# Integrable systems of particles and nonlinear equations. Lecture 13

A. Zabrodin*

## Equations of motion of integrable many-body systems as conditions of existence of meromorphic solutions to linear equations

This section is devoted to an alternative approach to derivation of equations of motion for integrable systems of particles. Its advantage is that it is rather direct and economic. This approach was suggested by Krichever (see [25, 26, 11]). Within this approach, the equations of motion are obtained as conditions of existence of meromorphic solutions to linear differential or difference equations, which serve as auxiliary linear problems for integrable nonlinear equations. A weak side of this method is that it does not allow one to find commutation representations for the equations of motion.

**The linear equation for KP** Consider the linear equation

$$(\partial_t - \partial_x^2 - 2u)\psi = 0,$$

which is one of the auxiliary linear problems for the KP equation. It is easy to see that if the function $u = u(x)$ has a pole at some point $a$ in the complex $x$-plane, then this pole has to be of the second order with zero residue. Expanding the left-hand side in a neighborhood of this pole, one can find a necessary condition for a meromorphic solution to the equation to exist in this neighborhood. The expansions are:

$$u = -\frac{1}{(x-a)^2} + u_0 + u_1(x-a) + \ldots,$$

$$\psi = \frac{\alpha}{x-a} + \beta + \gamma(x-a) + \delta(x-a)^2 + \ldots.$$

Plugging them into the linear equation, we see that the highest poles (of the third order) cancel identically. Equating the coefficients in front of $(x-a)^{-2}$, $(x-a)^{-1}$ and $(x-a)^0$

*e-mail: zabrodin@itep.ru

to zero, we obtain the conditions

$$\begin{cases} \dot{a}\alpha + 2\beta = 0, \\ \\ \dot{\alpha} + 2\gamma - 2u_0\alpha = 0, \\ \\ \dot{\beta} - \dot{a}\gamma - 2u_0\beta - 2u_1\alpha = 0, \end{cases}$$

where dot means the $t$-derivative, as usual. Taking the $t$-derivative of the first condition, substituting in the result $\dot{\alpha}$ and $\dot{\beta}$ from the second and the third ones, and then using the first one again, we obtain the necessary condition:

$$\ddot{a} + 4u_1 = 0.$$

It is possible to obtain the equations of motion for the CM system (in general elliptic) from it. Indeed, let $u(x)$ be a double-periodic function with poles at the points $x_i$:

$$u = -\sum_i \wp(x - x_i).$$

Its expansion near each pole at $x_i$ has the form given above, where

$$u_0 = -\sum_{j \neq i} \wp(x_i - x_j), \qquad u_1 = -\sum_{j \neq i} \wp'(x_i - x_j),$$

so the conditions of existence of a meromorphic solution in a neighborhood of each pole give the equations of motion for the elliptic CM system:

$$\ddot{x}_i = 4\sum_{j \neq i} \wp'(x_i - x_j).$$

**The linear equation for BKP.** Consider the linear equation

$$(\partial_t - \partial_x^3 - 6u\partial_x)\psi = 0,$$

which is one of the auxiliary linear problems for the BKP equation. Suppose that the function $u$ has a pole at some point $a$, and that there exists a meromorphic solution for $\psi$ in a neighborhood of this point. The expansions are:

$$u = -\frac{1}{(x-a)^2} + u_0 + u_1(x-a) + u_2(x-a)^2 + u_3(x-a)^3 + \dots,$$

$$\psi = \frac{\alpha}{x-a} + \beta + \gamma(x-a) + \delta(x-a)^2 + \varepsilon(x-a)^3 + \mu(x-a)^4 + \dots.$$

Plugging them into the left-hand side of the linear equation, we see that possible poles of the fourth and third order cancel identically. Equating to zero the coefficients in front of $(x-a)^{-2}$, $(x-a)^{-1}$, $(x-a)^0$ and $(x-a)$, we obtain the conditions

$$\begin{cases} \alpha\dot{a} + 6\alpha u_0 + 6\gamma = 0, \\ \\ \dot{\alpha} + 6\alpha u_1 + 12\delta = 0, \\ \\ \dot{\beta} - \gamma\dot{a} - 6\gamma u_0 + 6\alpha u_2 + 12\varepsilon = 0, \\ \\ \dot{\gamma} - 2\delta\dot{a} - 12\delta u_0 - 6\gamma u_1 + 6\alpha u_3 = 0. \end{cases}$$

