Integrable systems of particles and nonlinear equations. Lecture 11

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Two-dimensional Toda chain and its elliptic solutions

Similarly to the CM system, which is obtained as the dynamical system for poles of singular solutions to KP, the RS system also can be obtained as dynamics of poles of singular solutions to nonlinear equations. Instead of the KP equation (and the KP hierarchy) one should consider its “difference analog”, i.e., the equation of 2D Toda chain (and the corresponding hierarchy).

2D Toda chain

The hierarchy of 2D Toda chain is an infinite set of compatible nonlinear differential-difference equations. The set of independent variables includes continuous times $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ (“positive” times), $\bar{\mathbf{t}} = \{\bar t_1, \bar t_2, \bar t_3, \dots\}$ (“negative” times), which enter differential equations, and the “zeroth” time $t_0 = x$, which is usually regarded as a space variable, and the equations with respect to this variable are difference. If the negative times are frozen (put equal to 0), the equations that include the variables x and $\mathbf{t}$ form the modified KP hierarchy (mKP), which can be regarded as a subhierarchy of the 2D Toda chain.

Let us recall the main definitions and facts related to the 2D Toda chain. For more details see [21]. In the Lax-Sato formalism, the main objects are two Lax operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ which, in contrast to the KP hierarchy, are pseudo-difference rather than pseudodifferential operators, i.e., in general they are infinite linear combinations of shift operators of the form

$$\mathcal{L} = e^{\eta\partial_x} + \sum_{k\geq 0} U_k(x) e^{-k\eta\partial_x}, \quad \bar{\mathcal{L}} = c(x) e^{-\eta\partial_x} + \sum_{k\geq 0} \bar U_k(x) e^{k\eta\partial_x},$$

where η is a parameter (“a lattice spacing”), $e^{\eta\partial_x}$ is the shift operator defined by action to any function $f(x)$ as $e^{\pm\eta\partial_x} f(x) = f(x\pm\eta)$, and the coefficient functions $c(x)$, U_k, $\bar U_k$ are functions of x, $\mathbf{t}$ and $\bar{\mathbf{t}}$. Equations of the hierarchy (differential-difference equations for the functions $c(x)$, U_k, $\bar U_k$) are encoded by the Lax equations

$$\partial_{t_m}\mathcal{L} = [\mathcal{B}_m, \mathcal{L}], \quad \partial_{t_m}\bar{\mathcal{L}} = [\mathcal{B}_m, \bar{\mathcal{L}}] \qquad \mathcal{B}_m = (\mathcal{L}^m)_{\geq 0},$$

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$$\partial_{\bar t_m}\mathcal{L} = [\bar{\mathcal{B}}_m, \mathcal{L}], \quad \partial_{\bar t_m}\bar{\mathcal{L}} = [\bar{\mathcal{B}}_m, \bar{\mathcal{L}}] \qquad \bar{\mathcal{B}}_m = (\bar{\mathcal{L}}^m)_{<0},$$

where $\Bigl(\sum\limits_{k\in\mathbb{Z}} U_k e^{k\eta\partial_x}\Bigr)_{\geq 0} = \sum\limits_{k\geq 0} U_k e^{k\eta\partial_x}$, $\Bigl(\sum\limits_{k\in\mathbb{Z}} U_k e^{k\eta\partial_x}\Bigr)_{<0} = \sum\limits_{k<0} U_k e^{k\eta\partial_x}$. For example,

$$\mathcal{B}_1 = e^{\eta\partial_x} + U_0(x), \qquad \bar{\mathcal{B}}_1 = c(x)e^{-\eta\partial_x}.$$

There exist an equivalent formulation in the form of the Zakharov-Shabat equations for the difference operators $\mathcal{B}_m$, $\bar{\mathcal{B}}_m$:

$$\partial_{t_n}\mathcal{B}_m - \partial_{t_m}\mathcal{B}_n + [\mathcal{B}_m, \mathcal{B}_n] = 0,$$

$$\partial_{\bar t_n}\mathcal{B}_m - \partial_{t_m}\bar{\mathcal{B}}_n + [\mathcal{B}_m, \bar{\mathcal{B}}_n] = 0,$$

$$\partial_{\bar t_n}\bar{\mathcal{B}}_m - \partial_{\bar t_m}\bar{\mathcal{B}}_n + [\bar{\mathcal{B}}_m, \bar{\mathcal{B}}_n] = 0.$$

In particular, at $m = n = 1$ we have from the second equation:

$$\left\{\begin{array}{l} \partial_{t_1}\log c(x) = v(x) - v(x-\eta) \\[2ex] \partial_{\bar t_1} v(x) = c(x) - c(x+\eta), \end{array}\right.$$

where $v(x) = U_0(x)$. Excluding $v(x)$, we get the following differential-difference second order equation for the function $c(x)$:

$$\partial_{t_1}\partial_{\bar t_1}\log c(x) = 2c(x) - c(x+\eta) - c(x-\eta),$$

which is one of the forms of the 2D Toda equation. After the substitution $c(x) = e^{\varphi(x)-\varphi(x-\eta)}$ it acquires the familiar form

$$\partial_{t_1}\partial_{\bar t_1}\varphi(x) = e^{\varphi(x)-\varphi(x-\eta)} - e^{\varphi(x+\eta)-\varphi(x)}.$$

