Integrable systems of particles and nonlinear equations. Lecture 14

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The CM and RS systems in discrete time

As we argued in the previous sections, one can derive the dynamics of poles of elliptic solutions to nonlinear integrable equations from the auxiliary linear problems for these equations. For this, the wave function (the solution to the linear problem) should be parametrized by residues at the poles x_i, which obey a system of linear equations. An alternative approach is to parametrize the wave function by its zeros y_i, rather than by the residues, and try to derive equations of motion for these zeros. For example, in the rational case instead of the function ψ in the form

$$\psi = e^{kx}\Big(1 + \sum_i \frac{c_i}{x - x_i}\Big)$$

one can consider the function

$$\psi = e^{kx}\prod_i \frac{x - y_i}{x - x_i}$$

and substitute it into the linear equation in this form. As we shall see, this yields a system of equations connecting the zeros with the poles, and this system is symmetric under the exchange $x_i \leftrightarrow y_i$. Hence the zeros obey the same equations of motion as the poles do, i.e., the equations of motion of the CM or RS system. This fact allows us to regard the transformation $x_i \to y_i$ (from poles to zeros) as a Bäcklund transformation of the CM or RS systems. In its turn, such transformation can be interpreted as a shift of the discrete time $n \in \mathbb{Z}$ by one step. More precisely, denote $x_i = x_i^n$, $y_i = x_i^{n+1}$, then the Bäcklund transformation means evolution in the discrete time $x_i^n \to x_i^{n+1}$. This is the idea of constructing the integrable time discretization of the CM and RS systems. The equations of motion in discrete time connect x_i^n, x_i^{n+1} and x_i^{n-1}, and their properly taken continuous limit yields the CM or RS equations of motion. This construction is discussed in the works [27, 28, 29].

The CM system in discrete time

We begin with the CM system in its most general elliptic version.

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The Bäcklund transformation. Consider the linear equation

$$\partial_t \psi = \partial_x^2 \psi + 2\partial_x^2 \log \tau \, \psi$$

for the wave function ψ, where $t = t_2$. Let us represent the wave function as the ratio $\psi = \tilde{\tau}/\tau$, then the linear equation acquires the form

$$\partial_t \log \frac{\tilde{\tau}}{\tau} = \partial_x^2 \log(\tau\tilde{\tau}) + \Big(\partial_x \log \frac{\tilde{\tau}}{\tau}\Big)^2 .$$

For elliptic solutions the tau-function is

$$\tau = e^{Q(x,t)} \prod_i \sigma(x - x_i(t)),$$

where $Q(x,t)$ is some quadratic form in x, t. Its explicit form is not important for us here. Since ψ should be a double-Bloch function, the general form of $\tilde{\tau}$ is

$$\tilde{\tau} = C e^{Q(x,t)+\alpha x+\beta t} \prod_i \sigma(x - y_i(t)),$$

with some constants C, α, β, so

$$\frac{\tilde{\tau}}{\tau} = C e^{\alpha x + \beta t} \prod_i \frac{\sigma(x - y_i)}{\sigma(x - x_i)} .$$

Substituting this into our equation, we have:

$$\sum_i \Big(\dot{x}_i \zeta(x - x_i) - \dot{y}_i \zeta(x - y_i)\Big) = -\sum_i \Big(\wp(x - x_i) + \wp(x - y_i)\Big)$$

$$+ \left(\sum_i \Big(\zeta(x - x_i) - \zeta(x - y_i)\Big)\right)^2 + 2\alpha \sum_i \Big(\zeta(x - x_i) - \zeta(x - y_i)\Big) + \mathrm{const}\, .$$

Equating coefficients at the poles at the points $x = x_i$ and $x = y_i$, we get the following system of first order differential equations:

$$\left\{ \begin{array}{l} \dot{x}_i = 2\sum\limits_{j\neq i} \zeta(x_i - x_j) - 2\sum\limits_j \zeta(x_i - y_j) + 2\alpha, \\[2ex] \dot{y}_i = -2\sum\limits_{j\neq i} \zeta(y_i - y_j) + 2\sum\limits_j \zeta(y_i - x_j) + 2\alpha. \end{array} \right.$$

Redefining $x_i \to x_i + 2\alpha t$, $y_i \to y_i + 2\alpha t$, one can put $\alpha = 0$ without any loss of generality. $x_i \leftrightarrow y_i$ (with the simultaneous inversion of time $t \to -t$). Note that in the trigonometric and rational cases the number of the x_i's can be not necessarily equal to the number of y_i's because some of them may go to infinity.

Let us show that these equations imply the equations of motion for the CM system for the x_i's and for the y_i's, too. Indeed, taking the time derivative of the first equation, we have:

$$\begin{aligned} \ddot{x}_i &= -2\sum_{j\neq i} (\dot{x}_i - \dot{x}_j)\wp(x_i - x_j) + 2\sum_j (\dot{x}_i - \dot{y}_j)\wp(x_i - y_j) \\ &= -4\sum_{j\neq i}\Big(\sum_{k\neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) - \sum_{k\neq j} \zeta(x_j - x_k) + \sum_k \zeta(x_j - y_k)\Big)\wp(x_i - x_j) \\ &\quad +4\sum_j \Big(\sum_{k\neq i} \zeta(x_i - x_k) - \sum_k \zeta(x_i - y_k) + \sum_{k\neq j} \zeta(y_j - y_k) - \sum_k \zeta(y_j - x_k)\Big)\wp(x_i - y_j). \end{aligned}$$

It will be shown below that the right-hand side is equal to $4\sum_{j\neq i}\wp'(x_i-x_j)$, i.e., x_i's satisfy the CM equations of motion. By virtue of the symmetry, the same holds for y_i's. Therefore, the transformation $x_i \to y_i$ is a Bäcklund transformation for the CM system. It sends any solution to another solution of the same system of equations.

This statement is based on the identity

$$
\begin{aligned}
&-\sum_{j\neq i}\Bigl(\sum_{k\neq i}\zeta(x_i-x_k)-\sum_{k}\zeta(x_i-y_k)-\sum_{k\neq j}\zeta(x_j-x_k)+\sum_{k}\zeta(x_j-y_k)\Bigr)\wp(x_i-x_j)\\
&+\sum_{j}\Bigl(\sum_{k\neq i}\zeta(x_i-x_k)-\sum_{k}\zeta(x_i-y_k)+\sum_{k\neq j}\zeta(y_j-y_k)-\sum_{k}\zeta(y_j-x_k)\Bigr)\wp(x_i-y_j)\\
&-\sum_{j\neq i}\wp'(x_i-x_j)=0,
\end{aligned}
$$

where $x_1,\dots,x_N$, $y_1,\dots,y_N$ are arbitrary variables.

Proof of the identity. Let us prove this identity.

The first nontrivial case is $N=2$. Put $i=1$ and consider the left-hand side as a function of x_1. It is easy to see that it is an elliptic function of x_1. Its possible poles may be at $x_1=x_2$, $x_1=y_1$, $x_1=y_2$. Setting $x_1=x_2+\varepsilon$ and expanding the expression as $\varepsilon\to 0$, one can check that the left-hand side is actually regular at $x_1=x_2$ and, moreover, is of order $O(\varepsilon)$, so at $x_1=x_2$ it vanishes. In a similar way, we can prove that the left-hand side is regular at $x_1=y_1$, $x_1=y_2$. Therefore, it is identically equal to 0.

Passing to the general case, we denote the left-hand side by

$$F_N^{(i)}=F_N^{(i)}(x_1,\dots,x_N,y_1,\dots,y_N),$$

and consider it as a function of x_i. It is easy to see that it is an elliptic function of x_i. Its possible poles may be at $x_i=x_{i_0}$ ($i_0=1,\dots,N$, $i_0\neq i$) and $x_i=y_{i_0}$ ($i_0=1,\dots,N$). Setting $x_i=x_{i_0}+\varepsilon$, $x_i=y_{i_0}+\varepsilon$ and expanding it as $\varepsilon\to 0$, one can check that $F_N^{(i)}$ is regular, i.e., the singular terms cancel and $F_N^{(i)}=O(1)$ as $\varepsilon\to 0$. Therefore, $F_N^{(i)}$ is a function that does not depend on x_i. To find it, expand $F_N^{(i)}$ near x_{i_0} up to terms of order ε^0:

$$F_N^{(i)}=F_{N-1}^{(i_0)}(x_1,\dots,\hat{x}_i,\dots,x_N,y_1,\dots,\hat{y}_i,\dots,y_N)+G_{N-1}^{(i_0)}+O(\varepsilon),$$

where $\hat{x}_i$, $\hat{y}_i$ means that the arguments x_i, y_i are omitted, and

$$G_{N-1}^{(i_0)}=G_{N-1}^{(i_0)}(x_1,\dots,\hat{x}_i,\dots,x_N,y_1,\dots,y_N)$$

is given by

$$
\begin{aligned}
G_{N-1}^{(i_0)}=\frac{1}{2}\sum_{k}\wp'(x_{i_0}-y_k)-\frac{1}{2}\sum_{k\neq i,i_0}\wp'(x_{i_0}-x_k)+\zeta(x_{i_0}-y_i)\wp(x_{i_0}-y_i)\\
-\sum_{j\neq i,i_0}\Bigl(\zeta(x_{i_0}-x_j)-\zeta(x_{i_0}-y_i)+\zeta(x_j-y_i)\Bigr)\wp(x_{i_0}-x_j)
\end{aligned}
$$

$$+\sum_{j\neq i}\Big(\zeta(x_{i_0}-y_j)-\zeta(x_{i_0}-y_i)+\zeta(y_j-y_i)\Big)\wp(x_{i_0}-y_j)$$

$$+\Big(\sum_{k\neq i,i_0}\zeta(x_{i_0}-x_k)-\sum_{k\neq i}\zeta(x_{i_0}-y_k)+\sum_{k\neq i}\zeta(y_i-y_k)-\sum_{k\neq i,i_0}\zeta(y_i-x_k)\Big)\wp(x_{i_0}-y_i).$$

