# Integrable systems of particles and nonlinear equations. Lecture 4

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## The CM system with elliptic potential

The interaction potential of the trigonometric CM system admits a further deformation, for which it becomes a double-periodic (elliptic) function in the complex plane with second order poles. We will call such systems elliptic. In the limit when the second period tends to infinity, the elliptic system becomes trigonometric. Most of the properties of rational and trigonometric systems, including integrability, have their analogs for elliptic systems, although the formulations and proofs may be more involved.

**Elliptic functions.** Not assuming that the reader is familiar with elliptic functions, we give a short introduction to the theory, recalling the basic facts and fixing the notation. More details can be found in the books [6, 7, 8].

By elliptic functions we mean meromorphic functions of a vcomplex variable with two linearly independent periods (over $\mathbb{R}$). Such functions are also called double-periodic. If one of the periods tends to infinity, the elliptic functions degenerate to trigonometric (or hyperbolic). According to the Liouville theorem, meromorphic functions that are bounded in the whole complex plane are constants, so any non-constant elliptic function must have singularities. Moreover, only one simple pole in the parallelogramm of periods (the fundamental domain) is impossible since the sum of residues must be zero. So, the simplest elliptic functions have either two simple poles with opposite residues in the fundamental domain, or one second order pole with zero residue.

We will use the Weierstrass functions: the $\wp$-function, the $\zeta$-function and the $\sigma$-function. Among them, only the $\wp$-function is double-periodic, while the other two, closely related to it, have simple monodromy properties under shifts by periods. Let $\omega_1$, $\omega_2$ be complex numbers such that $\mathcal{I}m(\omega_2/\omega_1) > 0$. For instance, we can assume that $\omega_1$ is real positive while $\omega_2$ is purely imaginary with positive imaginary part, but from the algebraic point of view this is not necessary. The points $s = 2\omega_1 n_1 + 2\omega_2 n_2$, $n_1, n_2 \in \mathbb{Z}$ form a lattice $\Lambda$ in the complex plane. The factor space of the complex plane over this lattice, $\mathcal{E} = \mathbb{C}/\Lambda$, is a torus (an elliptic curve) realized as a parallelogramm with identified opposite sides.

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The Weierstrass function $\wp(z)$ is defined by the convergent series

$$\wp(z) = \frac{1}{z^2} + \sum_{s\neq 0}\Big(\frac{1}{(z-s)^2} - \frac{1}{s^2}\Big), \quad s = 2\omega_1 n_1 + 2\omega_2 n_2, \quad n_1, n_2 \in \mathbb{Z}.$$

It is an even double-periodic function with periods $2\omega_1$, $2\omega_2$: $\wp(z+2\omega_\alpha) = \wp(z)$, $\alpha = 1, 2$. At all points of the lattice it has second order poles, with no other singularities. In a small neighborhood of 0 it can be expanded as

$$\wp(z) = \frac{1}{z^2} + O(z^2), \quad z \to 0$$

(note the absence of the constant term of the expansion).

The function $\zeta(z)$ is a promitive of the $\wp$-function (with the sign minus): $\zeta'(z) = -\wp(z)$, i.e.,

$$\zeta(z) = \frac{1}{z} - \int_0^z \Big(\wp(x) - \frac{1}{x^2}\Big)dx.$$

It is defined by the series

$$\zeta(z) = \frac{1}{z} + \sum_{s\neq 0}\Big(\frac{1}{z-s} + \frac{1}{s} + \frac{z}{s^2}\Big), \quad s = 2\omega_1 n_1 + 2\omega_2 n_2, \quad n_1, n_2 \in \mathbb{Z}.$$