The Zakharov-Shabat equations are compatibility conditions for the linear problems

$$\partial_{t_m}\psi = \mathcal{B}_m(x)\psi, \quad \partial_{\bar t_m}\psi = \bar{\mathcal{B}}_m(x)\psi,$$

where the wave function ψ depends on a spectral parameter z: $\psi = \psi(z; \mathbf{t})$. It has the following expansion in powers of z:

$$\psi = z^{x/\eta} e^{\xi(\mathbf{t},z)} \left(1 + \frac{\xi_1(x, \mathbf{t}, \bar{\mathbf{t}})}{z} + \frac{\xi_2(x, \mathbf{t}, \bar{\mathbf{t}})}{z^2} + \dots\right),$$

where

$$\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k.$$

At integer x/η it is a meromorphic function, otherwise it has a ramification at 0 and ∞.

It is convenient to introduce a wave (dressing) operator as the pseudo-difference operator of the form

$$\mathcal{W}(x) = 1 + \xi_1(x)e^{-\eta\partial_x} + \xi_2(x)e^{-2\eta\partial_x} + \dots$$

with the same coefficients ξ_k, that enter the expansion of the wave function; then the wave function is represented in the form

$$\psi = \mathcal{W}(x) z^{x/\eta} e^{\xi(\mathbf{t},z)}.$$

The dual wave function ψ^* is defined by the formula

$$\psi^* = (\mathcal{W}^\dagger(x-\eta))^{-1} z^{-x/\eta} e^{-\xi(\mathbf{t},z)},$$

where the conjugate difference operator is defined by the rule $(f(x) \circ e^{n\eta\partial_x})^\dagger = e^{-n\eta\partial_x} \circ f(x)$. The dual wavr function has the expansion

$$\psi^* = z^{-x/\eta} e^{-\xi(\mathbf{t},z)} \left(1 + \frac{\xi_1^*(x,\mathbf{t},\bar{\mathbf{t}})}{z} + \frac{\xi_2^*(x,\mathbf{t},\bar{\mathbf{t}})}{z^2} + \dots\right).$$

The linear problems for the dual wave function have the form

$$-\partial_{t_m}\psi^* = \mathcal{B}_m^\dagger(x-\eta)\psi^*.$$

In particular,

$$\partial_{t_1}\psi(x) = \psi(x+\eta) + v(x)\psi(x),$$

$$-\partial_{t_1}\psi^*(x) = \psi^*(x-\eta) + v(x-\eta)\psi^*(x),$$

$$\partial_{\bar{t}_1}\psi(x) = c(x)\psi(x-\eta).$$

The general solution of the 2D Toda hierarchy is given by the tau-function $\tau = \tau(x,\mathbf{t},\bar{\mathbf{t}})$. In terms of it, the hierarchy is encoded by the generating bilinear relation

$$\oint_{C_\infty} z^{\frac{x-x'}{\eta}-1} e^{\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)} \tau\Big(x,\mathbf{t}-[z^{-1}],\bar{\mathbf{t}}\Big)\tau\Big(x'+\eta,\mathbf{t}'+[z^{-1}],\bar{\mathbf{t}}'\Big)dz$$

$$= \oint_{C_0} z^{\frac{x-x'}{\eta}-1} e^{\xi(\bar{\mathbf{t}},z^{-1})-\xi(\bar{\mathbf{t}}',z^{-1})} \tau\Big(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z]\Big)\tau\Big(x',\mathbf{t}',\bar{\mathbf{t}}'+[z]\Big)dz$$

which is valid for all $\mathbf{t},\mathbf{t}'$, $\bar{\mathbf{t}},\bar{\mathbf{t}}'$ and x,x' such that $(x-x')/\eta \in \mathbb{Z}$. We use the notation

$$\mathbf{t} \pm [z] = \Big\{t_1 \pm z, t_2 \pm \tfrac{1}{2}z^2, t_3 \pm \tfrac{1}{3}z^3, \dots\Big\},$$

which was already introduced in the context of the KP hierarchy. The integration contour in the left-hand side is a big circle around ∞, which separates the singularities that come from the exponential function (they are outside the contour) and the ones that come from the tau-functions. The integration contour in the right-hand side is a small circle around 0, which separates the singularities in the similar way.

The bilinear equations of the Hirota-Miwa type are corollaries of the integral equation. One of them is obtained if to put $x' = x$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}}$, $\mathbf{t} - \mathbf{t}' = [\lambda^{-1}] + [\mu^{-1}]$ in it, so that

$$e^{\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)} = \frac{\lambda\mu}{(\lambda-z)(\mu-z)}.$$

The integrals can be calculated using the residue calculus. This yields the functional relation

$$\mu\tau(x+\eta,\mathbf{t}+[\lambda^{-1}]-[\mu^{-1}],\bar{\mathbf{t}})\tau(x,\mathbf{t},\bar{\mathbf{t}})-\lambda\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})\tau(x,\mathbf{t}+[\lambda^{-1}]-[\mu^{-1}],\bar{\mathbf{t}})$$

$$+(\lambda-\mu)\tau(x+\eta,\mathbf{t}+[\lambda^{-1}],\bar{\mathbf{t}})\tau(x,\mathbf{t}-[\mu^{-1}],\bar{\mathbf{t}})=0.$$

