Integrable systems of particles and nonlinear equations. Lecture 5

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The Ruijsenaars-Schneider systems

The CM systems admit a deformation that is very different from the ones discussed in the previous section. The Hamiltonian of the deformed system is no longer a sum of the kinetic and potential terms. This deformation is often referred to as a relativistic generalization of the CM systems, and the corresponding systems are called the Ruijsenaars-Schneider (RS) systems. They were introduced in 1987 (see [9, 10]). These systems are integrable and exist in the rational, trigonometric and elliptic versions. We will start with the rational version, and then consider the elliptic ones as the most general case; all others are obtained from them via a degeneration (when one or two periods tend to infinity).

The rational RS system

Hamiltonian and equations of motion. The Hamiltonian has the form

$$H_+ = \eta^{-2} \sum_{i=1}^{N} e^{\eta p_i} \prod_{j\neq i} \frac{(x_{ij}+\eta)^{1/2}(x_{ij}-\eta)^{1/2}}{x_{ij}},$$

where p_i, x_i are canonically conjugate moments and coordinates, $x_{ij} = x_i - x_j$, and $\eta > 0$ is a parameter. In the limit $\eta \to 0$ the Hamiltonian of the rational CM system is reproduced. Indeed, expanding in powers of η, we have:

$$H_+ = \eta^{-2} N + \eta^{-1} \sum_i p_i + \frac{1}{2}\Big(\sum_i p_i^2 - \sum_{i\neq j} \frac{1}{x_{ij}^2}\Big) + O(\eta),$$

i.e., subtracting the singular terms, the first of which is just a constant, and the other one is the total momentum, we obtain, in the limit $\eta \to 0$, the CM Hamiltonian with $g = 1$. In fact there is no loss of generality because by rescaling of the coordinates one can achieve any value of the coupling constant.

To shorten the formulas below, let us denote

$$V_i = \prod_{j\neq i} \frac{(x_{ij}+\eta)^{1/2}(x_{ij}-\eta)^{1/2}}{x_{ij}}.$$

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The Hamiltonian equations of motion are as follows:

$$\dot{x}_i = \frac{\partial H_+}{\partial p_i} = \eta^{-1} e^{\eta p_i} V_i,$$

$$\dot{p}_i = -\frac{\partial H_+}{\partial x_i} = -\frac{1}{2\eta^2} \sum_{j \neq i} \Big(e^{\eta p_i} V_i + e^{\eta p_j} V_j \Big) \Big(\frac{1}{x_{ij} + \eta} + \frac{1}{x_{ij} - \eta} - \frac{2}{x_{ij}} \Big).$$

Problem. Show that the Newtonian equations of motion are of the form

$$\ddot{x}_i = -\sum_{j \neq i} \dot{x}_i \dot{x}_j \Big(\frac{1}{x_{ij} + \eta} + \frac{1}{x_{ij} - \eta} - \frac{2}{x_{ij}} \Big) = -2 \sum_{j \neq i} \frac{\eta^2 \dot{x}_i \dot{x}_j}{x_{ij}(x_{ij} + \eta)(x_{ij} - \eta)}.$$

The RS system is sometimes referred to as a relativistic generalization of the CM system. This is justified by the following reasoning. Along with H_+ introduce also

$$H_- = \eta^{-2} \sum_{i=1}^{N} e^{-\eta p_i} \prod_{j \neq i} \frac{(x_{ij} + \eta)^{1/2} (x_{ij} - \eta)^{1/2}}{x_{ij}}.$$

Problem. Prove that $\{H_+, H_-\} = 0$.

Hint: this is based on the identity

$$\sum_{i \neq k} \frac{1}{x_{ik}^3} \prod_{j \neq i,k} \Big(1 - \frac{\eta^2}{x_{ij}^2} \Big) = 0,$$

which can be proved by considering the left-hand side as a rational function of, say x_1 and verifying that it has no singularities and equals 0 at infinity.

Introduce the quantities

$$H = H_+ + H_- = 2\eta^{-2} \sum_i \cosh(\eta p_i) V_i,$$

$$P = H_+ - H_- = 2\eta^{-2} \sum_i \sinh(\eta p_i) V_i$$

(the Hamiltonian and momentum), and

$$B = \eta^{-1} \sum_i x_i.$$

Problem. Prove that these function satisfy the Poisson Poincare algebra

$$\{H, P\} = 0, \quad \{H, B\} = P, \quad \{P, B\} = H.$$

This algebra allows one to think about this system as a relativistic invariant one, and the parameter η has the meaning of the inverse velocity of light.

In order to get rid of the square roots, one can make a canonical transformation of the form

$$e^{\eta p_i} \to e^{\eta p_i} \prod_{j \neq i} \Big(\frac{x_{ij} + \eta}{x_{ij} - \eta} \Big)^{1/2}, \quad x_i \to x_i.$$

Problem. Check that this transformation is indeed canonical.