It is an odd function which is already not double-periodic, but it transforms in a very simple way under shifts by periods:

$$\zeta(z + 2\omega_\alpha) = \zeta(z) + 2\eta_\alpha, \quad \alpha = 1, 2,$$

where $\eta_\alpha$ are some constants depending on $\omega_1, \omega_2$, and $\eta_\alpha = \zeta(\omega_\alpha)$, as it can be easily seen by putting $z = -\omega_\alpha$ in this formula. It is not difficult to prove the following relation between them:

$$2\eta_1\omega_2 - 2\eta_2\omega_1 = \pi i.$$

The function $\zeta(z)$ has simple poles with residue 1 at all points of the lattice, and no other singularities. As $z \to 0$, the expansion is

$$\zeta(z) = \frac{1}{z} + O(z^3).$$

The function $\sigma(z)$ can be introduced by the relation $\sigma'(z)/\sigma(z) = \zeta(z)$, hence

$$\log(\sigma(z)/z) = \int_0^z \Big(\zeta(x) - \frac{1}{x}\Big)dx.$$

It is defined by the following infinite product over the lattice:

$$\sigma(z) = z\prod_{s\neq 0}\Big(1 - \frac{z}{s}\Big)e^{\frac{z}{s} + \frac{z^2}{2s^2}}, \quad s = 2\omega_1 n_1 + 2\omega_2 n_2, \quad n_1, n_2 \in \mathbb{Z}.$$

Its monodromy properties under shifts by periods are:

$$\sigma(z + 2\omega_\alpha) = -e^{2\eta_\alpha(z+\omega_\alpha)}\sigma(z).$$

The $\sigma$-function is an odd entire function with first order zeros at all points of the lattice. As $z \to 0$, the expansion is:

$$\sigma(z) = z + O(z^5).$$

The Weierstrass functions satisfy a lot of nice non-trivial identities. Here we note the following:

$$\wp(u) - \wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)},$$

$$\frac{\wp'(u)}{\wp(u) - \wp(v)} = \zeta(u-v) + \zeta(u+v) - 2\zeta(u),$$

$$\sigma(a+b)\sigma(a-b)\sigma(u+c)\sigma(u-c) + \sigma(b+c)\sigma(b-c)\sigma(u+a)\sigma(u-a)$$
$$+\sigma(c+a)\sigma(c-a)\sigma(u+b)\sigma(u-b) = 0.$$

The general method of proving such identities is as follows. One should bring everything to one side or divide one side by the other and consider the expression obtained in this way as a function of one argument (for example, $u$). After that one should check that this function is double-periodic and regular; therefore, by the Liouville theorem it is a constant. The constant can be found by putting $u$ equal to a particular value such that the value of the function can be easily calculated.

At $\omega_1 = \infty$, $\omega_2 = \pi i/\gamma$ the Weierstrass functions degenerate to hyperbolic ones:

$$\wp(x) = \frac{\gamma^2}{\sinh^2(\gamma x)} + \frac{\gamma^2}{3},$$

$$\zeta(x) = \gamma \coth(\gamma x) - \frac{\gamma^2}{3}\, x,$$

$$\sigma(x) = \gamma^{-1} e^{-\frac{1}{6}\gamma^2 x^2} \sinh(\gamma x).$$

If both periods tend to infinity, the Weierstrass functions become rational:

$$\wp(x) = \frac{1}{x^2}, \quad \zeta(x) = \frac{1}{x}, \quad \sigma(x) = x.$$

We will also need the function

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)}\, e^{-\zeta(\lambda)x},$$

which is called the Lamé-Hermite fucntion. It is double-periodic in $\lambda$ and quasi-periodic in $x$:

$$\Phi(x + 2\omega_\alpha, \lambda) = e^{2(\eta_\alpha \lambda - \zeta(\lambda)\omega_\alpha)} \Phi(x,\lambda).$$

The expansion as $x \to 0$ has the form

$$\Phi(x,\lambda) = \frac{1}{x} - \frac{1}{2}\,\wp(\lambda) x + O(x^2).$$

We note the following identities:

$$\partial_x \Phi(x,\lambda) = \Phi(x,\lambda)\Big(\zeta(x+\lambda) - \zeta(x) - \zeta(\lambda)\Big),$$

$$\Phi(x,\lambda)\Phi(y,\lambda) = \Phi(x+y,\lambda)\Big(\zeta(x)+\zeta(y)-\zeta(x+y+\lambda)+\zeta(\lambda)\Big),$$

$$\partial_x\Phi(x,\lambda)\Phi(y,\lambda) - \partial_y\Phi(y,\lambda)\Phi(x,\lambda) = \Phi(x+y,\lambda)\Big(\wp(y)-\wp(x)\Big),$$

which are used in what follows.