Note that the coefficient $\mu$, which could enter the last condition, cancels. Let us take the $t$-derivative of the first condition:

$$\ddot{a} + 6\dot{u}_0 + 6\frac{\dot{\gamma}}{\alpha} - 6\frac{\gamma\dot{\alpha}}{\alpha^2} = 0.$$

Substituting $\dot{\alpha}$ from the second condition and $\dot{\gamma}$ from the fourth one, we find the condition of existence of a meromorphic solution:

$$\ddot{a} + 6\dot{u}_0 - 12u_1(\dot{a} + 6u_0) - 36u_3 = 0.$$

This condition allows one to obtain the equations of motion of the system discussed before in connection with elliptic solutions to the BKP equation. Indeed, let $u(x)$ be the double-periodic function

$$u = -\sum_i \wp(x - x_i),$$

then its expansion in a neighborhood of $x_i$ has the form given above with

$$u_0 = -\sum_{j\neq i} \wp(x_i - x_j), \quad u_1 = -\sum_{j\neq i} \wp'(x_i - x_j), \quad u_3 = -\frac{1}{6}\sum_{j\neq i} \wp'''(x_i - x_j),$$

and

$$\dot{u}_0 = -\sum_{j\neq i} (\dot{x}_i - \dot{x}_j)\wp'(x_i - x_j).$$

Hence we see that our condition yields the equations of motion

$$\ddot{x}_i + 6\sum_{j\neq i} (\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{j\neq k\neq i} \wp(x_i - x_j)\wp'(x_i - x_k) = 0,$$

obtained earlier from other arguments.

**The linear problem for the 2D Toda chain in the time $t_1$.** Consider the differential-difference linear equation

$$\partial_t \psi(x) = \psi(x + \eta) + b(x)\psi(x),$$

which is the auxiliary linear problem for the 2D Toda chain for the $t = t_1$-flow. Suppose that the function $b(x)$ has a simple pole at some point $a$, then the equation implies that it has to have another pole at the point $a - \eta$:

$$b(x) = \begin{cases} \dfrac{\nu}{x - a} + \mu_0 + O(x - a), & x \to a \\[2ex] -\dfrac{\nu}{x - a + \eta} + \mu_1 + O(x - a + \eta), & x \to a - \eta. \end{cases}$$

The expansion of the function $\psi(x)$ in a neighborhood of the point $a$ is of the form

$$\psi(x) = \frac{\alpha}{x - a} + \beta + O(x - a),$$

then

$$\partial_t \psi(x) = \frac{\alpha\dot{a}}{(x - a)^2} + \frac{\dot{\alpha}}{x - a} + O(1).$$

Note that the function $\psi$ is regular at the point $a-\eta$. Substitute these expansions into the linear equation:

$$\frac{\alpha\dot a}{(x-a)^2}+\frac{\dot\alpha}{x-a}+O(1)=\Big(\frac{\nu}{x-a}+\mu_0+\ldots\Big)\Big(\frac{\alpha}{x-a}+\beta+\ldots\Big).$$

Equating the coefficients in front of the poles in the left- and right-hand sides, we obtain the conditions

$$\left\{\begin{array}{l}\nu=\dot a,\\[2ex] \dot\alpha=\nu\beta+\mu_0\alpha.\end{array}\right.$$

In a neighborhood of the point $a-\eta$ we have:

$$\partial_t\psi(a-\eta)+O(x-a+\eta)$$

$$=\frac{\alpha}{x-a+\eta}+\beta-\big(\frac{\nu}{x-a+\eta}+\mu_1+\ldots\big)\big(\psi(a-\eta)+(x-a+\eta)\psi'(a-\eta)+\ldots\big).$$