In the new variables the Hamiltonians $H_\pm$ are written in the form

$$H_\pm = \eta^{-2} \sum_i e^{\pm \eta p_i} \prod_{j\neq i} \frac{x_{ij} \pm \eta}{x_{ij}}.$$

Problem. Prove that for both these Hamiltonians the Newtonian equations of motion have the same form

$$\ddot{x}_i + \sum_{j\neq i} \dot{x}_i \dot{x}_j \Big(\frac{1}{x_{ij}+\eta} + \frac{1}{x_{ij}-\eta} - \frac{2}{x_{ij}} \Big) = 0$$

and take the limit to the CM equations of motion.

In what follows we will consider the Hamiltonians of the RS system without square roots (after the canonical transformation).

The Lax representation and integrals of motion. The equations of motion of the RS system have a Lax representation with the matrices

$$L_{ij} = \frac{\dot{x}_i}{x_i - x_j - \eta},$$

$$M_{ij} = \delta_{ij} \Big(\sum_{k\neq i} \frac{\dot{x}_k}{x_i - x_k} - \sum_k \frac{\dot{x}_k}{x_i - x_k + \eta} \Big) + \frac{(1-\delta_{ij})\dot{x}_i}{x_i - x_j}.$$

Of course the Lax pair can be expressed through the coordinates and moments if one makes the substitution

$$\dot{x}_i = \eta^{-1} e^{\eta p_i} \prod_{l\neq i} \frac{x_i - x_l + \eta}{x_i - x_l}.$$

Problem. Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the Newtonian equations of motion.

Note that the Lax matrix is characterized by the commutation relation

$$[X, L] - \eta L = \dot{\mathbf{x}} \mathbf{e}^{\mathrm{T}},$$

where $\mathbf{x} = (x_1, \ldots, x_N)^{\mathrm{T}}$. The matrix in the right-hand side has rank 1.

The Lax equation implies that the spectral invariants of the Lax matrix are integrals of motion. Consider, for example, the characteristic polynomial $\det(zI - L)$. It can be found explicitly, using the fact that the Lax matrix (and any its diagonal minor) is a Cauchy matrix multiplied by a diagonal matrix. The formula for determinant of Cauchy matrices is well known: известна:

$$\det_{1\leq i,j\leq n} \frac{1}{y_i - x_j} = \frac{\prod_{k<l}(y_k - y_l)(x_l - x_k)}{\prod_{k,l}(y_k - x_l)}.$$

In our case $y_i = x_i - \eta$. Hence we immediately find:

$$\det(zI - L) = z^N + \sum_{k=1}^{N} z^{N-k} I_k,$$

where the integrals of motion I_k have the form

$$I_k = \frac{\eta^{-k}}{k!} \sum_{[i_1,\dots,i_k]}^{N} \dot{x}_{i_1} \dots \dot{x}_{i_k} \prod_{\alpha<\beta}^{k} \frac{x^2_{i_\alpha i_\beta}}{x^2_{i_\alpha i_\beta} - \eta^2}.$$

The sum goes over all sets of distinct indices $i_1, \dots, i_k$. In terms of momenta we have:

$$I_k = \eta^{-2k} \sum_{\mathcal{I}\subset\{1,\dots,N\},|\mathcal{I}|=k} \exp\Big(\eta \sum_{i\in\mathcal{I}} p_i\Big) \prod_{i\in\mathcal{I},j\notin\mathcal{I}} \frac{x_{ij}+\eta}{x_{ij}}.$$

Here the sum goes over all subsets $\mathcal{I}$ of the set $\{1,\dots,N\}$ with k elements. The Hamiltonian is among these integrals: $I_1 = H_+$. Note that

$$I_N = \eta^{-2N} \exp\Big(\eta \sum_{i=1}^{N} p_i\Big).$$

One can also introduce integrals of motion I_{-k} by the formula

$$I_{-k} = \eta^{-4k} I_N^{-1} I_{N-k} = \eta^{-2k} \sum_{\mathcal{I}\subset\{1,\dots,N\},|\mathcal{I}|=k} \exp\Big(-\eta \sum_{i\in\mathcal{I}} p_i\Big) \prod_{i\in\mathcal{I},j\notin\mathcal{I}} \frac{x_{ij}-\eta}{x_{ij}}.$$

In particular, $I_{-1} = H_-$.

The fact that these integrals of motion are in involution was proved by Ruijsenaars in the most general elliptic case. His method used quantization of the RS system with the Planck's constant $\hbar$. It turns out that it is easier to prove that the integrals of motion after quantization (some shift operators) commute than to calculate the Poisson brackets between them in the classical limit. Their involutivity then follows in the limit $\hbar \to 0$.

The elliptic RS system

Hamiltonian end equations of motion. The Hamiltonians $H_\pm$ have the form

$$H_\pm = \sigma^{-2}(\eta) \sum_i e^{\pm\sigma(\eta)p_i} \prod_{j\neq i} \frac{\sigma(x_{ij}\pm\eta)}{\sigma(x_{ij})}.$$

Here and below in the trigonometric degeneration the sigma-function should be substituted by sinus. The limit $\eta \to 0$ can be found in the same way as in the rational case. After subtracting singular terms it coincides with the Hamiltonian of the elliptic CM system.