In the second and the third line we can use the identity

$$\zeta(x)-\zeta(y)-\zeta(x-y)=-\frac{1}{2}\,\frac{\wp'(x)+\wp'(y)}{\wp(x)-\wp(y)},$$

which allows us to transform sum of the expressions in these lines to the form

$$\sum_{j\neq i,i_0}\Big(\frac{1}{2}\,\wp'(x_{i_0}-x_j)-\frac{1}{2}\,\wp'(x_{i_0}-y_j)$$

$$+\Big(\zeta(x_{i_0}-y_j)+\zeta(y_j-y_i)-\zeta(x_{i_0}-x_j)-\zeta(x_j-y_i)\Big)\wp(x_{i_0}-y_i)\Big)$$

$$+\Big(\zeta(x_{i_0}-y_{i_0})-\zeta(x_{i_0}-y_i)+\zeta(y_{i_0}-y_i)\Big)\wp(x_{i_0}-y_{i_0}).$$

Substituting this back into the expression for $G^{(i_0)}_{N-1}$, we obtain, after some cancellations:

$$G^{(i_0)}_{N-1}=\frac{1}{2}\,\wp'(x_{i_0}-y_i)+\frac{1}{2}\,\wp'(x_{i_0}-y_{i_0})$$

$$+\Big(\zeta(x_{i_0}-y_{i_0})-\zeta(x_{i_0}-y_i)+\zeta(y_{i_0}-y_i)\Big)\Big(\wp(x_{i_0}-y_{i_0})-\wp(x_{i_0}-y_i)\Big).$$

Using the identity connecting the ζ- and$\wp$-functions again, we see that $G^{(i_0)}_{N-1}=0$.

To finish the proof, we use the induction: suppose that $F^{(i)}_{N-1}=0$ (as we know, this holds for $N=3$), then $F^{(i)}_N=O(\varepsilon)$ as $\varepsilon\to 0$ and, therefore, $F^{(i)}_N=0$.

Bäcklund transformations as dynamics in discrete time. The Bäcklund transformation $x_i\to y_i$ can be regarded as a shift in the discrete time $n\in\mathbb{Z}$ by one step. Having this in mind, denote $x_i=x_i^n$, $y_i=x_i^{n+1}$, then the equations that define the Bäcklund transformation acquire the form

$$\left\{\begin{array}{l}\dot x_i^n=2\sum\limits_{j\neq i}\zeta(x_i^n-x_j^n)-2\sum\limits_j\zeta(x_i^n-x_j^{n+1}),\\[2ex] \dot x_i^{n+1}=-2\sum\limits_{j\neq i}\zeta(x_i^{n+1}-x_j^{n+1})+2\sum\limits_j\zeta(x_i^{n+1}-x_j^n).\end{array}\right.$$

Shifting $n\to n-1$ in the second equation and subtracting them after that, we obtain equations of motion for the CM system in discrete time which connects x_i^n, x_i^{n+1} and x_i^{n-1}:

$$\sum_j\zeta(x_i^n-x_j^{n+1})+\sum_j\zeta(x_i^n-x_j^{n-1})-2\sum_{j\neq i}\zeta(x_i^n-x_j^n)=0. \tag{0.1}$$

Remarkably, these equations coincide with the system of nested Bethe ansatz equations for the elliptic quantum Gaudin model associated with the root system A_m. In this case the discrete time n may take only values $0,1,\dots,m+1$. At present it is not clear whether this coincidence is caused by some profound reasons.

The Hamiltonian approach to time discretization of integrable many-body systems is discussed in the book [5].

Continuum limit. The equations just obtained admit different continuum limits to continuous time. As is easy to see, the simplest naive limit gives trivial equations of motion $\ddot{x}_i = 0$. In order to obtain something less trivial, one should take the limit in a more clever way. To wit, put $t = n\delta$, $x_i^n \to x_i(t) + \varepsilon n$, so that

$$x_i^{n\pm 1} \to x_i \pm \varepsilon \pm \delta \dot{x}_i + \tfrac{1}{2}\delta^2 \ddot{x}_i + \dots$$

as $\varepsilon, \delta \to 0$, with $\delta = O(\varepsilon^2)$. Separating the terms with $j = i$ in the sums entering the equations of motion and those with $j \neq i$, and expanding them separately, we obtain in the first non-vanishing order:

$$\frac{1}{\varepsilon + \delta \dot{x}_i - \frac{1}{2}\delta^2 \ddot{x}_i} - \frac{1}{\varepsilon + \delta \dot{x}_i + \frac{1}{2}\delta^2 \ddot{x}_i} - \varepsilon^2 \sum_{j \neq i} \wp'(x_i - x_j) = 0,$$

or, after cancellation of singular terms,

$$\ddot{x}_i = 4g^2 \sum_{j \neq i} \wp'(x_i - x_j), \qquad g = \frac{\varepsilon^2}{2\delta}.$$

These are familiar equations of motion of the elliptic CM system in continuous time.

The RS system in discrete time

In this case the main idea is the same as for the CM system with the only difference that instead of the linear problem for the KP equation one should consider the one for the 2D Toda chain.

The Bäcklund transformation. Consider the first linear problem for the 2D Toda chain:

$$\partial_t \psi(x) = \psi(x + \eta) + \partial_t \log \frac{\tau(x + \eta)}{\tau(x)}\, \psi(x),$$

where $t = t_1$. Let us represent the wave function in the form $\psi = \tilde{\tau}/\tau$ and substitute it into the linear equation:

$$\partial_t \log \frac{\tilde{\tau}(x)}{\tau(x + \eta)} = \frac{\tilde{\tau}(x + \eta)\tau(x)}{\tau(x + \eta)\tilde{\tau}(x)}.$$

For elliptic solutions

$$\tau = e^{Q(x,t)} \prod_i \sigma(x - x_i(t)),$$

where $Q(x,t)$ is some quadratic form in x, t, which is not important for us here. Since ψ should be a double-Bloch function, the general form of $\tilde{\tau}$ is

$$\tilde{\tau} = A e^{Q(x,t) + \alpha x + \beta t} \prod_i \sigma(x - y_i(t)),$$

with some constants A, α, β, so that

$$\frac{\tilde{\tau}}{\tau} = A e^{\alpha x + \beta t} \prod_i \frac{\sigma(x - y_i)}{\sigma(x - x_i)}.$$

Substituting this into the equation, we have:

$$\sum_i \Big(\dot{x}_i \zeta(x - x_i + \eta) - \dot{y}_i \zeta(x - y_i)\Big) = e^{\alpha\eta} \prod_i \frac{\sigma(x - x_i)\sigma(x - y_i + \eta)}{\sigma(x - y_i)\sigma(x - x_i + \eta)} + \text{const}.$$

Equating residues at the poles at $x = x_i - \eta$ and $x = y_i$, we get the system of equations of the following form:

$$\begin{cases} \dot{x}_i = C \displaystyle\prod_{k \neq i} \frac{\sigma(x_i - x_k - \eta)}{\sigma(x_i - x_k)} \prod_j \frac{\sigma(x_i - y_j)}{\sigma(x_i - y_j - \eta)}, \\[2ex] \dot{y}_i = C \displaystyle\prod_{k \neq i} \frac{\sigma(y_i - y_k + \eta)}{\sigma(y_i - y_k)} \prod_j \frac{\sigma(y_i - x_j)}{\sigma(y_i - x_j + \eta)} \end{cases}$$

with some constant C, which can be put equal to 1 without loss of generality (this can be achieved by a rescaling of time). These equations are symmetric with respect to the simultaneous exchange $x_i \leftrightarrow y_i$ and $\eta \to -\eta$. As before, in the trigonometric and rational cases the number of x_i's may be not necessarily equal to the number of y_i's.

The equations just obtained imply that the both sets of variables, x_i's and y_i's, satisfy the RS equations of motion, i.e., the transformation $x_i \to y_i$ is indeed a Bäcklund transformation. To prove this, we need some special technique. Introduce the function

$$\phi(x, y) = \frac{\sigma(x + y)}{\sigma(x)\sigma(y)},$$

which differs from the Laḿ-Hermite function only by an exponential multiplier. After a proper rescaling of time, the system obtained above can be written in the form

$$\begin{cases} \dot{x}_i = \displaystyle\prod_{k \neq i} \phi(x_i - x_k, -\eta) \prod_j \phi(x_i - y_j - \eta, \eta), \\[2ex] \dot{y}_i = -\displaystyle\prod_{k \neq i} \phi(y_i - y_k, \eta) \prod_j \phi(y_i - x_j + \eta, -\eta), \end{cases}$$

while the RS equations of motion acquire the form

$$\frac{\ddot{x}_i}{\dot{x}_i} = \sum_{k \neq i} \dot{x}_k \left(\frac{\phi'(x_k - x_i, -\eta)}{\phi(x_k - x_i, -\eta)} - \frac{\phi'(x_i - x_k, -\eta)}{\phi(x_i - x_k, -\eta)} \right),$$

where $\phi'(x, y) = \partial_x \phi(x, y)$. Note that the rescaling of time does not make any effect since the equations are homogeneous in time. Taking the time derivative of the first equation, we have:

$$\begin{aligned} \frac{\ddot{x}_i}{\dot{x}_i} &= \sum_{k \neq i} (\dot{x}_i - \dot{x}_k) \frac{\phi'(x_i - x_k, -\eta)}{\phi(x_i - x_k, -\eta)} + \sum_k (\dot{x}_i - \dot{y}_k) \frac{\phi'(x_i - y_k - \eta, \eta)}{\phi(x_i - y_k - \eta, \eta)} \\ &= \sum_{k \neq i} \dot{x}_k \left(\frac{\phi'(x_k - x_i, -\eta)}{\phi(x_k - x_i, -\eta)} - \frac{\phi'(x_i - x_k, -\eta)}{\phi(x_i - x_k, -\eta)} \right) \end{aligned}$$

$$- \sum_{k \neq i} \dot{x}_k \frac{\phi'(x_k - x_i, -\eta)}{\phi(x_k - x_i, -\eta)} + \sum_{k \neq i} \dot{x}_i \frac{\phi'(x_i - x_k, -\eta)}{\phi(x_i - x_k, -\eta)} + \sum_k (\dot{x}_i - \dot{y}_k) \frac{\phi'(x_i - y_k - \eta, \eta)}{\phi(x_i - y_k - \eta, \eta)}.$$

