at $x = x_i$. It easy to see that poles of the third order cancel identically. The condition of cancellation of the second and first order poles have the form

$$c_i \dot{x}_i = -2zc_i - 2\sum_{j\neq i} c_j \Phi(x_i - x_j),$$

$$\dot{c}_i = (c + \wp(\lambda))c_i - 2\sum_{j\neq i} c_j \Phi'(x_i - x_j) - 2c_i \sum_{j\neq i} \wp(x_i - x_j).$$

Introducing $N \times N$ matrices

$$L_{ij}(\lambda) = -\tfrac{1}{2}\dot{x}_i \delta_{ij} - 2(1-\delta_{ij})\Phi(x_i - x_j, \lambda),$$

$$M_{ij}(\lambda) = \delta_{ij}(\wp(\lambda) + c) - 2\delta_{ij}\sum_{k\neq i} \wp(x_i - x_k) - 2(1-\delta_{ij})\Phi'(x_i - x_j, \lambda),$$

we can rewrite these conditions in the matrix form as a system of linear equations for the vector $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$:

$$\left\{ \begin{array}{l} L(\lambda)\mathbf{c} = z\mathbf{c}, \\ \\ \dot{\mathbf{c}} = M(\lambda)\mathbf{c}. \end{array} \right.$$

We have obtained the Lax pair for the elliptic CM system (the extra term in the matrix M proportional to the unity matrix is irrelevant because it cancels in the Lax equation). Therefore, we conclude that the dynamics of poles of elliptic solutions to the KP equation is isomorphic to the CM dynamics with the elliptic potential and the Hamiltonian

$$H_2 = \sum_i p_i^2 - \sum_{i \neq j} \wp(x_i - x_j).$$

The coupling constant g (equal to 1 here) can be easily restored by a rescaling of the variables and periods.

Problem. Consider the dynamics of poles in the variable t and show that it is isomorphic to the CM Hamiltonian flow with the Hamiltonian

$$H_3 = -\sum_i p_i^3 + 3 \sum_{i \neq j} p_i \wp(x_i - x_j).$$

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