Problem. Prove that $\{H_+, H_-\} = 0$.

Problem. Obtain the Newtonian equations of motion

$$\ddot{x}_i + \sum_{j\neq i} \dot{x}_i \dot{x}_j \Big(\zeta(x_{ij}+\eta) + \zeta(x_{ij}-\eta) - 2\zeta(x_{ij})\Big) = 0.$$

Using an identity for elliptic functions, they can be also represented in the form

$$\ddot{x}_i = \sum_{j\neq i} \dot{x}_i \dot{x}_j \frac{\wp'(x_{ij})}{\wp(\eta) - \wp(x_{ij})}.$$

The Lax representation and integrals of motion. As in the case of the elliptic CM system, the elliptic RS system admits a whole family of Lax pairs which depend on a spectral parameter λ. The matrices L, M are expressed in terms of the Lamé-Hermite function $\Phi(x,\lambda)$ in the following way:

$$L_{ij}(\lambda) = \dot{x}_i \Phi(x_{ij}-\eta,\lambda),$$

$$M_{ij}(\lambda) = \delta_{ij}\Big(\sum_{k\neq i} \dot{x}_k \zeta(x_{ik}) - \sum_k \dot{x}_k \zeta(x_{ik}+\eta)\Big) + (1-\delta_{ij})\dot{x}_i \Phi(x_{ij},\lambda).$$

Problem. Prove that the Lax equation $\dot{L}(\lambda) + [L(\lambda), M(\lambda)] = 0$ is equivalent to the equations of motion.

The characteristic polynomial of the Lax matrix

$$R(z,\lambda) = \det(zI - L(\lambda))$$

is conserved and serves as a generating function for integrals of motion. Ruijsenaars proved that all of them are in involution. The equation $R(z,\lambda)=0$ defines a complex curve which is called the spectral curve. The curve is an integral of motion.

These integrals can be found in an explicit form. The Lax matrix (and any its diagonal minor) is an elliptic Cauchy matrix multiplied by a diagonal matrix. The formula for determinant of an elliptic Cauchy matrix is known:

$$\det_{1\le i,j\le n}\left(\frac{\sigma(y_i - x_j + \lambda)}{\sigma(\lambda)\sigma(y_i - x_j)}\right) = \frac{\sigma\Big(\lambda + \sum_{k=1}^n (y_k - x_k)\Big)}{\sigma(\lambda)} \frac{\prod_{k<l}\sigma(y_k - y_l)\sigma(x_l - x_k)}{\prod_{k,l}\sigma(y_k - x_l)}.$$

Problem. Prove this formula.

In our case $y_i = x_i - \eta$. Hence we immediately find:

$$\det(zI - L(\lambda)) = z^N + \sum_{k=1}^N z^{N-k} \frac{\sigma(\lambda - k\eta)}{\sigma(\lambda)} I_k,$$

where the integrals of motion I_k have the form

$$I_k = \frac{\sigma^{-k}(\eta)}{k!} \sum_{[i_1,\dots,i_k]}^N \dot{x}_{i_1}\dots\dot{x}_{i_k} \prod_{\alpha<\beta}^k \frac{\sigma^2(x_{i_\alpha i_\beta})}{\sigma(x_{i_\alpha i_\beta}+\eta)\sigma(x_{i_\alpha i_\beta}-\eta)}.$$

The sum goes over all sets of distinct indices $i_1,\dots,i_k$. In terms of momenta we have:

$$I_k = \sigma^{-2k}(\eta) \sum_{\mathcal{I}\subset\{1,\dots,N\}, |\mathcal{I}|=k} \exp\Big(\sigma(\eta)\sum_{i\in\mathcal{I}} p_i\Big) \prod_{i\in\mathcal{I}, j\notin\mathcal{I}} \frac{\sigma(x_{ij}+\eta)}{\sigma(x_{ij})}.$$

Here the summation goes over all subsets $\mathcal{I}$ of the set $\{1,\dots,N\}$ with k elements. The Hamiltonian is among these integrals: $I_1 = H_+$. Note that

$$I_N = \sigma^{-2N}(\eta)\exp\Big(\sigma(\eta)\sum_{i=1}^N p_i\Big).$$

One can also introduce integrals of motion I_{-k}:

$$I_{-k} = \sigma^{-4k}(\eta) I_N^{-1} I_{N-k} = \sigma^{-2k}(\eta) \sum_{\mathcal{I}\subset\{1,\dots,N\}, |\mathcal{I}|=k} \exp\Big(-\sigma(\eta)\sum_{i\in\mathcal{I}} p_i\Big) \prod_{i\in\mathcal{I}, j\notin\mathcal{I}} \frac{\sigma(x_{ij}-\eta)}{\sigma(x_{ij})}.$$

In particular, $I_{-1} = H_-$.

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