Comparing with the RS equations of motion, we see that it is necessary to prove that the sum in the third line is actually zero. This follows from the identity

$$\sum_i \dot{x}_i = \sum_i \dot{y}_i$$

or

$$\sum_i \left(\prod_{k\neq i} \phi(x_i - x_k, -\eta) \prod_j \phi(x_i - y_j - \eta, \eta) + \prod_{k\neq i} \phi(y_i - y_k, \eta) \prod_j \phi(y_i - x_j + \eta, -\eta) \right) = 0,$$

which is proved below. Indeed, taking the derivative of this identity with respect to x_i and using the equations of the system, we just see that the sum in the third line is zero. Our identity is a particular case of a more general one, which has the form

$$\prod_{i=1}^{n} \phi(w_i, z_i) = \sum_{i=1}^{n} \phi\Big(w_i, \sum_{k=1}^{n} z_k\Big) \prod_{j\neq i}^{n} \phi(w_j - w_i, z_j).$$

It will be proved below. The initial identity can be obtained from this more general one by putting $n = 2N - 1$. It is then convenient to take the sum over $i = 2, \ldots, 2N$. Let us choose the variables in the following way:

$$z_2 = \ldots = z_N = -\eta, \quad z_{N+1} = \ldots = z_{2N} = \eta, \ \text{ so that } \sum_{k=2}^{2N} z_k = \eta,$$

$$w_2 = x_1 - x_2, \ldots, w_N = x_1 - x_N, \quad w_{N+1} = x_1 - y_1 - \eta, \ldots, w_{2N} = x_1 - y_N - \eta.$$

Then the identity yields: $\sum_{i=1}^{N} \dot{x}_i = \sum_{i=1}^{N} \dot{y}_i$ в форме $\dot{x}_1 = -\sum_{i=2}^{N} \dot{x}_i + \sum_{i=1}^{N} \dot{y}_i$.

The fact that the y_i's satisfy the same RS equations of motion follow from the symmetry $x_i \leftrightarrow y_i$.

Proof of the identity. We begin with the simplest case $n = 2$.

Problem. Prove the identity

$$\phi(w_1, z_1)\phi(w_2, z_2) = \phi(w_1, z_1 + z_2)\phi(w_2 - w_1, z_2) + \phi(w_2, z_1 + z_2)\phi(w_1 - w_2, z_1).$$

In the general case the proof can be done by induction. Suppose that the identity holds at some n and prove that this implies that it holds also for $n \to n + 1$:

$$\prod_{i=1}^{n+1} \phi(w_i, z_i) = \sum_{i=1}^{n+1} \phi\Big(w_i, \sum_{k=1}^{n+1} z_k\Big) \prod_{j\neq i}^{n+1} \phi(w_j - w_i, z_j).$$

By the induction assumption, we can transform the left-hand side:

$$\prod_{i=1}^{n+1} \phi(w_i, z_i) = \sum_{i=1}^{n} \phi(w_{n+1}, z_{n+1})\phi\Big(w_i, \sum_{k=1}^{n} z_k\Big) \prod_{j\neq i}^{n} \phi(w_j - w_i, z_j)$$

$$= \sum_{i=1}^{n} \phi(w_{n+1} - w_i, z_{n+1})\phi\Big(w_i, \sum_{k=1}^{n+1} z_k\Big) \prod_{j\neq i}^{n} \phi(w_j - w_i, z_j)$$

$$+\sum_{i=1}^{n}\phi\Big(w_i-w_{n+1},\sum_{k=1}^{n}z_k\Big)\phi\Big(w_{n+1},\sum_{k=1}^{n}z_k\Big)\prod_{j\neq i}^{n}\phi(w_j-w_i,z_j),$$

where we have used the identity from the problem above. Consider the right-hand side of the identity and write the first n terms of the sum separately:

$$\sum_{i=1}^{n+1}\phi\Big(w_i,\sum_{k=1}^{n+1}z_k\Big)\prod_{j\neq i}^{n+1}\phi(w_j-w_i,z_j)$$

$$=\sum_{i=1}^{n}\phi\Big(w_i,\sum_{k=1}^{n+1}z_k\Big)\prod_{j\neq i}^{n+1}\phi(w_j-w_i,z_j)+\phi\Big(w_{n+1},\sum_{k=1}^{n+1}z_k\Big)\prod_{j=1}^{n}\phi(w_j-w_{n+1},z_j).$$

Comparing with the left-hand side, we see that the first terms in these expressions coincide. Therefore, we should prove that

$$\sum_{i=1}^{n}\phi\Big(w_i-w_{n+1},\sum_{k=1}^{n}z_k\Big)\phi\Big(w_{n+1},\sum_{k=1}^{n+1}z_k\Big)\prod_{j\neq i}^{n}\phi(w_j-w_i,z_j)$$

$$=\phi\Big(w_{n+1},\sum_{k=1}^{n+1}z_k\Big)\prod_{j=1}^{n}\phi(w_j-w_{n+1},z_j).$$

After cancellation of the common multiplier $\phi\Big(w_{n+1},\sum_{k=1}^{n+1}z_k\Big)$ we see that the equality follows from the induction assumption after the change $w_i\to w_i-w_{n+1}$.

The Bäcklund transformation as a dynamics in discrete time. As in the case of the CM system, the Bäcklund transformation $x_i\to y_i$ can be regarded as a shift of the discrete time $n\in\mathbb{Z}$ by one step. Denote $x_i=x_i^n$, $y_i=x_i^{n+1}$, then the equations that determine the Bäcklund transformation acquire the form

$$\left\{\begin{array}{l}\dot x_i^n=\displaystyle\prod_{k\neq i}\frac{\sigma(x_i^n-x_k^n-\eta)}{\sigma(x_i^n-x_k^n)}\prod_{j}\frac{\sigma(x_i^n-x_j^{n+1})}{\sigma(x_i^n-x_j^{n+1}-\eta)},\\[4ex] \dot x_i^{n+1}=\displaystyle\prod_{k\neq i}\frac{\sigma(x_i^{n+1}-x_k^{n+1}+\eta)}{\sigma(x_i^{n+1}-x_k^{n+1})}\prod_{j}\frac{\sigma(x_i^{n+1}-x_j^{n})}{\sigma(x_i^{n+1}-x_j^{n}+\eta)}.\end{array}\right.$$

Shifting $n\to n-1$ in the second equation (then the left-hand side become identical) and equating the right-hand sides after that, we obtain the equations of motion for the RS system in discrete time:

$$\prod_{k=1}^{N}\frac{\sigma(x_i^n-x_k^{n-1})}{\sigma(x_i^n-x_k^{n-1}+\eta)}\,\frac{\sigma(x_i^n-x_k^n+\eta)}{\sigma(x_i^n-x_k^n-\eta)}\,\frac{\sigma(x_i^n-x_k^{n+1}-\eta)}{\sigma(x_i^n-x_k^{n+1})}=-1.$$

It is easy to see that in the limit $\eta\to 0$ they turn into the CM equations of motion in discrete time.

Remarkably, the obtained equations coincide with the nested Bethe ansatz equations for a generalized quantun spin chain with elliptic R-matrix (first found by Belavin) associated with the root system A_m. In this case the discrete time n take values $0,1,\dots,m+1$, while x_i^n are Bethe roots at nth level of Bethe ansatz.

Problem. Perform the continuum limit of these equations and show that they turn into the RS equations of motion.

Discrete Lax representation. The equations of motion for the RS system in discrete time has a commutation representation of the Lax type, which is a discrete time version of the differential one, and has the same meaning: the Lax matrix undergoes an isospectral transformation under the time evolution. The existence of a Lax-type representation means integrability of the RS system in discrete time (as well as the CM system in discrete time as its limiting case). The discrete Lax representation can be obtained from analysis of elliptic solutions to the 2D Toda chain with discrete time.