Another important equation is obtained in a similar way by setting $x'=x-\eta$, $\mathbf{t}-\mathbf{t}'=[\lambda^{-1}]$, $\bar{\mathbf{t}}-\bar{\mathbf{t}}'=[\nu]$. It has the form

$$\tau(x,\mathbf{t}-[\lambda^{-1}],\bar{\mathbf{t}})\tau(x,\mathbf{t},\bar{\mathbf{t}}-[\nu])-\tau(x,\mathbf{t},\bar{\mathbf{t}})\tau(x,\mathbf{t}-[\lambda^{-1}],\bar{\mathbf{t}}-[\nu])$$

$$=\ \nu\lambda^{-1}\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[\nu])\tau(x-\eta,\mathbf{t}-[\lambda^{-1}],\bar{\mathbf{t}}).$$

The equations of the hierarchy are obtained by expansion of these relations in powers of λ, μ, ν.

The coefficient functions in the Lax operators are expressed through the tau-function. For example:

$$U_0(x)=v(x)=\partial_{t_1}\log\frac{\tau(x+\eta)}{\tau(x)},\quad c(x)=\frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}.$$

In terms of the tau-function, the Toda equation acquires the form

$$\partial_{t_1}\partial_{\bar{t}_1}\log\tau(x)=-\frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}.$$

The wave functions are expressed through the tau-function as follows:

$$\psi=z^{x/\eta}e^{\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}-[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})},$$

$$\psi^*=z^{-x/\eta}e^{-\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}+[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}.$$

One can also introduce additional wave functions $\bar{\psi}$, $\bar{\psi}^*$ by the formulas

$$\bar{\psi}=z^{x/\eta}e^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})},$$

$$\bar{\psi}^*=z^{-x/\eta}e^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}.$$

They satisfy the same linear equations as ψ, ψ^* do. It will be convenient to normalize them in a different way:

$$\phi(x)=\frac{\tau(x)}{\tau(x+\eta)}\,\bar{\psi}(x)=z^{x/\eta}e^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})},$$

$$\phi^*(x)=\frac{\tau(x)}{\tau(x-\eta)}\,\bar{\psi}^*(x)=z^{-x/\eta}e^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}})}.$$

They satisfy the linear equations

$$\partial_{\bar t_1}\phi(x)=\phi(x-\eta)-\bar v(x)\phi(x), \quad -\partial_{\bar t_1}\phi^*(x)=\phi^*(x+\eta)-\bar v(x-\eta)\phi^*(x),$$

where $\bar v(x)=\partial_{\bar t_1}\log\dfrac{\tau(x+\eta)}{\tau(x)}$.

At last, let us mention some useful corollaries of the integral bilinear relation which will be used later. Differentiating it with respect to t_m and putting $x=x'$, $\mathbf{t}=\mathbf{t}'$ $\bar{\mathbf{t}}=\bar{\mathbf{t}}'$ after that, we obtain:

$$\frac{1}{2\pi i}\oint_{C_\infty} z^{m-1}\tau\Big(x,\mathbf{t}-[z^{-1}],\bar{\mathbf{t}}\Big)\tau\Big(x+\eta,\mathbf{t}+[z^{-1}],\bar{\mathbf{t}}\Big)dz$$

$$=\partial_{t_m}\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})\tau(x,\mathbf{t},\bar{\mathbf{t}})-\partial_{t_m}\tau(x,\mathbf{t},\bar{\mathbf{t}})\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})$$

or

$$\operatorname*{res}_{\infty}\Big(z^m\psi(x)\psi^*(x+\eta)\Big)=-\partial_{t_m}\log\frac{\tau(x+\eta)}{\tau(x)}.$$

Equivalently, this relation can be written as

$$\psi(x)\psi^*(x+\eta)=1+\sum_{m\geq 1} z^{-m-1}\partial_{t_m}\log\frac{\tau(x+\eta)}{\tau(x)}.$$

Similarly, differentiating the bilinear integral relation with respect to $\bar t_m$ and putting $x=x'$, $\mathbf{t}=\mathbf{t}'$, $\bar{\mathbf{t}}=\bar{\mathbf{t}}'$ after that, we obtain:

$$\operatorname*{res}_{0}\Big(z^{-m}\phi(x)\phi^*(x+\eta)\Big)=-\partial_{\bar t_m}\log\frac{\tau(x+\eta)}{\tau(x)}.$$

Here $\operatorname*{res}_{\infty}$, $\operatorname*{res}_{0}$ of a Laurent series is defined as $\operatorname*{res}_{\infty}(z^{-n})=-\delta_{n1}$, $\operatorname*{res}_{0}(z^{-n})=\delta_{n1}$.

The RS system from dynamics of poles of singular solutions

Among all solutions to the 2D Toda chain, of a special interest are solutions that have a finite number of poles in x in a fundamental domain in the complex plane. In particular, one can consider rational, trigonometric or elliptic solutions with poles depending on the times $\mathbf{t},\bar{\mathbf{t}}$.