Equating the coefficients at $(x-a+\eta)^{-1}$ and $(x-a+\eta)^0$, we obtain the conditions

$$\left\{\begin{array}{l}\alpha=\nu\psi(a-\eta),\\[2ex] \partial_t\psi(a-\eta)=\beta-\mu_1\psi(a-\eta)-\nu\psi'(a-\eta).\end{array}\right.$$

Taking the $t$-derivative of the first equation and using the conditions obtained earlier, we find:

$$\dot\alpha=\ddot a\psi(a-\eta)+\dot a\dot\psi(a-\eta),$$

where

$$\dot\psi(a-\eta)=\partial_t\psi(a-\eta)+\dot a\psi'(a-\eta)$$

is the full $t$-derivative of $\psi(a-\eta)$ (here $\psi'(a-\eta)$ is the $x$-derivative at $a-\eta$). Combining these conditions, we arrive at the condition of existence of a meromorphic solution:

$$\ddot a-\dot a(\mu_0+\mu_1)=0.$$

Suppose now that $b(x)$ is an elliptic function of $x$, with simple poles at the points $x_j$, then it is of the form

$$b(x)=\sum_j\dot x_j\Big(\zeta(x-x_j)-\zeta(x-x_j+\eta)\Big).$$

Put $a=x_i$ for some $i$, then

$$\mu_0=\sum_{k\neq i}\dot x_k\zeta(x_i-x_k)-\sum_k\dot x_k\zeta(x_i-x_k+\eta),$$

$$\mu_1=\sum_{k\neq i}\dot x_k\zeta(x_i-x_k)-\sum_k\dot x_k\zeta(x_i-x_k-\eta),$$

and the conditions obtained above yield the equations of motion for the elliptic RS system:

$$\ddot x_i+\sum_{k\neq i}\dot x_i\dot x_k\Big(\zeta(x_i-x_k+\eta)+\zeta(x_i-x_k-\eta)-2\zeta(x_i-x_k)\Big)=0.$$

**The linear problem for the 2D Toda chain in the $\bar{t}_1$-flow.** Consider now the linear equation

$$\partial_t \psi(x) = v(x)\psi(x-\eta),$$

which is the linear problem for the 2D Toda chain in the time $t = \bar{t}_1$. Suppose that the function $v(x)$ has a second order pole at a point $a$ with the expansion

$$v(x) = \frac{\nu}{(x-a)^2} + \frac{\mu}{x-a} + O(1)$$

in a neighborhood of this point. Suppose also that $v(x)$ has zero at $a-\eta$: $v(a-\eta) = 0$. Then $\psi(x)$ has a simple pole at the point $a$:

$$\psi(x) = \frac{\alpha}{x-a} + O(1), \quad x \to a.$$

Plugging these expansions in the linear equation, we write:

$$\frac{\alpha \dot{a}}{(x-a)^2} + \frac{\dot{\alpha}}{x-a} = \Big(\frac{\nu}{(x-a)^2} + \frac{\mu}{x-a} + O(1)\Big)\Big(\psi(a-\eta) + (x-a)\psi'(a-\eta) + \dots\Big).$$

Equating the coefficients at the poles, we obtain the conditions

$$\left\{\begin{array}{l} \alpha \dot{a} = \nu \psi(a-\eta), \\ \\ \dot{\alpha} = \nu \psi'(a-\eta) + \mu \psi(a-\eta). \end{array}\right.$$

At $x = a-\eta$ our linear equation yields: $\partial_t \psi(a-\eta) = 0$, so the full $t$-derivative of $\psi(a-\eta)$ is

$$\dot{\psi}(a-\eta) = \dot{a}\psi'(a-\eta).$$

Combining the equations obtained, we find the condition

$$\nu \ddot{a} + \mu \dot{a}^2 - \dot{\nu}\dot{a} = 0.$$

Suppose now that $v(x)$ is an elliptic function of the form

$$v(x) = \prod_{j=1}^{N} \frac{\sigma(x-x_j+\eta)\sigma(x-x_j-\eta)}{\sigma^2(x-x_j)},$$

then we have, putting $a = x_i$:

$$\nu = -\sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(x_i-x_j+\eta)\sigma(x_i-x_j-\eta)}{\sigma^2(x_i-x_j)},$$

$$\mu = \nu \sum_{k \neq i} \Big(\zeta(x_i-x_k+\eta) + \zeta(x_i-x_k-\eta) - 2\zeta(x_i-x_k)\Big),$$

and this condition yields the same equations of motion for the elliptic RS system.