The tau-function of the 2D Toda chain in discrete time can be defined be means of the rule

$$\tau^n(x) = \tau(x, \mathbf{t} - n[\lambda^{-1}], \bar{\mathbf{t}}),$$

where λ^{-1} plays the role of inverse lattice spacing. Values of the contonuous times $\mathbf{t}, \bar{\mathbf{t}}$ are assumed to be fixed. In the case of two discrete times n, m we set

$$\tau^{n,m}(x) = \tau(x, \mathbf{t} - n[\lambda^{-1}] - m[\mu^{-1}], \bar{\mathbf{t}}).$$

Then the bilinear equation for the tau-function of the 2D Toda chain

$$\lambda\tau(x+\eta, \mathbf{t})\tau(x, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}]) - \mu\tau(x+\eta, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}])\tau(x, \mathbf{t})$$

$$= (\lambda - \mu)\tau(x+\eta, \mathbf{t} + [\lambda^{-1}])\tau(x, \mathbf{t} - [\mu^{-1}]),$$

where we have omitted the times $\bar{\mathbf{t}}$ for brevity, can be rewritten as the following equation in the discrete times:

$$\lambda\tau^{n+1,m}(x+\eta)\tau^{n,m+1}(x) - \mu\tau^{n,m+1}(x+\eta)\tau^{n+1,m}(x) = (\lambda-\mu)\tau^{n,m}(x+\eta)\tau^{n+1,m+1}(x).$$

The wave function depending on the spectral parameter z is introduced by the formula

$$\psi^n(x) = z^{x/\eta}\Big(1 - \frac{z}{\lambda}\Big)^n e^{\xi(\mathbf{t},z)}\frac{\tau(x, \mathbf{t} - n[\lambda^{-1}] - [z^{-1}])}{\tau(x, \mathbf{t})}.$$

It is easy to verify that the bilinear functional relation for the tau-function is equivalent to the following linear equation for the wave function:

$$\psi^{n+1}(x) = -\lambda^{-1}\psi^n(x+\eta) + V^n(x)\psi^n(x), \qquad V^n(x) = \frac{\tau^n(x)\tau^{n+1}(x+\eta)}{\tau^{n+1}(x)\tau^n(x+\eta)}.$$

For elliptic solutions in x we have

$$\tau^n(x) = \prod_{j=1}^{N}\sigma(x - x_j^n),$$

and

$$V^n(x) = \prod_j \frac{\sigma(x - x_j^n)\sigma(x - x_j^{n+1} + \eta)}{\sigma(x - x_j^{n+1})\sigma(x - x_j^n + \eta)}$$

is an elliptic function of x. Solutions for $\psi^n(x)$ should be found among double-Bloch functions

$$\psi^n(x) = z^{x/\eta}\sum_i c_i^n \Phi(x - x_i^n, u),$$

where the second spectral paramemter is denoted by u here. The substitution into the linear equation gives:

$$\sum_i c_i^{n+1}\Phi(x-x_i^{n+1},u)+z\lambda^{-1}\sum_i c_i^n\Phi(x-x_i^n+\eta,u)$$

$$-\prod_k \frac{\sigma(x-x_i^n)\sigma(x-x_i^{n+1}+\eta)}{\sigma(x-x_i^{n+1})\sigma(x-x_i^n+\eta)}\sum_i c_i^n\Phi(x-x_i^n,u)=0.$$

Possible poles of the left-hand side are at $x=x_i^n-\eta$ and $x=x_i^{n+1}$. Their cancellation leads to the following over-determined system of linear equations for components of the vector $\mathbf{c}^n=(c_1^n,\ldots,c_N^n)^{\mathrm{T}}$:

$$\left\{\begin{array}{l} L^n\mathbf{c}^n=z\lambda^{-1}\mathbf{c}^n, \\ \\ \mathbf{c}^{n+1}=M^n\mathbf{c}^n. \end{array}\right.$$

Matrix elements of the matrices L^n, M^n are:

$$L_{ij}^n=f_i^n\Phi(x_i^n-x_j^n-\eta,u),$$

$$M_{ij}^n=g_i^n\Phi(x_i^{n+1}-x_j^n,u),$$

where

$$f_i^n=\frac{\prod\limits_j \sigma(x_i^n-x_j^n-\eta)\sigma(x_i^n-x_j^{n+1})}{\prod\limits_j \sigma(x_i^n-x_j^{n+1}-\eta)\prod\limits_{l\neq i}\sigma(x_i^n-x_l^n)},$$

$$g_i^n=\frac{\prod\limits_j \sigma(x_i^{n+1}-x_j^n)\sigma(x_i^{n+1}-x_j^{n+1}+\eta)}{\prod\limits_j \sigma(x_i^{n+1}-x_j^n+\eta)\prod\limits_{l\neq i}\sigma(x_i^{n+1}-x_l^{n+1})}.$$

The compatibility condition has the form

$$L^{n+1}M^n=M^nL^n.$$

This is the discrete Lax equation, which means that the shift by one step of the discrete time is an isospectral transformation, and so the spectrum of the Lax matrix is conserved.

Let us show that the discrete Lax equation is equivalent to the RS equations of motion in discrete time obtained above. First of all we notice that

$$\sum_i f_i^n+\sum_i g_i^n=0,$$

because in the left-hand side there is sum of residues of the elliptic function $V^n(x)$. Denote $R^n=L^{n+1}M^n-M^nL^n$, then

$$\begin{aligned} R_{ij}^n &= f_i^{n+1}\sum_k g_k^n\Phi(x_i^{n+1}-x_k^{n+1}-\eta,u)\Phi(x_k^{n+1}-x_j^{n+1},u) \\ &\quad -g_i^n\sum_k f_k^n\Phi(x_i^{n+1}-x_k^n,u)\Phi(x_k^n-x_j^n-\eta,u). \end{aligned}$$

The equality $R_{ij}^n = 0$ (the discrete Lax equation) in the limit $u \to 0$ implies the equality

$$f_i^{n+1} \sum_k g_k^n - g_i^n \sum_k f_k^n = 0,$$

or, if we recall that $\sum\limits_k f_k^n + \sum\limits_k g_k^n = 0$,

$$f_i^{n+1} = -g_i^n.$$

These are just the equations of motion in discrete time obtained above:

$$\prod_{k=1}^N \frac{\sigma(x_i^n - x_k^{n-1})}{\sigma(x_i^n - x_k^{n-1} + \eta)} \frac{\sigma(x_i^n - x_k^n + \eta)}{\sigma(x_i^n - x_k^n - \eta)} \frac{\sigma(x_i^n - x_k^{n+1} - \eta)}{\sigma(x_i^n - x_k^{n+1})} = -1.$$

Problem. Show that these equations are equivalent to the equality $R_{ij}^n = 0$ for all u.

The deformed RS system in discrete time

The Bäcklund transformation. Let us apply the same method to the case of the Toda lattice of type B. The first linear problem has the form

$$\partial_t \psi(x) = v(x)\Big(\psi(x+\eta) - \psi(x-\eta)\Big),$$

where $v(x)$ is expressed through the tau-function $\tau(x)$:

$$v(x) = \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}.$$

For elliptic solutions

$$\tau(x) = C \prod_{i=1}^N \sigma(x - x_i),$$

where it is assumed that all its zeros x_j are distinct, so $v(x)$ is an elliptic function with periods $2\omega_1$, $2\omega_2$. As usual in such a case, we search for solutions among double-Bloch functions. Poles of ψ are zeros of the tau-function, so we can represent the solutions in the form

$$\psi(x) = \mu^{x/\eta} e^{(\mu - \mu^{-1})t} \frac{\hat{\tau}(x)}{\tau(x)},$$

where

$$\hat{\tau}(x) = \prod_{i=1}^N \sigma(x - y_i)$$

with some y_i's. Then the ψ-function is indeed double-Bloch with Bloch multipliers

$$B_1 = \mu^{2\omega_1/\eta} \exp\Big(2\zeta(\omega_1) \sum_{j=1}^N (x_j - y_j)\Big), \quad B_2 = \mu^{2\omega_2/\eta} \exp\Big(2\zeta(\omega_2) \sum_{j=1}^N (x_j - y_j)\Big).$$

It is possible to show that

$$\sum_{j=1}^N (\dot{x}_j - \dot{y}_j) = 0,$$

so the Bloch multipliers do not depend on time. Earlier we saw that poles of the wave function satisfy the equations of motion for the deformed RS system for arbitrary μ.

Substituting the wave function into the linear equation, we have:

$$\frac{\partial_t \hat{\tau}(x)}{\hat{\tau}(x)} - \frac{\partial_t \tau(x)}{\tau(x)} + \mu - \mu^{-1} = \mu \frac{\hat{\tau}(x+\eta)\tau(x-\eta)}{\hat{\tau}(x)\,\tau(x)} - \mu^{-1}\frac{\tau(x+\eta)\hat{\tau}(x-\eta)}{\tau(x)\,\hat{\tau}(x)}.$$

This equation is invariant under the simultaneous exchange $\tau \leftrightarrow \hat{\tau}$, $\mu \leftrightarrow \mu^{-1}$, hence y_j's satisfy the same equations as x_j's. Both sides of the equation have simple poles at $x = x_j$ and $x = y_j$. Equating the residues, we come to equations connecting zeros y_i's and poles x_i's of the wave function:

$$\dot{x}_i = \mu\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}\prod_k \frac{\sigma(x_i - y_k + \eta)}{\sigma(x_i - y_k)}$$

$$+ \mu^{-1}\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}\prod_k \frac{\sigma(x_i - y_k - \eta)}{\sigma(x_i - y_k)},$$

$$\dot{y}_i = \mu\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(y_i - y_j + \eta)}{\sigma(y_i - y_j)}\prod_k \frac{\sigma(y_i - x_k - \eta)}{\sigma(y_i - x_k)}$$

$$+ \mu^{-1}\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(y_i - y_j - \eta)}{\sigma(y_i - y_j)}\prod_k \frac{\sigma(y_i - x_k + \eta)}{\sigma(y_i - x_k)}.$$

They are symmetric with respect to the exchange $x_j \leftrightarrow y_j$, $\mu \leftrightarrow \mu^{-1}$. The map $x_j \to y_j$ can be regarded as a Bäcklund transformation of the deformed RS system.

It should be noted that the equations for the x_j's and y_j's in principle can be derived from the equations obtained. For this, one should apply to them the time derivative and use them again, substituting the expressions for $\dot{x}_j$'s, $\dot{y}_j$'s through x_j's, y_j's. They will turn out to be equivalent to a nontrivial identity for elliptic functions of many variables. This identity seems to be too complicated to prove it directly. However, we do not need the direct proof since the equations for x_j's follow from the previous results while those for y_j's follow from the symmetry $x_j \leftrightarrow y_j$. Note that the Bäcklund transformation for the RS system differs from its analog for the deformed RS system by absence of the second terms in the right-hand sides. In this sense it is formally contained in the latter as a limiting case ($\mu \to \infty$ or $\mu \to 0$).

