Integrable systems of particles and nonlinear equations. Lecture 12

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Nonabelian Toda chain and spin RS system

Elliptic solutions of the nonabelian (matrix) 2D Toda chain were studied in the paper [18]. The hierarchy of nonabelian 2D Toda chain can be constructed in a similar way as it was done for the matrix KP hierarchy, using pseudo-difference Lax operators with matrix coefficients. Here we will not discuss the general theory, concentrating on the first nontrivial equation which has the form

$$\partial_t((\partial_{\bar t} g(x))g^{-1}(x)) = g(x)g^{-1}(x-\eta) - g(x+\eta)g^{-1}(x),$$

where $g(x)$ is an $n \times n$ matrix. This equation is equivalent to compatibility of the over-determined system of linear problems

$$\partial_t \Psi(x) = \Psi(x+\eta) + V(x)\Psi(x),$$

$$\partial_{\bar t} \Psi(x) = C(x)\Psi(x-\eta),$$

where

$$V(x) = (\partial_t g(x))g^{-1}(x), \quad C(x) = g(x)g^{-1}(x-\eta).$$

The analysis of elliptic solutions is similar to the one done before for the matrix KP equation, so here we omit some details. Consider the first linear problem for the matrix function Ψ and the conjugated problem for the dual matrix function Ψ^*:

$$\partial_t \Psi_{\alpha\beta}(x) = \Psi_{\alpha\beta}(x+\eta) + V_{\alpha\gamma}(x)\Psi_{\gamma\beta}(x),$$

$$-\partial_t \Psi^*_{\alpha\beta}(x) = \Psi^*_{\alpha\beta}(x-\eta) + \Psi^*_{\alpha\gamma}(x)V_{\gamma\beta}(x),$$

where, as usual, summation over repeated Greek indices is assumed. The pole ansatz for elliptic solutions has the form

$$V_{\alpha\beta}(x) = \sum_i a_i^\alpha b_i^\beta \Big(\zeta(x-x_i) - \zeta(x-x_i+\eta)\Big).$$

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It is an elliptic function of x. Therefore, the wave functions can be found among double-Bloch functions:

$$\Psi_{\alpha\beta}(x) = z^{x/\eta} \sum_i a_i^\alpha c_i^\beta \Phi(x - x_i, \lambda),$$

$$\Psi^*_{\alpha\beta}(x) = z^{-x/\eta} \sum_i c_i^{*\alpha} b_i^\beta \Phi(x - x_i + \eta, -\lambda).$$

Substituting this into the linear problem, we arrive at an expression which has simple poles at the points $x_i - \eta$ and also poles at x_i (up to second order). Cancellation of the second order poles gives the constraint

$$\dot{x}_i = b_i^\gamma a_i^\gamma.$$

Cancellation of the poles at $x_i - \eta$ gives the linear equations

$$\sum_j c_j^\beta b_i^\gamma a_j^\gamma \Phi(x_{ij} - \eta) = z c_i^\beta.$$

Cancellation of the poles at x_i gives the equations

$$\partial_t(a_i^\alpha c_i^\beta) + \zeta(\eta)\dot{x}_i a_i^\alpha c_i^\beta = c_i^\beta \sum_{j\neq i} a_j^\alpha a_i^\gamma b_j^\gamma \Big(\zeta(x_{ij}) - \zeta(x_{ij} + \eta)\Big) + a_i^\alpha \sum_{j\neq i} c_j^\beta b_i^\gamma a_j^\gamma \Phi(x_{ij}, \lambda),$$

which can be rewritten in the form

$$\frac{\dot{a}_i^\alpha}{a_i^\alpha} - \sum_{j\neq i} \frac{\dot{a}_j^\alpha}{a_i^\alpha} a_i^\gamma b_j^\gamma \Big(\zeta(x_{ij}) - \zeta(x_{ij} + \eta)\Big) = -\frac{\dot{c}_i^\beta}{c_i^\beta} + \sum_{j\neq i} \frac{\dot{c}_j^\beta}{c_i^\beta} b_i^\gamma a_j^\gamma \Phi(x_{ij}, \lambda) - \zeta(\eta)\dot{x}_i.$$