**The linear equation for the 2D Toda chain of type B.** Consider the linear equation

$$\partial_t \psi(x) = v(x)(\psi(x+\eta) - \psi(x-\eta)),$$

which is the auxiliary linear problem for the 2D Toda chain of type B. Suppose that the function $v(x)$ has a second order pole at some point $a$ with the expansion

$$v(x) = \frac{\nu}{(x-a)^2} + \frac{\mu}{x-a} + O(1)$$

in a neighborhood of this point. Suppose also that $v(x)$ has zeros at the points $a-\eta$ и $a+\eta$:

$$v(x) = \left\{ \begin{array}{l} (x-a-\eta)V^+(a) + O((x-a-\eta)^2), \quad x \to a+\eta, \\ \\ (x-a+\eta)V^-(a) + O((x-a+\eta)^2), \quad x \to a-\eta. \end{array} \right.$$

Then the function $\psi(x)$ has a simple pole at $a$:

$$\psi(x) = \frac{\alpha}{x-a} + O(1), \quad x \to a.$$

As $x \to a$, the linear equation with these expansions gives:

$$\frac{\alpha \dot a}{(x-a)^2} + \frac{\dot\alpha}{x-a}$$

$$= \Big(\frac{\nu}{(x-a)^2} + \frac{\mu}{x-a} + O(1)\Big)\Big(\psi(a+\eta) - \psi(a-\eta) + (x-a)(\psi'(a+\eta) - \psi'(a-\eta)) + \ldots\Big).$$

Equating the coefficients in front of the poles, we obtain the conditions

$$\left\{ \begin{array}{l} \alpha \dot a = \nu(\psi(a+\eta) - \psi(a-\eta)), \\ \\ \dot\alpha = \mu(\psi(a+\eta) - \psi(a-\eta)) + \nu(\psi'(a+\eta) - \psi'(a-\eta)). \end{array} \right.$$

At $x = a \pm \eta$ the linear equation yields:

$$\partial_t \psi(a \pm \eta) = \mp \alpha\, V^{\pm}(a).$$

Therefore,

$$\dot\psi(a \pm \eta) = \mp \alpha\, V^{\pm}(a) + \dot a \psi'(a \pm \eta).$$

Taking the $t$-derivative $\alpha \dot a = \nu(\psi(a+\eta) - \psi(a-\eta))$ and combining with the other equations, we obtain the condition

$$\nu \ddot a + \mu \dot a^2 - \dot\nu \dot a + \nu^2 (V^+(a) + V^-(a)) = 0.$$

Suppose that $v(x)$ is an elliptic function of the form

$$v(x) = \prod_{j=1}^{N} \frac{\sigma(x - x_j + \eta)\sigma(x - x_j - \eta)}{\sigma^2(x - x_j)},$$

then we have, putting $a = x_i$:

$$\nu = -\sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)\sigma(x_i - x_j - \eta)}{\sigma^2(x_i - x_j)},$$

$$\mu = \nu \sum_{k\neq i}\Big(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k)\Big),$$

$$V^{\pm}(x_i) = \pm\frac{\sigma(2\eta)}{\sigma^2(\eta)}\prod_{j\neq i}\frac{\sigma(x_i - x_j \pm 2\eta)\sigma(x_i - x_j)}{\sigma^2(x_i - x_j \pm \eta)},$$

and our condition gives the equations

$$\ddot{x}_i + \sum_{k\neq i}\dot{x}_i\dot{x}_k\Big(\zeta(x_i-x_k+\eta) + \zeta(x_i-x_k-\eta) - 2\zeta(x_i - x_k)\Big)$$

$$-\sigma(2\eta)\left[\prod_{j\neq i}\frac{\sigma(x_i-x_j+2\eta)\sigma(x_i-x_j-\eta)}{\sigma(x_i-x_j+\eta)\sigma(x_i-x_j)} - \prod_{j\neq i}\frac{\sigma(x_i-x_j-2\eta)\sigma(x_i-x_j+\eta)}{\sigma(x_i-x_j-\eta)\sigma(x_i-x_j)}\right] = 0,$$

which are equations of motion for the deformed RS system.