The dynamics in discrete time. Denote the discrete time variable by $n \in \mathbb{Z}$ and put $x_i = x_i^n$, $y_i = x_i^{n+1}$. Shifting $n \to n-1$ in the second equation, we see that the left-hand sides become the same. Equating the right-hand sides, we arrive at the equations

$$\mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1} - \eta)$$

$$+ \mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} + \eta)\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1})$$

$$= \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1})$$

$$+ \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta),$$

or

$$\prod_{j=1}^{N} \frac{\sigma(x_i^n - x_j^{n+1})\sigma(x_i^n - x_j^n + \eta)\sigma(x_i^n - x_j^{n-1} - \eta)}{\sigma(x_i^n - x_j^{n+1} + \eta)\sigma(x_i^n - x_j^n - \eta)\sigma(x_i^n - x_j^{n-1})}$$

$$= -1 + \mu^{-2} \prod_{j=1}^{N} \frac{\sigma(x_i^n - x_j^{n+1})\sigma(x_i^n - x_j^{n-1} + \eta)}{\sigma(x_i^n - x_j^{n+1} + \eta)\sigma(x_i^n - x_j^{n-1})} + \mu^{-2} \prod_{j=1}^{N} \frac{\sigma(x_i^n - x_j^{n+1} - \eta)\sigma(x_i^n - x_j^n + \eta)}{\sigma(x_i^n - x_j^{n+1} + \eta)\sigma(x_i^n - x_j^n - \eta)}.$$

Continuum limit. The equations just obtained admit different continuum limits. One possibility is to introduce new variables

$$X_j^n = x_j^n - n\eta$$

and suppose that they behave smoothly as the time changes, i.e., $X_j^{n+1} = X_j^n + O(\varepsilon)$ as $\varepsilon \to 0$, where we have introduced the lattice spacing ε in the time lattice, so the continuous time variable is $t = n\varepsilon$. In terms of the X_j^n's the equations are rewritten as

$$\prod_{j=1}^{N} \frac{\sigma(X_i^n - X_j^{n+1} - \eta)\sigma(X_i^n - X_j^n + \eta)\sigma(X_i^n - X_j^{n-1})}{\sigma(X_i^n - X_j^{n+1})\sigma(X_i^n - X_j^n - \eta)\sigma(X_i^n - X_j^{n-1} + \eta)}$$

$$= -1 + \mu^{-2} \prod_{j=1}^{N} \frac{\sigma(X_i^n - X_j^{n+1} - \eta)\sigma(X_i^n - X_j^{n-1} + 2\eta)}{\sigma(X_i^n - X_j^{n+1})\sigma(X_i^n - X_j^{n-1} + \eta)}$$

$$+ \mu^{-2} \prod_{j=1}^{N} \frac{\sigma(X_i^n - X_j^{n+1} - 2\eta)\sigma(X_i^n - X_j^n + \eta)}{\sigma(X_i^n - X_j^{n+1})\sigma(X_i^n - X_j^n - \eta)}.$$

We should expand them in powers of ε, taking into account that

$$X_j^{n\pm1} = X_j \pm \varepsilon \dot{X}_j + \frac{1}{2}\varepsilon^2 \ddot{X}_j + O(\varepsilon^3)$$

as $\varepsilon \to 0$. It is enough to expand up to order ε. For consistency of the procedure we should require that μ^{-1} be of order ε.

Problem. Show that at $\mu^{-1}=\varepsilon$ the leading order gives the equations of motion of the deformed RS system for the X_j's.

Another possibility is to assume the smooth behavior for the original variables:

$$x_j^{n\pm 1}=x_j\pm \varepsilon \dot x_j +\frac{1}{2}\,\varepsilon^2 \ddot x_j +O(\varepsilon^3).$$

It is easy to see that in the case of general position, if $\mu^{-2}-1=O(1)$ as $\varepsilon\to 0$, the leafing order is ε, and the expansion gives the equations of motion of the RS system (non-deformed). However, if $\mu^{-2}-1=O(\varepsilon)$, say, $\mu^{-2}=1+\alpha\varepsilon+O(\varepsilon^2)$, then the first order yields the identity $0=0$, and one has to expand further, up to the second order in ε. This leads to the equations derived in [32] for dynamics of poles of elliptic solutions to the semi-discrete BKP equation. These equations are not resolved with respect to the second order derivatives $\ddot x_i$. We do not present them here because of their rather complicated form.

The spin CM system in discrete time

According to our approach, the spin CM system in discrete time can be obtained as dynamics of poles of singular solutions of the semi-discrete matrix KP equation. For simplicity, we will be restricted by rational solutions. To this end, we need some preparations first. To wit, we should represent the matrix KP hierarchy in the framework of the bilinear formalism.

The semi-discrete matrix KP hierarchy. As was already mentioned, the matrix KP heirarchy is a subhierarchy of the multi-component KP. The independent variables in the latter are n infinite sets of continuous times

$$\mathbf{t}=\{\mathbf{t}_1,\mathbf{t}_2,\dots,\mathbf{t}_n\},\qquad \mathbf{t}_\alpha=\{t_{\alpha,1},t_{\alpha,2},t_{\alpha,3},\dots\},\qquad \alpha=1,\dots,n.$$

It is convenient to introduce also the variable x such that

$$\partial_x=\sum_{\alpha=1}^{n}\partial_{t_{\alpha,1}}.$$

In the matrix hierarchy, the variables $\mathbf{t}$ are restricted as $t_{\alpha,m}=t_m$ for all α,m, and, in particular, $x=t_1$, but in order to write the bilinear relation, we will have to shift the times $t_{\alpha,m}$ separately.

The tau-function of the matrix KP hierarchy is the matrix $\tau_{\alpha\beta}(x,\mathbf{t})$, in which all diagonal elements are the same: $\tau_{\alpha\alpha}(x,\mathbf{t})=\tau(x,\mathbf{t})$. The integral bilinear equation for the tau-function has the form

$$\sum_{\gamma=1}^{n}\epsilon_{\alpha\gamma}\epsilon_{\beta\gamma}\oint_{C_\infty}dz\, z^{\delta_{\alpha\gamma}+\delta_{\beta\gamma}-2}e^{\xi(\mathbf{t}_\gamma-\mathbf{t}'_\gamma,\,z)}\tau_{\alpha\gamma}\Big(\mathbf{t}-[z^{-1}]_\gamma\Big)\,\tau_{\gamma\beta}\Big(\mathbf{t}'+[z^{-1}]_\gamma\Big)=0,$$

where $\epsilon_{\alpha\gamma}=1$ если $\alpha\le\gamma$, $\epsilon_{\alpha\gamma}=-1$ если $\alpha>\gamma$, and

$$\xi(\mathbf{t}_\gamma,z)=\sum_{k\ge 1}t_{\gamma,k}z^k,\qquad \Big(\mathbf{t}\pm[z^{-1}]_\gamma\Big)_{\alpha k}=t_{\alpha,k}\pm\delta_{\alpha\gamma}\frac{z^{-k}}{k}.$$

Problem. Prove the following functional relations as corollaries of the integral bilinear equation:

а)

$$\tau_{\alpha\beta}(\mathbf{t}-[\mu^{-1}]_\beta)\partial_{t_{\gamma,1}}\tau(\mathbf{t}) - \tau(\mathbf{t})\partial_{t_{\gamma,1}}\tau_{\alpha\beta}(\mathbf{t}-[\mu^{-1}]_\beta) + \frac{\epsilon_{\alpha\gamma}\epsilon_{\gamma\beta}}{\epsilon_{\alpha\beta}}\,\tau_{\alpha\gamma}(\mathbf{t})\tau_{\gamma\beta}(\mathbf{t}-[\mu^{-1}]_\beta) = 0$$

(no summation over repeated Greek indices!),

б)

$$\partial_{t_{\beta,1}}\tau_{\alpha\beta}(\mathbf{t}-[\nu^{-1}]_\beta)\tau(\mathbf{t}-[\mu^{-1}]_\alpha) - \partial_{t_{\beta,1}}\tau(\mathbf{t}-[\mu^{-1}]_\alpha)\tau_{\alpha\beta}(\mathbf{t}-[\nu^{-1}]_\beta)$$

$$+\nu\tau_{\alpha\beta}(\mathbf{t}-[\nu^{-1}]_\beta)\tau(\mathbf{t}-[\mu^{-1}]_\alpha) - \nu\tau_{\alpha\beta}(\mathbf{t})\tau(\mathbf{t}-[\mu^{-1}]_\alpha-[\nu^{-1}]_\beta) = 0,$$

в)

$$\epsilon_{\beta\alpha}\mu\tau(\mathbf{t})\tau_{\alpha\beta}\Big(\mathbf{t}+[\mu^{-1}]-[\nu^{-1}]_\beta\Big) + \epsilon_{\beta\alpha}\partial_{t_{\alpha,1}}\tau(\mathbf{t})\tau_{\alpha\beta}\Big(\mathbf{t}+[\mu^{-1}]-[\nu^{-1}]_\beta\Big)$$

$$-\epsilon_{\beta\alpha}\tau(\mathbf{t})\partial_{t_{\alpha,1}}\tau_{\alpha\beta}\Big(\mathbf{t}+[\mu^{-1}]-[\nu^{-1}]_\beta\Big)$$

$$+\epsilon_{\alpha\beta}(\mu-\nu)\tau_{\alpha\beta}\Big(\mathbf{t}-[\nu^{-1}]_\beta\Big)\tau\Big(\mathbf{t}+[\mu^{-1}]\Big) + \epsilon_{\alpha\beta}\nu\tau_{\alpha\beta}(\mathbf{t})\tau\Big(\mathbf{t}+[\mu^{-1}]-[\nu^{-1}]_\beta\Big)$$

$$-\sum_{\gamma\neq\alpha,\beta}\epsilon_{\alpha\gamma}\epsilon_{\beta\gamma}\tau_{\alpha\gamma}(\mathbf{t})\tau_{\gamma\beta}\Big(\mathbf{t}+[\mu^{-1}]-[\nu^{-1}]_\beta\Big) = 0,$$

where we have omitted the argument x for brevity.