Trigonometric solutions of the 2D Toda chain: correspondence on the level of hierarchies. We begin with trigonometric solutions in the space variable x. It will be shown that their pole dynamics in the times $t_k,\bar t_k$ coincides with that of the trigonometric RS system. Let us present this result in some more details. The tau-function of the 2D Toda chain for trigonometric solutions has the form

$$\tau(x,\mathbf{t},\bar{\mathbf{t}})=\exp\Big(-\sum_{k\geq 1}kt_k\bar t_k\Big)\prod_{i=1}^{N}\Big(e^{2\gamma x}-e^{2\gamma x_i(\mathbf{t},\bar{\mathbf{t}})}\Big),$$

where γ is related to the period. (Zeros x_i of the tau-function are poles of the solution.) The limit $\gamma \to 0$ corresponds to rational solutions. We will show that the time evolution of the x_i's in the time t_m is described by the Hamiltonian flow with the Hamiltonian

$$H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\,\mathrm{tr}\, L^m,$$

where

$$L_{ij} = \frac{\gamma\eta\, e^{\eta p_i}}{\sinh(\gamma(x_i - x_j - \eta))} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))}$$

is the Lax matrix of the trigonometric RS system. In particular,

$$H_1 = \sum_i e^{\eta p_i} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))}$$

is the standard Hamiltonian of the RS system. Similarly, the evolution of the x_i's in the time $\bar{t}_m$ is described by the Hamiltonian flow with the Hamiltonian

$$\bar{H}_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\,\mathrm{tr}\, L^{-m}.$$

The derivation of these results is given below.

The dynamics in t_1. Consider first the dynamics in positive times, putting $\bar{\mathbf{t}} = 0$. The tau-function is then of the form

$$\tau(x, \mathbf{t}) = \prod_{i=1}^{N} \Big(e^{2\gamma x} - e^{2\gamma x_i(\mathbf{t})} \Big).$$

As before, it is convenient to pass to the variables

$$w = e^{2\gamma x}, \quad w_i = e^{2\gamma x_i}.$$

In these variables, the tai-function becomes a polynomial of w of degree N with the roots w_i, which are assumed to be distinct: $\tau = \prod\limits_i (w - w_i)$. The function $v(x)$ is then given by the formula

$$v(x) = \partial_{t_1} \log \frac{\tau(x+\eta)}{\tau(x)} = \sum_i \left(\frac{\dot{w}_i}{w - w_i} - \frac{\dot{w}_i}{qw - w_i} \right),$$

where

$$q = e^{2\gamma\eta}.$$

Here and below in this section dt under letters denotes the derivative with respect to t_1.

First, we will study the t_1-dynamics. The pole ansatz for the wave function has the form

$$\psi = z^{x/\eta} e^{t_1 z} \left(1 + \sum_i \frac{2\gamma c_i}{w - w_i} \right),$$

where we have put $t_k = 0$ at $k \geq 2$. The coefficients c_i may depend on $\mathbf{t}$ and on z. Substituting ψ and v into the linear equation

$$-\partial_{t_1}\psi(x) + \psi(x+\eta) + v(x)\psi(x) = 0,$$

we get:

$$-z\sum_i \frac{c_i}{w-w_i} - \sum_i \frac{\dot c_i}{w-w_i} - \sum_i \frac{\dot w_i c_i}{(w-w_i)^2} + \sum_i \frac{q^{-1}c_i}{w-q^{-1}w_i}$$
$$+\frac{1}{2\gamma}\sum_i\left(\frac{\dot w_i}{w-w_i} - \frac{\dot w_i q^{-1}}{w-q^{-1}w_i}\right) + \sum_i\left(\frac{\dot w_i}{w-w_i} - \frac{\dot w_i q^{-1}}{w-q^{-1}w_i}\right)\sum_k \frac{c_k}{w-w_k} = 0.$$

The left-hand side is a rational function of w vanishing at infinity. Possible (simple) poles are at $w = w_i$ and $w = q^{-1}w_i$ (the second order poles cancel identically). Therefore, to obtain equality, one should equate all the residues to zero. This gives the following system of linear equations for the coefficients c_i:

$$\begin{cases} zc_i - q\displaystyle\sum_k \frac{\dot w_i c_k}{w_i - qw_k} = \frac{1}{2\gamma}\,\dot w_i \\[2ex] \dot c_i = c_i\left(\displaystyle\sum_{k\neq i}\frac{\dot w_k}{w_i-w_k} - \sum_k \frac{\dot w_k}{qw_i - w_k}\right) + \displaystyle\sum_{k\neq i}\frac{\dot w_i c_k}{w_i-w_k} - q\sum_k \frac{\dot w_i c_k}{w_i - qw_k}. \end{cases}$$

In a similar way, the conjugated linear equation

$$\partial_{t_1}\psi^*(x) + \psi^*(x-\eta) + v(x-\eta)\psi^*(x) = 0$$

with the pole ansatz for the ψ^*-function of the form

$$\psi^* = z^{-x/\eta}e^{-t_1 z}\left(1 + \sum_i \frac{2\gamma c_i^*}{w-w_i}\right)$$

leads to the system

$$\begin{cases} zc_i^* - \displaystyle\sum_k \frac{\dot w_i c_k^*}{w_k - qw_i} = -\frac{1}{2\gamma}\,\dot w_i \\[2ex] \dot c_i^* = c_i^*\left(\displaystyle\sum_{k\neq i}\frac{\dot w_k}{w_i-w_k} + \sum_k \frac{q\dot w_k}{w_i - qw_k}\right) + \displaystyle\sum_{k\neq i}\frac{\dot w_i c_k^*}{w_i-w_k} - \sum_k \frac{\dot w_i c_k^*}{qw_i - w_k}. \end{cases}$$