## Список литературы


[1] F. Calogero, *Solution of the one-dimensional $N$-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971) 419—436.

[2] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16** (1975) 197–220.

[3] A. Perelomov, *Integrable systems of classical mechanics and Lie algebras*, Birkhäuser Basel, 1990.

[4] M.A. Olshanetsky, A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71** (1981) 313–400.

[5] Yu. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Springer Basel AG, 2003.

[6] N.I. Akhiezer, *Elements of the theory of elliptic functions*, “Nauka”, Moscow, 1970.

[7] E.T. Whittaker, G.N. Watson, *A course of modern analysis*, Cambridge At the University Press, 1927 (Russian translation: Э.Т. Уиттекер, Дж.Н. Ватсон, *Курс современного анализа*, том II, Государственное издательство физико-математической литературы, Москва, 1963).

[8] T. Takebe, *Elliptic integrals and elliptic functions*, Springer, 2023.

[9] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986) 370–405.

[10] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987) 191–213.

[11] I. Krichever, A. Zabrodin, *Monodromy free linear equations and many-body systems*, Letters in Mathematical Physics 113:75 (2023).

[12] A. Zabrodin, *On integrability of the deformed Ruijsenaars-Schneider system*, Uspekhi Mat. Nauk **78:2** (2023) 149–188.

[13] T. Shiota, *Calogero-Moser hierarchy and KP hierarchy*, J. Math. Phys. **35** (1994) 5844–5849.

[14] A. Zabrodin, *KP hierarchy and trigonometric Calogero-Moser hierarchy*, J. Math. Phys. **61** (2020) 043502.

[15] V. Prokofev, A. Zabrodin, *Elliptic solutions to the KP hierarchy and elliptic Calogero-Moser model*, Journal of Physics A: Math. Theor., **54** (2021) 305202.

[16] J. Gibbons, T. Hermsen, *A generalization of the Calogero-Moser system*, Physica D **11** (1984) 337–348.

[17] I. Krichever, O. Babelon, E. Billey and M. Talon, *Spin generalization of the Calogero-Moser system and the matrix KP equation*, Amer. Math. Soc. Transl. Ser. 2 **170** (1995) 83–119.

[18] I. Krichever, A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian two-dimensional Toda chain and representations of the Sklyanin algebra*, Uspekhi Mat. Nauk **50:6** (1995) 3–56.

[19] D. Rudneva, A. Zabrodin, *Dynamics of poles of elliptic solutions to BKP equation*, Journal of Physics A: Math. Theor. **53** (2020) 075202.

[20] A. Zabrodin, *How Calogero-Moser particles can stick together*, J. Phys. A: Math. Theor. **54** (2021) 225201.

[21] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, Adv. Studies in Pure Math. **4** (1984) 1–95.

[22] P. Iliev, *Rational Ruijsenaars-Schneider hierarchy and bispectral difference operators*, Physica D **229** (2007), no. 2, 184–190.

[23] V. Prokofev, A. Zabrodin, *Elliptic solutions to Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model*, Teor. Mat. Fys., **208** (2021), No. 2, 282—309 (English translation: Theoretical and Mathematical Physics, **208** (2021) 1093–1115).

V. Prokofev, A. Zabrodin, *Elliptic solutions of the Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model*, ТМФ **208** (2021) 282—309.

[24] I. Krichever, A. Zabrodin, *Toda lattice with constraint of type B*, Physica D **453** (2023) 133827.

[25] I. Krichever, *Integrable linear equations and the Riemann-Schottky problem*, In: Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday. Ed. by Ginzburg, Victor. Basel: Birkhäuser. Progress in Mathematics **253** (2006) 497–514.

[26] I. Krichever, *Characterizing Jacobians via trisecants of the Kummer variety*, Annals of Mathematics **172** (2010) 485–516.