The wave functions are given by the formulas

$$\Psi_{\alpha\beta}(x,\mathbf{t};z) = \epsilon_{\alpha\beta}\frac{\tau_{\alpha\beta}(x,\mathbf{t}-[z^{-1}]_\beta)}{\tau(x,\mathbf{t})}\,z^{\delta_{\alpha\beta}-1}e^{\xi(\mathbf{t}_\beta,z)},$$

$$\Psi^*_{\alpha\beta}(x,\mathbf{t};z) = \epsilon_{\beta\alpha}\,\frac{\tau_{\alpha\beta}(x,\mathbf{t}+[z^{-1}]_\alpha)}{\tau(x,\mathbf{t})}\,z^{\delta_{\alpha\beta}-1}e^{-\xi(\mathbf{t}_\alpha,z)}.$$

The function $\Psi_{\alpha\beta}$ in the matrix hierarchy has the expansion

$$\Psi_{\alpha\beta}(x,\mathbf{t};z) = \Big(\delta_{\alpha\beta} + w^{(1)}_{\alpha\beta}(x,\mathbf{t})z^{-1} + O(z^{-2})\Big)\,e^{xz+\xi(\mathbf{t},z)},$$

where $\xi(\mathbf{t},z) = \sum\limits_{k\geq 1} t_k z^k$. The coefficient $w^{(1)}_{\alpha\beta}(x,\mathbf{t})$ plays an important role. It is expressed through the tau-function in the following way:

$$w^{(1)}_{\alpha\beta}(x,\mathbf{t}) = \begin{cases} \epsilon_{\alpha\beta}\,\dfrac{\tau_{\alpha\beta}(x,\mathbf{t})}{\tau(x,\mathbf{t})} & \text{если } \alpha\neq\beta, \\[2ex] -\,\dfrac{\partial_{t_{\alpha,1}}\tau(x,\mathbf{t})}{\tau(x,\mathbf{t})} & \text{если } \alpha=\beta. \end{cases}$$

Evolution in the discrete time $p\in\mathbb{Z}$ is defined as shifting the continuous times according to the rule [33]

$$\tau^p = \tau\left(\mathbf{t} - p\sum_{\alpha=1}^{N}[\mu^{-1}]_\alpha\right), \qquad \Psi^p = \Psi\left(\mathbf{t} - p\sum_{\alpha=1}^{N}[\mu^{-1}]_\alpha; z\right),$$

where we again omit the variable x identifying it with t_1. Here μ is a continuous parameter which plays the role of inverse lattice spacing. Each value of μ defined its own flow in discrete time. This hierarchy is called semi-discrete because the space variable x (and t_1) remain continuous. One can show, using the bilinear equations from the problem above, that the linear equations for the wave functions have the form

$$\mu\Psi^p - \mu\Psi^{p+1} = \partial_x\Psi^p + \left(w^{(1)}(p+1) - w^{(1)}(p)\right)\Psi^p,$$

$$\mu\Psi^{*p} - \mu\Psi^{*\,p-1} = -\partial_x\Psi^{*p} + \Psi^{*p}\left(w^{(1)}(p) - w^{(1)}(p-1)\right),$$

or, in components,

$$\mu\Psi^p_{\alpha\beta} - \mu\Psi^{p+1}_{\alpha\beta} = \partial_x\Psi^p_{\alpha\beta} + \sum_\gamma\left(w^{(1)}_{\alpha\gamma}(p+1) - w^{(1)}_{\gamma\alpha}(p)\right)\Psi^p_{\gamma\beta},$$

$$\mu\Psi^{*p}_{\alpha\beta} - \mu\Psi^{*\,p-1}_{\alpha\beta} = -\partial_x\Psi^{*p}_{\alpha\beta} + \sum_\gamma\Psi^{*p}_{\alpha\gamma}\left(w^{(1)}_{\gamma\beta}(p) - w^{(1)}_{\gamma\beta}(p-1)\right).$$

Below we will use them for analysis of the dynamics of poles of rational in $x = t_1$ solutions of the semi-discrete matrix KP hierarchy.

Rational solutions. We study solutions of the semi-discrete matrix KP hierarchy that are rational functions of x. The tau-function for such solutions is a polynomial of x:

$$\tau^p = C\prod_{i=1}^{\mathcal{N}}(x - x_i(p)).$$

The roots x_i of this polynomial depend on the times $\mathbf{t}$ and on the discrete time p. In this section, we will denote the dependence on the discrete time p as an argument in brackets rather than an upper index. The matrix wave functions Ψ, Ψ^* (and, therefore, the coefficient $w^{(1)}$) as functions of x have simple poles at $x = x_i$. From the general theory of algebraic-geometrical solutions [17] it follows (see also the work [34], where this is derived from the bilinear equations for the tau-function), that residues at the poles are matrices of rank 1. We can parametrize them by the vectors $\mathbf{a}_i = (a_i^1, \dots, a_i^n)^{\mathrm{T}}$, $\mathbf{b}_i = (b_i^1, \dots, b_i^n)^{\mathrm{T}}$:

$$\operatorname*{res}_{x=x_i} w^{(1)}_{\alpha\beta} = -a_i^\alpha b_i^\beta \qquad \text{or} \qquad \operatorname*{res}_{x=x_i} w^{(1)} = -\mathbf{a}_i\mathbf{b}_i^{\mathrm{T}}.$$

For the residues we have [34]:

$$\operatorname*{res}_{x=x_i}\Psi_{\alpha\beta} = e^{x_i z + \xi(\mathbf{t},z)}a_i^\alpha c_i^\beta, \qquad \operatorname*{res}_{x=x_i}\Psi^*_{\alpha\beta} = e^{-x_i z - \xi(\mathbf{t},z)}c_i^{*\alpha}b_i^\beta,$$

where c_i^α, $c_i^{*\alpha}$ are components of the vectors $\mathbf{c}_i = (c_i^1, \dots, c_i^n)^{\mathrm{T}}$, $\mathbf{c}_i^* = (c_i^{*1}, \dots, c_i^{*n})$. The vectors $\mathbf{a}_i$, $\mathbf{b}_i$ depend on the times t_k c $k \geq 2$, while the vectors $\mathbf{c}_i$, $\mathbf{c}_i^*$ depend also on z. The dependence of the vectors on the discrete time will be denoted as $a_i^\alpha = a_i^\alpha(p)$, $b_i^\alpha = b_i^\alpha(p)$. We have the following representation for the matrix wave functions:

$$\Psi_{\alpha\beta} = e^{xz+\xi(\mathbf{t},z)}\left(C_{\alpha\beta} + \sum_{i=1}^N \frac{a_i^\alpha c_i^\beta}{x - x_i}\right),$$

$$\Psi^*_{\alpha\beta} = e^{-xz-\xi(\mathbf{t},z)} \left(C^{-1}_{\alpha\beta} + \sum_{i=1}^{N} \frac{c_i^{*\alpha} b_i^{\beta}}{x - x_i} \right),$$

where the matrix $C_{\alpha\beta}$ does not depend on x, and $C^{-1}_{\alpha\beta}$ are matrix elements of the inverse matrix. The matrices $w^{(1)}$ и $V = -2\partial_x w^{(1)}$ have the form

$$w^{(1)}_{\alpha\beta} = S_{\alpha\beta} - \sum_{i=1}^{\mathcal{N}} \frac{a_i^{\alpha} b_i^{\beta}}{x - x_i}, \qquad V_{\alpha\beta} = -2 \sum_{i=1}^{\mathcal{N}} \frac{a_i^{\alpha} b_i^{\beta}}{(x - x_i)^2},$$

where the matrix S does not depend on x. Tending $x \to \infty$ in the relation

$$\frac{1}{2\pi i} \sum_{\gamma=1}^{n} \oint_{C_\infty} dz\, z^m \Psi_{\alpha\gamma}(\mathbf{t}; z) \Psi^*_{\gamma\beta}(\mathbf{t}; z) = -\partial_{t_m} w^{(1)}_{\alpha\beta}(\mathbf{t}),$$

which follows from the integral bilinear relation, we conclude that $\partial_{t_m} S = 0$ for all $m \geq 1$, hence the matrix S does not depend on all the times.