After the transformation $\tilde c_i = c_i w_i^{-1/2}$, $\tilde c_i^* = c_i^* w_i^{-1/2}$ these linear system can be written in the matrix form

$$\left(zI - q^{1/2}L\right)\tilde{\mathbf{c}} = \dot X W^{1/2}\mathbf{e}, \quad \partial_{t_1}\tilde{\mathbf{c}} = M\tilde{\mathbf{c}},$$
$$\tilde{\mathbf{c}}^*\dot X^{-1}\left(zI - q^{-1/2}L\right) = -\mathbf{e}^T W^{1/2}, \quad \partial_{t_1}\tilde{\mathbf{c}}^* = -\tilde{\mathbf{c}}^*\tilde M,$$

where $\tilde{\mathbf{c}} = (\tilde c_1, \ldots, \tilde c_N)^{\mathrm{T}}$ is the column vector, $\tilde{\mathbf{c}}^* = (\tilde c_1^*, \ldots, \tilde c_N^*)$ is the row-vector, $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$, and the matrices X, W, L, M, $\tilde M$ are of the form

$$X = \mathrm{diag}\,(x_1, x_2, \ldots, x_N), \quad W = \mathrm{diag}\,(w_1, w_2, \ldots, w_N),$$

$$L_{ij} = 2\gamma q^{1/2}\,\frac{\dot x_i w_i^{1/2} w_j^{1/2}}{w_i - q w_j},$$

$$M_{ij} = \gamma\delta_{ij}\left(\sum_{k\neq i}\frac{w_i+w_k}{w_i-w_k}\,\dot x_k - \sum_k \frac{q w_i + w_k}{q w_i - w_k}\,\dot x_k\right) + 2\gamma\frac{\dot x_i w_i^{1/2} w_j^{1/2}}{w_i - w_j}\,(1-\delta_{ij}) - 2\gamma q\frac{\dot x_i w_i^{1/2} w_j^{1/2}}{w_i - q w_j},$$

$$\tilde M_{ji} = -\gamma\delta_{ij}\left(\sum_{k\neq i}\frac{w_i+w_k}{w_i-w_k}\,\dot x_k - \sum_k \frac{w_i + q w_k}{w_i - q w_k}\,\dot x_k\right) + 2\gamma\frac{\dot x_i w_i^{1/2} w_j^{1/2}}{w_j - w_i}\,(1-\delta_{ij}) - 2\gamma\frac{\dot x_i w_i^{1/2} w_j^{1/2}}{w_j - q w_i}.$$

The following commutation relation can be checked directly:

$$q^{-1/2} W L - q^{1/2} L W = W^{-1/2}\dot W E W^{1/2}.$$

Here $E = \mathbf{e}\mathbf{e}^T$ is the matrix of rank 1 with all matrix elements equal to 1, as before. This commutation relation will be used later.

The linear system for the vector $\tilde{\mathbf{c}}$ is overdetermined. Taking the t_1-derivative of the first equation, and substituting into the second one, we obtain the compatibility condition for the system:

$$\Big(\dot L + [L, M]\Big)\tilde{\mathbf{c}} + q^{-1/2}\Big(\ddot X + \gamma \dot X^2 - M\dot X\Big) W^{1/2}\mathbf{e} = 0.$$

A direct calculation shows that

$$\dot L + [L, M] = RL,$$

$$\Big(\ddot X + \gamma \dot X^2 - M\dot X\Big) W^{1/2}\mathbf{e} = R\dot X W^{1/2}\mathbf{e},$$

where

$$R = \ddot X \dot X^{-1} + D^+ + D^- - 2D^0$$

and the diagonal matrices $D^\pm$, D^0 have the form

$$D^\pm_{ij} = \delta_{ij}\gamma\sum_{k\neq i}\frac{q^{\pm 1} w_i + w_k}{q^{\pm 1} w_i - w_k}\,\dot x_k, \quad D^0_{ij} = \delta_{ij}\gamma\sum_{k\neq i}\frac{w_i + w_k}{w_i - w_k}\,\dot x_k.$$

Therefore, the compatibility condition acquires the form $R\tilde{\mathbf{c}} = 0$, that means that $R_{ii} = 0$ for all i. This gives the equations of motion of the trigonometric RS system

$$\begin{aligned}\ddot x_i &= -\gamma\sum_{k\neq i}\dot x_i \dot x_k\Big(\coth(\gamma(x_{ik}+\eta)) + \coth(\gamma(x_{ik}-\eta)) - 2\coth(\gamma x_{ik})\Big)\\ &= \sum_{k\neq i}\dot x_i\dot x_k\frac{2\gamma\sinh^2(\gamma\eta)\cosh(\gamma x_{ik})}{\sinh(\gamma x_{ik})\sinh(\gamma(x_{ik}+\eta))\sinh(\gamma(x_{ik}-\eta))},\end{aligned}$$