The right-hand side does not depend on the index α, hence

$$\Lambda_i = \frac{\dot{a}_i^\alpha}{a_i^\alpha} - \sum_{j\neq i} \frac{\dot{a}_j^\alpha}{a_i^\alpha} a_i^\gamma b_j^\gamma \Big(\zeta(x_{ij}) - \zeta(x_{ij} + \eta)\Big)$$

does not depend on α, and so we obtain equations of motion for the vectors a_i^α, b_i^α:

$$\dot{a}_i^\alpha = \Lambda_i a_i^\alpha + \sum_{j\neq i} a_j^\alpha a_i^\gamma b_j^\gamma \Big(\zeta(x_{ij}) - \zeta(x_{ij} + \eta)\Big)$$

and evolution equations for c_i^α:

$$\dot{c}_i^\beta = -(\Lambda_i + \zeta(\eta)\dot{x}_i) c_i^\beta + \sum_{j\neq i} c_j^\beta b_i^\gamma a_j^\gamma \Phi(x_{ij}, \lambda).$$

In the matrix form, we have:

$$L(\lambda)\mathbf{c}^\beta = z\mathbf{c}^\beta, \quad \partial_t \mathbf{c}^\beta = M(\lambda)\mathbf{c}^\beta$$

with the matrices

$$L_{ij}(\lambda) = b_i^\gamma a_j^\gamma \Phi(x_{ij} - \eta, \lambda),$$

$$M_{ij}(\lambda) = -\delta_{ij}(\Lambda_i + \zeta(\eta)\dot{x}_i) + (1 - \delta_{ij}) b_i^\gamma a_j^\gamma \Phi(x_{ij}, \lambda).$$

We recognize the Lax pair for the spin RS system.

Similar calculations for the conjugated problem lead to the equations

$$\mathbf{c}^{*\alpha} L(\lambda) = z\mathbf{c}^{\alpha}, \quad -\partial_t \mathbf{c}^{*\alpha} = \mathbf{c}^{*\alpha} M^*(\lambda),$$

where $\mathbf{c}^{*\alpha}$ is a row-vector, and

$$M^*_{ij}(\lambda) = -\delta_{ij}(\Lambda^*_i + \zeta(\eta)\dot{x}_i) + (1-\delta_{ij}) b^\gamma_i a^\gamma_j \Phi(x_{ij}, \lambda)$$

with

$$-\Lambda^*_i = \frac{\dot{b}^\beta_i}{b^\beta_i} - \sum_{j\neq i} \frac{\dot{b}^\beta_j}{b^\beta_i} b^\gamma_i a^\gamma_j \Big(\zeta(x_{ij}) - \zeta(x_{ij}-\eta)\Big).$$

Compatibility conditions for these systems have the form of the Lax equations

$$\dot{L} + [L, M] = 0, \quad \dot{L} + [L, M^*] = 0,$$

from which we conclude that $\Lambda^*_i = \Lambda_i$. In this way we get the second group of equations for the vectors a^α_i, b^α_i in the form

$$\dot{b}^\beta_i = -\Lambda_i b^\beta_i + \sum_{j\neq i} b^\beta_j b^\gamma_i a^\gamma_j \Big(\zeta(x_{ij}) - \zeta(x_{ij}-\eta)\Big).$$

As was already mentioned, we can put $\Lambda_i = 0$ without loss of generality.

2D Toda chain of type B and the deformed RS system

The deformed RS system introduced earlier can be obtained as dynamics of poles of singular solutions to the 2D Toda chain of type B, which was introduced in [24].

2D Toda chain of type B

The hierarchy of 2D Toda chain of type B is a subhierarchy of the 2D Toda one, which is obtained by imposing certain constraints on the Lax operators of the latter. It can be regarded as a difference analog of the BKP hierarchy.

We will use the conventions and notations introduced in the previous discussion of the deformed RS system. Let us introduce the shift operator

$$T = e^{-\varphi(x)} e^{\eta\partial_x},$$

where

$$e^{\varphi(x)} = \frac{\tau(x+\eta)}{\tau(x)},$$

and impose the following constraint on the Lax operators:

$$(T - T^\dagger)\bar{\mathcal{L}} = \mathcal{L}^\dagger (T - T^\dagger).$$

The conjugation of difference operators (the †-operation) is defined in accordance with the rule $(f(x) \circ e^{\eta\partial_x})^\dagger = e^{-\eta\partial_x} \circ f(x)$.