First we consider the pole dynamics in t_2. For this, consider the linear problems

$$\partial_{t_2} \Psi_{\alpha\beta} = \partial_x^2 \Psi_{\alpha\beta} - 2 \sum_{i=1}^{N} \sum_{\gamma} \frac{a_i^{\alpha} b_i^{\gamma}}{(x - x_i)^2} \Psi_{\gamma\beta},$$

$$-\partial_{t_2} \Psi^*_{\alpha\beta} = \partial_x^2 \Psi^*_{\alpha\beta} - 2 \sum_{\gamma} \Psi^*_{\alpha\gamma} \sum_{i=1}^{N} \frac{a_i^{\gamma} b_i^{\beta}}{(x - x_i)^2}$$

and substitute into them the pole ansatz for the wave functions. First consider the equation for Ψ. Comparing the behavior of both sides as $x \to \infty$, we concludde that $\partial_{t_2} C_{\alpha\beta} = 0$, hence $C_{\alpha\beta}$ does not depend on t_2. (In a similar way, considering the linear problems for higher times, one can see that $C_{\alpha\beta}$ does not depend on all t_k's.) Equating the coefficients at the poles $x = x_i$ of different orders, we obtain the following conditions:

- at $\frac{1}{(x-x_i)^3}$: $\quad b_i^{\gamma} a_i^{\gamma} = 1$ or $\mathbf{b}_i^{\mathrm{T}} \mathbf{a}_i = 1$;
- at $\frac{1}{(x-x_i)^2}$: $\quad a_i^{\alpha} c_i^{\beta} \dot{x}_i = -2z a_i^{\alpha} c_i^{\beta} - 2a_i^{\alpha} \tilde{b}_i^{\beta} - 2 \sum_{k \neq i} \frac{a_i^{\alpha} b_i^{\gamma} a_k^{\gamma} c_k^{\beta}}{x_i - x_k}, \qquad \tilde{b}_i^{\beta} = b_i^{\gamma} C_{\gamma\beta};$
- at $\frac{1}{x-x_i}$: $\quad \partial_{t_2}(a_i^{\alpha} c_i^{\beta}) = -2 \sum_{k \neq i} \frac{a_k^{\alpha} b_k^{\gamma} a_i^{\gamma} c_i^{\beta} - a_i^{\alpha} b_i^{\gamma} a_k^{\gamma} c_k^{\beta}}{(x_i - x_k)^2}.$

Hereafter summation over repeated Greek indices is assumed and dor means the t_2-derivative. The conditions coming from cancellation of third order poles are already familiar constraints for the vectors $\mathbf{a}_i$, $\mathbf{b}_i$. The conditions coming from poles of the second order can be written in the matrix form:

$$\sum_{k=1}^{N} (zI - L)_{ik} c_k^{\alpha} = -\tilde{b}_i^{\alpha}, \qquad L_{ik} = -\frac{\dot{x}_i}{2} \delta_{ik} - (1 - \delta_{ik}) \frac{b_i^{\gamma} a_k^{\gamma}}{x_i - x_k}.$$

Here L is the Lax matrix of the spin CM system. The conditions coming from simple poles yield evolution equations in the time t_2. Similar calculations for the linear problem for Ψ^* give the same constraints $\mathbf{b}_i^{\mathrm{T}}\mathbf{a}_i = 1$ and linear equations for the components $\mathbf{c}_i^*$ with the same Lax matrix L:

$$\sum_{k=1}^{N} c_k^{*\alpha}(zI - L)_{ki} = \tilde a_i^{\alpha}, \qquad \tilde a_i^{\alpha} = C^{-1}_{\alpha\gamma} a_i^{\gamma}.$$

For completeness, we give also already familiar equations of motion in the time t_2:

$$\dot a_i^{\alpha} = -2\sum_{k\neq i}\frac{b_k^{\gamma}a_i^{\gamma}a_k^{\alpha}}{(x_i-x_k)^2}, \qquad \dot b_i^{\alpha} = 2\sum_{k\neq i}\frac{b_i^{\gamma}a_k^{\gamma}b_k^{\alpha}}{(x_i-x_k)^2},$$

$$\ddot x_i = -8\sum_{k\neq i}\frac{b_i^{\gamma}a_k^{\gamma}b_k^{\gamma'}a_i^{\gamma'}}{(x_i-x_k)^3}.$$

Now we are ready to study evolution in discrete time. The wave functions are of the form

$$\Psi^p_{\alpha\beta} = \Big(1-\frac{z}{\mu}\Big)^{p} e^{xz+\xi(\mathbf{t},z)}\left(C_{\alpha\beta} + \sum_i \frac{a_i^{\alpha}(p)c_i^{\beta}(p)}{x-x_i(p)}\right),$$

$$\Psi^{*p}_{\alpha\beta} = \Big(1-\frac{z}{\mu}\Big)^{-p} e^{-xz-\xi(\mathbf{t},z)}\left(C^{-1}_{\alpha\beta} + \sum_i \frac{c_i^{*\alpha}(p)b_i^{\beta}(p)}{x-x_i(p)}\right).$$

We should substitute them into the linear problems for discrete time and equate coefficients in front of poles at $x = x_i(p)$ and $x = x_i(p+1)$. Note that the constant term $S_{\alpha\beta}$ in $w^{(1)}_{\alpha\beta}(p)$ cancels in the combination $w^{(1)}_{\alpha\beta}(p+1) - w^{(1)}_{\alpha\beta}(p)$ because the shift $p \to p+1$ is equivalent to a shift of continuous times, but $S_{\alpha\beta}$ does not depend on them. Cancellation of the poles yield the conditions:

- at $\frac{1}{(x-x_i(p))^2}$: $\quad b_i^{\gamma}(p)a_i^{\gamma}(p) = 1$;
- at $\frac{1}{x-x_i(p+1)}$:

$$(z-\mu)a_i^{\alpha}(p+1)c_i^{\beta}(p+1) = -a_i^{\alpha}(p+1)\tilde b_i^{\beta}(p+1) - \sum_j \frac{a_i^{\alpha}(p+1)b_i^{\gamma}(p+1)a_j^{\gamma}(p)c_j^{\beta}(p)}{x_i(p+1)-x_j(p)};$$

- at $\frac{1}{x-x_i(p)}$:

$$(z-\mu)a_i^{\alpha}(p)c_i^{\beta}(p) + a_i^{\alpha}(p)\tilde b_i^{\beta}(p) - \sum_j \frac{a_j^{\alpha}(p+1)b_j^{\gamma}(p+1)a_i^{\gamma}(p)c_i^{\beta}(p)}{x_i(p)-x_j(p+1)}$$

$$+\sum_{j\neq i}\frac{a_i^{\alpha}(p)b_i^{\gamma}(p)a_j^{\gamma}(p)c_j^{\beta}(p)}{x_i(p)-x_j(p)} + \sum_{j\neq i}\frac{a_j^{\alpha}(p)b_j^{\gamma}(p)a_i^{\gamma}(p)c_i^{\beta}(p)}{x_i(p)-x_j(p)} = 0.$$

In a similar way, from the linear problem for Ψ^* we obtain:

- at $\frac{1}{(x-x_i(p))^2}$: $\quad b_i^{\gamma}(p)a_i^{\gamma}(p) = 1$;

- at $\frac{1}{x-x_i(p-1)}$:

$$(z-\mu)c_i^{*\alpha}(p-1)b_i^\beta(p-1) = \tilde{a}_i^\alpha(p-1)b_i^\beta(p-1) + \sum_j \frac{c_j^{*\alpha}(p)b_j^\gamma(p)a_i^\gamma(p-1)b_i^\beta(p-1)}{x_i(p-1)-x_j(p)};$$

- at $\frac{1}{x-x_i(p)}$:

$$(z-\mu)c_i^{*\alpha}(p)b_i^\beta(p) - \tilde{a}_i^\alpha(p)b_i^\beta(p) + \sum_j \frac{c_i^{*\alpha}(p)b_i^\gamma(p)a_j^\gamma(p-1)b_j^\beta(p-1)}{x_i(p)-x_j(p-1)}$$

$$-\sum_{j\neq i} \frac{c_i^{*\alpha}(p)b_i^\gamma(p)a_j^\gamma(p)b_j^\beta(p)}{x_i(p)-x_j(p)} + \sum_{j\neq i} \frac{c_j^{*\alpha}(p)b_j^\gamma(p)a_i^\gamma(p)b_i^\beta(p)}{x_j(p)-x_i(p)} = 0.$$

The conditions coming from second order poles are the same constraints, as before. Introduce the matrices

$$L_{ij}(p) = -\delta_{ij}\frac{\dot{x}_i(p)}{2} - (1-\delta_{ij})\,\frac{b_i^\gamma(p)a_j^\gamma(p)}{x_i(p)-x_j(p)}$$

(the Lax matrix) and

$$M_{ij}(p) = \frac{b_i^\gamma(p+1)a_j^\gamma(p)}{x_i(p+1)-x_j(p)},$$

then the conditions coming from simple poles at $x_i(p)$ and $x_i(p\pm 1)$ can be written as

$$\begin{cases} (z-\mu)c_i^\beta(p+1) = -\tilde{b}_i^\beta(p+1) - \sum_j M_{ij}(p)c_j^\beta(p) \\ \\ a_i^\alpha(p)\underbrace{\left[\sum_j \Big(z\delta_{ij} - L_{ij}(p)\Big)c_j^\beta(p) + \tilde{b}_i^\beta(p)\right]}_{=0} \\ \\ \qquad\qquad + c_i^\beta(p)\left[\sum_j a_j^\alpha(p+1)M_{ji}(p) + \sum_j a_j^\alpha(p)L_{ji}(p) - \mu a_i^\alpha(p)\right] = 0, \end{cases}$$

$$\begin{cases} (z-\mu)c_i^{*\alpha}(p-1) = \tilde{a}_i^\alpha(p-1) - \sum_j c_j^{*\alpha}(p)M_{ji}(p-1) \\ \\ b_i^\beta(p)\underbrace{\left[\sum_j c_j^{*\alpha}(p)\Big(z\delta_{ij} - L_{ji}(p)\Big) - \tilde{a}_i^\alpha(p)\right]}_{=0} \\ \\ \qquad\qquad + c_i^{*\alpha}(p)\left[\sum_j M_{ij}(p-1)b_j^\beta(p-1) + \sum_j L_{ij}(p)b_j^\beta(p) - \mu b_i^\beta(p)\right] = 0. \end{cases}$$