The matrix equation $\dot L + [L, M'] = 0$ with

$$L_{ij} = \frac{\gamma\dot x_i}{\sinh(\gamma(x_{ij}-\eta))},$$

$$M'_{ij} = \gamma\delta_{ij}\left(\sum_{k\neq i}\dot x_k\coth(\gamma x_{ik}) - \sum_k \dot x_k\coth(\gamma(x_{ik}+\eta))\right) + (1-\delta_{ij})\,\frac{\gamma\dot x_i}{\sinh(\gamma x_{ij})}$$

is the Lax representation for them. These equations are Hamiltonian with the Hamiltonian

$$H_1 = \sum_i e^{\eta p_i} \prod_{k\neq i} \frac{\sinh(\gamma(x_{ik}+\eta))}{\sinh(\gamma x_{ik})}$$

The Lax equation implies that the conserved quantities have the form $\mathrm{tr}\, L^m$. In the paper [10] it is shown that they are in involution.

The dynamics in positive times. For the analysis of the dynamics in higher positive times t_k, $k \geq 2$ we employ the earlier obtained relation

$$\operatorname*{res}_{\infty} \Big(z^m \psi(x)\psi^*(x+\eta)\Big) = -\partial_{t_m} \log \frac{\tau(x+\eta)}{\tau(x)},$$

which for trigonometric solutions acquires the form

$$\frac{1}{2\pi i}\oint_{C_\infty} z^{m-1}\left(1+\sum_i \frac{2\gamma c_i}{w-w_i}\right)\left(1+\sum_k \frac{2\gamma c_k^*}{qw-w_k}\right) dz = \sum_i \left(\frac{\partial_{t_m} w_i}{w-w_i} - \frac{\partial_{t_m} w_i}{qw-w_i}\right).$$

The both sides are rational functions of w with simple poles at $w=w_i$ and $w=q^{-1}w_i$ vanishing at infinity. Equating residues at the poles, we get:

$$\partial_{t_m} x_i = -2\gamma \operatorname*{res}_{\infty}\Big(z^m \tilde{c}_i^* \dot{w}_i^{-1} \tilde{c}_i\Big).$$

The solution to the linear equations for the vectors $\tilde{\mathbf{c}}$, $\tilde{\mathbf{c}}^*$, yield:

$$\tilde{\mathbf{c}} = \frac{1}{2\gamma}(zI - q^{1/2}L)^{-1}\dot{W}W^{-1/2}\mathbf{e}, \quad \tilde{\mathbf{c}}^* = -\frac{1}{2\gamma}\,\mathbf{e}^T W^{1/2}(zI-q^{-1/2}L)^{-1}\dot{W}W^{-1}.$$

Plugging this into the expression for $\partial_{t_m} x_i$, we obtain:

$$\partial_{t_m} x_i = -\frac{1}{2\gamma}\operatorname*{res}_{\infty}\sum_{k,k'}\left[z^m w_k^{1/2}\Big(\frac{1}{zI-q^{-1/2}L}\Big)_{ki} w_i^{-1}\Big(\frac{1}{zI-q^{1/2}L}\Big)_{ik'}\dot{w}_{k'}w_{k'}^{-1/2}\right]$$

$$= -\frac{1}{2\gamma}\operatorname*{res}_{\infty}\,\mathrm{tr}\left(z^m \dot{W}W^{-1/2}EW^{1/2}\frac{1}{zI-q^{-1/2}L}E_iW^{-1}\frac{1}{zI-q^{1/2}L}\right),$$

where E_i is the diagonal matrix with matrix elements $(E_i)_{jk} = \delta_{ij}\delta_{ik}$. Using the commutation relation

$$q^{-1/2}WL - q^{1/2}LW = W^{-1/2}\dot{W}EW^{1/2},$$

we have:

$$\partial_{t_m} x_i = -\frac{1}{2\gamma}\operatorname*{res}_{\infty}\,\mathrm{tr}\left(z^m(q^{-1/2}WL - q^{1/2}LW)\frac{1}{zI-q^{-1/2}L}\,E_iW^{-1}\frac{1}{zI-q^{1/2}L}\right)$$

$$= -\frac{1}{2\gamma}\operatorname*{res}_{\infty}\,\mathrm{tr}\left(z^m\Big(E_i\,\frac{1}{zI-q^{-1/2}L} - E_i\,\frac{1}{zI-q^{1/2}L}\Big)\right).$$

Further, we use the identity

$$E_iL = \dot{x}_i\frac{\partial L}{\partial \dot{x}_i} = \eta^{-1}\frac{\partial L}{\partial p_i}$$

which is easy to check, to continue the chain of equalities:

$$\partial_{t_m} x_i = -\frac{1}{2\gamma\eta} \operatorname*{res}_{\infty} \mathrm{tr} \left(z^m \left(\frac{\partial L}{\partial p_i} \frac{L^{-1}}{zI - q^{-1/2}L} - \frac{\partial L}{\partial p_i} \frac{L^{-1}}{zI - q^{1/2}L} \right) \right)$$

$$= \frac{1}{2\gamma\eta} (q^{-m/2} - q^{m/2})\, \mathrm{tr} \left(\frac{\partial L}{\partial p_i} L^{m-1} \right) = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \frac{\partial}{\partial p_i} \mathrm{tr}\, L^m.$$