Problem. Show that this constraint is invariant under the flows $\partial_{t_k} - \partial_{\bar{t}_k}$ of the Toda chain цепочки Тоды for all $k \geq 1$ and is destroyed by the flows $\partial_{t_k} + \partial_{\bar{t}_k}$).

Therefore, to define the dynamics of the Toda chain of the type B (or simply B-Toda) we should restrict the independent variables by setting $\bar{t}_k = -t_k$. So, in the B-Toda there is only one set of times $\mathbf{t} = \{t_1, t_2, \dots\}$ rather than two.

Here we will not discuss the general theory. Instead, we will show how to obtain the first equation (more precisely, the system of equations) of the hierarchy. Consider the difference operators

$$\mathcal{A}_1 = v(x)(e^{\eta\partial_x} - e^{-\eta\partial_x}),$$

$$\mathcal{A}_2 = \Big(f_0(x) + f_1(x)e^{\eta\partial_x} + e^{-\eta\partial_x} f_1(x)\Big)(e^{\eta\partial_x} - e^{-\eta\partial_x})$$

with some functions $v(x), f_0(x), f_1(x)$. Impose the Zakharov-Shabat equations

$$\partial_{t_2}\mathcal{A}_1 - \partial_{t_1}\mathcal{A}_2 + [\mathcal{A}_1, \mathcal{A}_2] = 0.$$

This gives the following system of three equations for the three unknown functions:

$$\left\{\begin{array}{l} v(x)f_1(x+\eta) = v(x+2\eta)f_1(x), \\[2ex] \partial_{t_1} f_1(x) + v(x+\eta)f_0(x) - v(x)f_0(x+\eta) = 0, \\[2ex] \partial_{t_2} v(x) - \partial_{t_1} f_0(x) + 2v(x)\Big(f_1(x) - f_1(x-\eta)\Big) = 0. \end{array}\right.$$

The first equation allows us to exclude the function f_1:

$$f_1(x) = v(x)v(x+\eta),$$

then the system acquires the form

$$\left\{\begin{array}{l} \partial_{t_1} \log\Big(v(x)v(x+\eta)\Big) = \dfrac{f_0(x+\eta)}{v(x+\eta)} - \dfrac{f_0(x)}{v(x)}, \\[3ex] \partial_{t_2} v(x) - \partial_{t_1} f_0(x) + 2v^2(x)\Big(v(x+\eta) - v(x-\eta)\Big) = 0. \end{array}\right.$$

One can introduce the tau-function of the B-Toda hierarchy. The functions $v(x), f_0(x)$ are expressed through it as follows:

$$v(x) = \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}, \qquad f_0(x) = v(x)\partial_{t_1} \log\frac{\tau(x+\eta)}{\tau(x-\eta)}.$$

With this substitution, the first equation becomes an identity while the second one turns into a bilinear equation for the tai-function.

Elliptic solutions

Let us consider solutions that are elliptic functions of x, and find the dynamics of their poles as functions of $t = t_1$. As usual, we should address the corresponding linear problem $\partial_t\psi = \mathcal{A}_1\psi$, which is written as the differential-difference equation

$$\partial_t\psi(x) = v(x)\Big(\psi(x+\eta) - \psi(x-\eta)\Big).$$

The function $v(x)$ is expressed through the tau-function as is mentioned above. For our purposes, it is convenient to rewrite this equation in terms of the function

$$\Psi(x) = \frac{\tau(x+\eta)}{\tau(x)}\,\psi(x+\eta).$$

Then the equation acquires the form

$$\partial_t \Psi(x-\eta) = \Psi(x) + b(x)\Psi(x-\eta) - u^-(x)\Psi(x-2\eta),$$

where

$$b(x) = \partial_t \log \frac{\tau(x)}{\tau(x-\eta)}, \quad u^-(x) = \frac{\tau(x-2\eta)\tau(x+\eta)}{\tau(x-\eta)\tau(x)}.$$