Introduce N-component column vectors $\mathbf{C}^\alpha = (c_1^\alpha, \dots, c_N^\alpha)^{\mathrm{T}}$, $\mathbf{C}^{*\alpha} = (c_1^{*\alpha}, \dots, c_N^{*\alpha})^{\mathrm{T}}$, $\mathbf{A}^\alpha = (a_1^\alpha, \dots, a_N^\alpha)^{\mathrm{T}}$, $\mathbf{B}^\alpha = (b_1^\alpha, \dots, b_N^\alpha)^{\mathrm{T}}$ and $\tilde{\mathbf{A}}^\alpha = (\tilde{a}_1^\alpha, \dots, \tilde{a}_N^\alpha)^{\mathrm{T}}$, $\tilde{\mathbf{B}}^\alpha = (\tilde{b}_1^\alpha, \dots, \tilde{b}_N^\alpha)^{\mathrm{T}}$. In this notation, our equations acquire the form of linear equations for the vectors $\mathbf{A}^\alpha$ and $\mathbf{B}^\beta$:

$$\begin{cases} \mathbf{A}^{\alpha T}(p+1)M(p) + \mathbf{A}^{\alpha \mathrm{T}}(p)L(p) = \mu \mathbf{A}^{\alpha \mathrm{T}}(p) \\ M(p-1)\mathbf{B}^\beta(p-1) + L(p)\mathbf{B}^\beta(p) = \mu \mathbf{B}^\beta(p) \end{cases}$$

and linear equations for the vectors $\mathbf{C}^\alpha$ и $\mathbf{C}^{*\alpha}$:

$$\begin{cases} (z-\mu)\mathbf{C}^\beta(p+1) = -\tilde{\mathbf{B}}^\beta(p+1) - M(p)\mathbf{C}^\beta(p) \\ (z-\mu)\mathbf{C}^\beta(p) = -\tilde{\mathbf{B}}^\beta(p) + L(p)\mathbf{C}^\beta(p) - \mu\mathbf{C}^\beta(p), \end{cases}$$

$$\begin{cases} (z-\mu)\mathbf{C}^{*\alpha \mathrm{T}}(p-1) = \tilde{\mathbf{A}}^{\alpha \mathrm{T}}(p-1) - \mathbf{C}^{*\alpha \mathrm{T}}(p)M(p-1) \\ (z-\mu)\mathbf{C}^{*\alpha \mathrm{T}}(p) = \tilde{\mathbf{A}}^{\alpha \mathrm{T}}(p) + \mathbf{C}^{*\alpha \mathrm{T}}(p)L(p) - \mu\mathbf{C}^{*\alpha \mathrm{T}}(p). \end{cases}$$

From these equations we have:

$$M(p)\mathbf{C}^\beta(p) + \Big(L(p+1) - \mu I\Big)\mathbf{C}^\beta(p+1) = 0,$$

$$\mathbf{C}^{*\alpha \mathrm{T}}(p+1)M(p) + \mathbf{C}^{*\alpha \mathrm{T}}(p)\Big(L(p) - \mu I\Big) = 0.$$

Combining these equations, we obtain:

$$M(p)\Big(-\tilde{\mathbf{B}}^\beta(p) + L(p)\mathbf{C}^\beta(p) - \mu\mathbf{C}^\beta(p)\Big) - \Big(L(p+1) - \mu I\Big)\Big(\tilde{\mathbf{B}}^\beta(p+1) + M(p)\mathbf{C}^\beta(p)\Big) = 0,$$

$$\Big(\tilde{\mathbf{A}}^{\alpha T}(p) + \mathbf{C}^{*\alpha T}(p)L(p) - \mu\mathbf{C}^{*\alpha T}(p)\Big)M(p-1) \\ + \Big(\tilde{\mathbf{A}}^{\alpha T}(p-1) - \mathbf{C}^{*\alpha T}(p)M(p-1)\Big)\Big(L(p-1) - \mu I\Big) = 0,$$

or, after simplifications,

$$\Big(M(p)L(p) - L(p+1)M(p)\Big)\mathbf{C}^\beta(p) = 0,$$

$$\mathbf{C}^{*\alpha \mathrm{T}}(p+1)\Big(M(p)L(p) - L(p+1)M(p)\Big) = 0.$$

This implies the compatibility condition

$$L(p+1)M(p) = M(p)L(p)$$

or $L(p+1) = M(p)L(p)M^{-1}(p)$, i.e., the discrete Lax equation. By a direct calculation it is possible to show that it is satisfied if the already mentioned equations hold:

$$\begin{cases} \mathbf{A}^{\alpha T}(p+1)M(p) + \mathbf{A}^{\alpha \mathrm{T}}(p)L(p) = \mu \mathbf{A}^{\alpha \mathrm{T}}(p) \\ M(p-1)\mathbf{B}^\beta(p-1) + L(p)\mathbf{B}^\beta(p) = \mu \mathbf{B}^\beta(p). \end{cases}$$

In the explicit form these equations read:

$$\sum_j \frac{a_j^\alpha(p+1)b_j^\gamma(p+1)a_i^\gamma(p)}{x_j(p+1)-x_i(p)} = \frac{\dot x_i(p)}{2}\, a_i^\alpha(p) + \sum_{j\neq i} \frac{a_j^\alpha(p)b_j^\gamma(p)a_i^\gamma(p)}{x_j(p)-x_i(p)} + \mu a_i^\alpha(p),$$

$$\sum_j \frac{b_i^\gamma(p)a_j^\gamma(p-1)b_j^\beta(p-1)}{x_i(p)-x_j(p-1)} = \frac{\dot x_i(p)}{2}\, b_i^\beta(p) + \sum_{j\neq i} \frac{b_i^\gamma(p)a_j^\gamma(p)b_j^\beta(p)}{x_i(p)-x_j(p)} + \mu b_i^\beta(p).$$

Multiply the first equation by $b_i^\alpha(p)$ and sum over α, after that multiply the second equation by $a_i^\beta(p)$ and sum over β. Taking into account that the constraints $b_i^\gamma a_i^\gamma = 1$ hold, and subtracting the equations obtained above from each other, we obtain the following equations of motion:

$$\sum_j \frac{b_i^\gamma(p)a_j^\gamma(p+1)b_j^\beta(p+1)a_i^\beta(p)}{x_i(p)-x_j(p+1)} + \sum_j \frac{b_i^\gamma(p)a_j^\gamma(p-1)b_j^\beta(p-1)a_i^\beta(p)}{x_i(p)-x_j(p-1)}$$

$$= 2\sum_{j\neq i} \frac{b_i^\gamma(p)a_j^\gamma(p)b_j^\beta(p)a_i^\beta(p)}{x_i(p)-x_j(p)},$$

which generalize the equations of motion for the CM system in discrete time to the spin case. The previous two equations are not closed since they contain $\dot x_i$, i.e., the derivative of x_i with respect to the "alien" time t_2. Nevertheless, we will show that their continuum limit gives the required equations of motion of the spin CM system for the spin variables.

Continuum limit. Set $x_i(p) = \lambda p + y_i(p)$ and expand $x_i(p\pm 1) = \pm\lambda + x_i(p) \pm \varepsilon\partial_t y_i + \frac{\varepsilon^2}{2}\,\partial_t^2 y_i + O(\varepsilon^3)$, $a_i^\alpha(p\pm 1) = a_i^\alpha \pm \varepsilon\partial_t a_i^\alpha + O(\varepsilon^2)$, $b_i^\alpha(p\pm 1) = b_i^\alpha \pm \varepsilon\partial_t b_i^\alpha + O(\varepsilon^2)$, where $\lambda = O(\sqrt{\varepsilon})$. Separating the terms with $j=i$ in the sums that enter the equations of motion, expand it up to the first non-vanishing order $O(\varepsilon)$ as $\varepsilon\to 0$. This yields the equations of motion for the spin CM system in continuous time:

$$\partial_t^2 y_i = -2g\sum_{j\neq i} \frac{b_i^\gamma a_j^\gamma b_j^\beta a_i^\beta}{(y_i-y_j)^3}, \qquad g = \lambda^4/\varepsilon^2 = O(1),$$

Hence we see that it is necessary to put $\lambda^4 = 4\varepsilon^2$.

Next, let us consider the continuum limit of the other two equations. Expanding the second equation as $\lambda,\varepsilon\to 0$, we obtain:

$$\frac{b_i^\beta}{\lambda} - \frac{\varepsilon}{\lambda^2}\, b_i^\beta \partial_t y_i - \frac{\varepsilon}{\lambda}\, b_i^\gamma \partial_t a_i^\gamma b_i^\beta - \frac{\varepsilon}{\lambda}\,\partial_t b_i^\beta - \lambda\sum_{j\neq i}\frac{b_i^\gamma a_j^\gamma b_j^\beta}{(y_i-y_j)^2} + O(\varepsilon) = \frac{\dot y_i}{2}\, b_i^\beta + \mu b_i^\beta.$$

Comparison of the leading terms gives $\lambda = \mu^{-1}$. Terms of the next order lead to the relation

$$\partial_t y_i = -\frac{\lambda^2}{2\varepsilon}\,\dot y_i - \lambda b_i^\gamma \partial_t a_i^\gamma + O(\varepsilon).$$

We expect that in the continuum limit the time t that corresponds to p tends to t_2. That is why we should require $\lambda^2 = -2\varepsilon$, which agrees with the relation $\lambda^4 = 4\varepsilon^2$ between them mentioned above. Then in the order $O(\lambda)$ we have:

$$\partial_t b_i^\beta = 2\sum_{j\neq i} \frac{b_i^\gamma a_j^\gamma b_j^\beta}{(y_i - y_j)^2},$$

which is the equation of motion for the spin variables in the continuous time. The continuum limit of the first equation is considered in the similar way.

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