This gives the first half of the Hamiltonian equations for the flow in t_m:

$$\partial_{t_m} x_i = \frac{\partial H_m}{\partial p_i}, \quad H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \mathrm{tr}\, L^m.$$

The derivation of the second half of Hamiltonian equations requires more efforts. The idea was suggested in the work [22] for rational solutions. First of all we note that the first half of the Hamiltonian equations can be written as

$$\partial_{t_m} x_i = -m\eta\kappa_m \mathrm{tr}\,(E_i L^m), \quad \kappa_m = \frac{\sinh(m\gamma\eta)}{m\gamma\eta}.$$

Differentiating with respect to t_1 and using the Lax equations, we have:

$$\partial_{t_m} \dot{x}_i = -m\eta\kappa_m \mathrm{tr}\,(E_i[M', L^m]) = -m\eta\kappa_m \mathrm{tr}\,(L^m[E_i, M']).$$

Now, let us act be the differential operator ∂_{t_m} to the equation

$$\log \dot{x}_i = \eta p_i + \sum_{k\neq i} \log \frac{\sinh(\gamma(x_{ik} + \eta))}{\sinh(\gamma x_{ik})} + \log \eta.$$

We get:

$$\partial_{t_m} p_i = \eta^{-1}\partial_{t_m} \log \dot{x}_i - \eta^{-1} \sum_j \sum_{l\neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il} + \eta))}{\sinh(\gamma x_{il})}\, \partial_{t_m} x_j$$

$$= -m\kappa_m \dot{x}_i^{-1} \mathrm{tr}\,(L^m[E_i, M']) + m\kappa_m \sum_j \sum_{l\neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il} + \eta))}{\sinh(\gamma x_{il})}\, \mathrm{tr}\,(E_j L^m)$$

$$= -m\kappa_m\, \mathrm{tr} \left(A^{(i)} L^{m-1} \right),$$

where the matrix $A^{(i)}$ is of the form

$$A^{(i)} = \dot{x}_i^{-1}(LE_iM' - M'E_iL) - \sum_j \sum_{l\neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il} + \eta))}{\sinh(\gamma x_{il})} E_j L.$$

Note that the diagonal part of the matrix M' does not contribute, so instead of M' we can substitute its off-diagonal part

$$A_{ij} = 2\gamma(1 - \delta_{ij}) \frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - w_j}.$$

The matrix elements are:

$$(LE_iA)_{jk} = \gamma \dot{x}_i L_{jk} \left(\frac{w_i + w_k}{w_i - w_k} - \frac{qw_i + w_k}{qw_i - w_k} \right) (1 - \delta_{ik}),$$

$$(AE_iL)_{jk} = -\gamma\dot{x}_i L_{jk}\left(\frac{w_i+w_j}{w_i-w_j} - \frac{w_i+qw_k}{w_i-qw_k}\right)(1-\delta_{ij}),$$

$$\sum_l \sum_{r\neq i} \frac{\partial}{\partial x_l} \log \frac{\sinh(\gamma(x_{ir}+\eta))}{\sinh(\gamma x_{ir})}\,(E_lL)_{jk}$$

$$= \gamma\delta_{ij}L_{jk}\sum_{r\neq i}\left(\frac{qw_i+w_r}{qw_i-w_r} - \frac{w_i+w_r}{w_i-w_r}\right) - \gamma(1-\delta_{ij})L_{jk}\left(\frac{qw_i+w_j}{qw_i-w_j} - \frac{w_i+w_j}{w_i-w_j}\right).$$

Collecting everything together, we find matrix elements of the matrix $A^{(i)}$:

$$\begin{aligned} A^{(i)}_{jk} \;&=\; \gamma L_{jk}\left(\frac{w_i+w_k}{w_i-w_k}(1-\delta_{ik}) - \frac{w_i+qw_k}{w_i-qw_k}(1-\delta_{ij}) + \frac{qw_i+w_j}{qw_i-w_j}(\delta_{ik}-\delta_{ij})\right. \\ &\left. -\delta_{ij}\sum_{r\neq i}\left(\frac{qw_i+w_r}{qw_i-w_r} - \frac{w_i+w_r}{w_i-w_r}\right)\right). \end{aligned}$$

A direct calculation shows that

$$A^{(i)} = -\frac{\partial L}{\partial x_i} - [C^{(i)}, L],$$

where $C^{(i)}$ is the following matrix:

$$C^{(i)} = \gamma\sum_l \frac{qw_l+w_i}{qw_l-w_i}\,E_l - \gamma\sum_{l\neq i}\frac{w_l+w_i}{w_l-w_i}\,E_l.$$