**The Hamiltonian and equations of motion.** The Hamiltonian of the elliptic CM system has the form

$$H = \sum_i p_i^2 - g^2 \sum_{i\neq j} \wp(x_i - x_j).$$

The scaling of the coordinates and periods $x_i \to gx_i$, $\omega_\alpha \to g\omega_\alpha$ allows one to put $g = 1$ without loss of generality. The equations of motion are:

$$\dot{x}_i = 2p_i,$$

$$\dot{p}_i = -2g^2 \sum_{j\neq i} \wp'(x_i - x_j),$$

or, in the Newtonian form,

$$\ddot{x}_i = 4g^2 \sum_{j\neq i} \wp'(x_i - x_j).$$

**The Lax representation.** In the elliptic case a whole family of Lax pairs can be constructed in a natural way. They depend on a complex parameter $\lambda$ which is called the spectral parameter. Such families exist in the degenerate cases (rational and trigonometric) as well; the previously discussed Lax pairs are obtained at $\lambda = \infty$. In the elliptic case there is no distinguished point on the torus like $\infty$, that is why it is natural to consider the whole family.

So, the matrices $L = L(\lambda)$, $M = M(\lambda)$ now depend on a parameter $\lambda$. Their matrix elements are expressed in terms of the Laḿ-Hermite function

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)}\, e^{-\zeta(\lambda)x}$$

and its derivative $\partial_x\Phi(x,\lambda) = \Phi'(x,\lambda)$ in the following way:

$$L_{ij}(\lambda) = -\delta_{ij}p_i - g(1-\delta_{ij})\Phi(x_i - x_j, \lambda),$$

$$M_{ij}(\lambda) = -2g\delta_{ij}\sum_{k\neq i}\wp(x_i - x_k) - 2g(1-\delta_{ij})\Phi'(x_i - x_j, \lambda).$$

**Problem.** Find the trigonometric and rational degenerations of this Lax pair.

**Problem.** Prove that the Lax equation

$$\dot{L}(\lambda) + [L(\lambda), M(\lambda)] = 0$$

is equivalent to the equations of motion.

As before, the Lax equation implies that the time evolution of the Lax matrix is an isospectral transformation, and we can conclude that the spectral invariants $\mathrm{tr} L^k(\lambda)$ are integrals of motion.

**Problem.** Find $\mathrm{tr} L^2(\lambda)$ and $\mathrm{tr} L^3(\lambda)$ in explicit form.

The characteristic polynomial of the Lax matrix

$$R(z,\lambda) = \det(zI - L(\lambda))$$

is also an integral of motion. The equation $R(z,\lambda) = 0$ defines a complex curve which is called the spectral curve. It is an integral of motion.

**Problem.** Prove that all these integrals of motion are in involution (it is enough to prove this for eigenvalues of the Lax matrix).

**Properties of the spectral curve.** Let us investigate the spectral curve defined by the equation

$$R(z,\lambda) = \det\Big(zI - L(\lambda)\Big) = 0$$

in some detail. The matrix $L = L(\lambda)$, which has essential singularity at $\lambda = 0$, can be represented in the form $L = G\tilde{L}G^{-1}$, where matrix elements of $\tilde{L}$ do not have essential singularities and $G$ is the diagonal matrix $G_{ij} = \delta_{ij} e^{-\zeta(\lambda)x_i}$. Therefore,

$$R(z,\lambda) = \sum_{k=0}^{N} R_k(\lambda) z^k,$$

where the coefficients $R_k(\lambda)$ are elliptic functions of $\lambda$ with poles at $\lambda = 0$. The functions $R_k(\lambda)$ can be represented as linear combinations of the $\wp$-function and its derivatives. Coefficients of this expansion are integrals of motion. Fixing values of these integrals, we obtain via the equation $R(z,\lambda) = 0$ the algebraic curve $\Gamma$ which is an $N$-sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega_1$, $2\omega_2$.