For elliptic solutions

$$\tau(x) = C\prod_{j=1}^{N}\sigma(x-x_j),$$

then

$$b(x) = \sum_j \dot{x}_j\Big(\zeta(x-x_j-\eta) - \zeta(x-x_j)\Big),$$

$$u^-(x) = \prod_j \frac{\sigma(x-x_j-2\eta)\sigma(x-x_j+\eta)}{\sigma(x-x_j-\eta)\sigma(x-x_j)}$$

are elliptic functions of x with periods $2\omega_1$, $2\omega_2$. Therefore, solutions for $\Psi(x)$ should be found among double-Bloch functions of the form

$$\Psi(x) = k^{x/\eta}\sum_{i=1}^{N} c_i\Phi(x-x_i,\lambda),$$

where the coefficients c_i do not depend on x. The spectral parameters k,λ are to be connected by equation of the spectral curve. Substituting into the linear equation, we get:

$$k^{-1}\sum_i \dot{c}_i\Phi(x-x_i-\eta) - k^{-1}\sum_i c_i\dot{x}_i\Phi'(x-x_i-\eta)$$

$$= \sum_i c_i\Phi(x-x_i) + k^{-1}\sum_j \dot{x}_j\Big(\zeta(x-x_j-\eta) - \zeta(x-x_j)\Big)\sum_i c_i\Phi(x-x_i-\eta)$$

$$- k^{-2}\prod_j \frac{\sigma(x-x_j-2\eta)\sigma(x-x_j+\eta)}{\sigma(x-x_j-\eta)\sigma(x-x_j)}\sum_i c_i\Phi(x-x_i-2\eta),$$

where $\Phi'(x) = \partial_x\Phi(x,\lambda)$. The both sides have poles at $x = x_i$ и $x = x_i+\eta$ (possible poles at $x = x_i + 2\eta$ in the last terms cancel by zeros of the numerator). The second order poles at $x = x_i + \eta$ cancel identically. Comparing the simple poles at $x = x_i$, we obtain the equations

$$c_i - k^{-1}\dot{x}_i\sum_j c_j\Phi(x_{ij}-\eta) - k^{-2}\sigma(2\eta)U_i^-\sum_j c_j\Phi(x_{ij}-2\eta) = 0,$$

where

$$U_i^- = \prod_{j\neq i} \frac{\sigma(x_{ij}-2\eta)\sigma(x_{ij}+\eta)}{\sigma(x_{ij}-\eta)\sigma(x_{ij})}.$$

Below we will also encounter the function U_i^+, which differs from U_i^- by the change of sign $\eta \to -\eta$. Let us introduce the $N\times N$ matrix $L = L(k, \lambda)$ with matrix elements

$$L_{ij}(k,\lambda) = \dot{x}_i \Phi(x_{ij}-\eta, \lambda) + k^{-1}\sigma(2\eta) U_i^- \Phi(x_{ij}-2\eta, \lambda)$$

and the vector $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$. Then the system of linear equations can be written in the matrix form:

$$L(k,\lambda)\mathbf{c} = k\mathbf{c},$$

which gives the equation of the spectral curve

$$\det\Big(kI - L(k,\lambda)\Big) = 0.$$

Comparing the poles at $x = x_i + \eta$, we obtain the equations

$$\dot{c}_i = \sum_j M_{ij} c_j \quad \text{или} \quad \dot{\mathbf{c}} = M\mathbf{c},$$

where the matrix $M = M(k, \lambda)$ has the form

$$M_{ij}(k,\lambda) = \dot{x}_i(1-\delta_{ij})\Phi(x_{ij},\lambda) + k^{-1}\sigma(2\eta)U_i^+\Phi(x_{ij}-\eta,\lambda)$$

$$- \delta_{ij}\Big(\sum_k \dot{x}_k \zeta(x_{ik}+\eta) - \sum_{k\neq i}\zeta(x_{ik})\Big).$$

We recognize the matrices L, M of the deformed RS system. Compatibility of the over-determined system of equations for the vector $\mathbf{c}$ is expressed as the Manakov's triple equation, which generalizes the Lax equation. Therefore, the dynamics of poles in the time t_1 is isomorphic to the dynamics of the deformed RS system.

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