*Example* ($N = 2$):

$$R(z,\lambda) = \det_{2\times 2}\Big(zI - L(\lambda)\Big) = z^2 + \frac{1}{2}z(\dot{x}_1 + \dot{x}_2) + \frac{1}{4}\dot{x}_1\dot{x}_2 + \wp(x_1 - x_2) - \wp(\lambda) = 0.$$

*Problem.* Write the equation of the spectral curve at $N = 3$.

In a small neighborhood of the point $\lambda = 0$ the matrix $\tilde{L}$ can be written as

$$\tilde{L} = -\lambda^{-1}(E - I) + O(1),$$

where $E$ is the rank 1 matrix with matrix elements $E_{ij} = 1$ for all $i, j = 1, \ldots, N$. The matrix $E$ has eigenvalue 0 with multiplicity $N-1$ and another eigenvalue equal to $N$. Therefore, we can write $R(z,\lambda)$ in the form

$$\begin{aligned} R(z,\lambda) &= \det\Big(z + \lambda^{-1}(E - I) + O(1)\Big) \\ &= \Big(z + (N-1)\lambda^{-1} - f_N(\lambda)\Big) \prod_{i=1}^{N-1} (z - \lambda^{-1} - f_i(\lambda)), \end{aligned}$$

where $f_i$ are regular functions of $\lambda$ at $\lambda = 0$: $f_i(\lambda) = O(1)$ as $\lambda \to 0$. This means that the function $z$ has simple poles on all sheets at the points $P_j$ ($j = 1, \dots, N$) of the curve $\Gamma$ located above $\lambda = 0$. So we have the following expansions of the function $z$ near the "points at infinity" $P_j$:

$$z = \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \dots, N-1,$$

$$z = -(N-1)\lambda^{-1} + f_N(\lambda) \quad \text{near } P_N.$$

From these formulas we see that the $N$-th sheet is distinguished. We call it the upper sheet.

In order to find genus $g$ of the spectral curve $\Gamma$, we recall the Riemann-Hurwitz formula. For a ramified covering $X \to Y$ of degree $n$ with $m$ ramification points $R_j$, where $X, Y$ are two Riemann surfaces (algebraic curves), let $e_j$ be the ramification index at the point $R_j$, then the Riemann-Hurwitz formula states that

$$2g(X) - 2 = n(2g(Y) - 2) + \sum_{j=1}^{m} (e_j - 1).$$

Let us apply it to the covering $\Gamma \to \mathcal{E}$. In this case $g(\mathcal{E}) = 1$, so the first term in the right-hand side vanishes, and in general position we have simple ramification, i.e., $e_j = 2$ for all $j$. Therefore, in our case we have $2g-2 = m$, where $m$ is the number of ramification points. The ramification points are zeros on $\Gamma$ of the function $\partial R/\partial z$. Differentiating the equation

$$R(z, \lambda) = \Big(z + (N-1)\lambda^{-1} - f_N(\lambda)\Big) \prod_{i=1}^{N-1} (z - \lambda^{-1} - f_i(\lambda))$$

with respect to $z$, we can conclude that the function $\partial R/\partial z$ has simple poles at the points $P_1, \dots, P_{N-1}$ on all sheets except the upper one, where it has a pole of order $N-1$. The number of poles of any meromorphic function is equal to the number of its zeros. Therefore, $m = 2(N-1)$ and so $g = N$.

We note that the spectral curve $\Gamma$ is not smooth because in the case of general position genus $g$ of the curve which is an $N$-sheet covering of an elliptic curve is $g = \frac{1}{2}N(N-1)+1$.

The spectral curve can be defined for the trigonometric and rational CM systems as well, if one uses the degenerations of the Lax matrix depending on the spectral parameter. In these cases the elliptic functions degenerate to trigonometric and rational ones, and the spectral curve becomes a curve of genus 0 with singularities (for example, double points and cusps).

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