Indeed,

$$\begin{aligned} \frac{\partial L_{jk}}{\partial x_i} &= \gamma L_{jk}\left(\frac{w_j+qw_k}{w_j-qw_k}\,(\delta_{ik}-\delta_{ij}) + \frac{w_i+qw_j}{w_i-qw_j}\,(1-\delta_{ij}) - \frac{w_i+w_j}{w_i-w_j}\,(1-\delta_{ij})\right. \\ &\left. +\,\delta_{ij}\sum_{r\neq i}\left(\frac{qw_i+w_r}{qw_i-w_r} - \frac{w_i+w_r}{w_i-w_r}\right)\right), \end{aligned}$$

$$[C^{(i)}, L]_{jk} = \gamma L_{jk}\left(\frac{qw_j+w_i}{qw_j-w_i} - \frac{qw_k+w_i}{qw_k-w_i} - \frac{w_j+w_i}{w_j-w_i}(1-\delta_{ij}) + \frac{w_k+w_i}{w_k-w_i}(1-\delta_{ik})\right),$$

hence $A^{(i)} + \partial L/\partial x_i + [C^{(i)}, L] = 0$. From the relation

$$A^{(i)} = -\frac{\partial L}{\partial x_i} - [C^{(i)}, L]$$

we conclude that

$$\partial_{t_m} p_i = -m\kappa_m \operatorname{tr}\left(A^{(i)}L^{m-1}\right) = m\kappa_m \operatorname{tr}\left(\frac{\partial L}{\partial x_i}\,L^{m-1}\right) = \kappa_m\frac{\partial}{\partial x_i}\operatorname{tr} L^m.$$

This gives the second half of Hamiltonian equations for the highest flows:

$$\partial_{t_m} p_i = -\frac{\partial H_m}{\partial x_i}.$$

The dynamics in negative times. For analysis of the dynamics of zeros of the tau-function in the negative times, we first consider the evolution in $\bar t_1$. We will use the additional wave functions ϕ, ϕ^*. The ansatz for them is

$$\phi(x) = z^{x/\eta} e^{\bar t_1 z^{-1}} \left(1 + \sum_i \frac{2\gamma b_i}{qw - w_i}\right),$$

$$\phi^*(x) = z^{-x/\eta} e^{-\bar t_1 z^{-1}} \left(1 + \sum_i \frac{2\gamma b_i^*}{q^{-1}w - w_i}\right),$$

where b_i, b_i^* are some coefficients that depend on z and on the times but not on x. Similarly to the previous case, we substitute the wave functions into the linear problems and obtain a system of linear equations for $\mathbf{b}$, $\mathbf{b}^*$ as the condition of cancellation of the poles. The equations for b_i's are the same as the ones for c_i^*, with the change $z \to -z^{-1}$, $w \to qw$ and $\partial_{t_1} \to \partial_{\bar t_1}$, with b_i^* и c_i being connected in the similar way. Passing to $\tilde b_i = w_i^{-1/2} b_i$, $\tilde b_i^* = w_i^{-1/2} b_i^*$, we have, after some calculations:

$$\tilde{\mathbf{b}}^{\mathrm{T}} (\partial_{\bar t_1} X)^{-1} (z^{-1} I + q^{-1/2} \bar L) = \mathbf{e}^{\mathrm{T}} W^{1/2},$$

$$(z^{-1} I + q^{1/2} \bar L) \tilde{\mathbf{b}}^* = -\partial_{\bar t_1} X\, W^{1/2} \mathbf{e},$$

where the matrix $\bar L$ has the form

$$\bar L_{ij} = 2\gamma q^{1/2} \frac{\partial_{\bar t_1} x_i\, w_i^{1/2} w_j^{1/2}}{w_i - q w_j}.$$

This matrix satisfies the commutaton relation

$$q^{-1/2} W \bar L - q^{1/2} \bar L W = W^{-1/2} \partial_{\bar t_1} W\, E W^{1/2}.$$

Using the relation

$$\operatorname*{res}_0 \Big(z^{-m} \phi(x) \phi^*(x+\eta) \Big) = -\partial_{\bar t_m} \log \frac{\tau(x+\eta)}{\tau(x)},$$

we find, similarly to the above,

$$\partial_{\bar t_m} x_i = -2\gamma \operatorname*{res}_0 \Big(z^{-m-2} \tilde b_i^* (\partial_{\bar t_1} w_i)^{-1} \tilde b_i \Big).$$

Plugging here the solutions of the linear systems and repeating the calculations done for positive times, we will have:

$$\partial_{\bar t_m} x_i = (-1)^m \frac{\sinh(m\gamma\eta)}{\gamma} \operatorname{tr}(E_i \bar L^m).$$

Let us derive a relation between the matrices L and $\bar L$. For this, we need the relation between the velocities $\dot x_i = \partial_{t_1} x_i$ and $\partial_{\bar t_1} x_i$ which can be obtained from the Toda equation

$$\partial_{t_1} \partial_{\bar t_1} \log \tau(x) = -\frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}.$$

Plugging here the tau-function in the form $\tau = \prod\limits_i (w-w_i)$, we obtain:

$$\sum_i \frac{\partial_{t_1}\partial_{\bar t_1} w_i}{w-w_i} + \sum_i \frac{\partial_{t_1} w_i\, \partial_{\bar t_1} w_i}{(w-w_i)^2} = \prod_k \frac{(qw-w_k)(q^{-1}w-w_k)}{(w-w_k)^2}.$$

Comparing the coefficients in front of the highest poles, we arrive at the relation

$$\partial_{t_1} w_i\, \partial_{\bar t_1} w_i = \frac{\prod_k (qw_i-w_k)(q^{-1}w_i-w_k)}{\prod_{l\neq i}(w_i-w_l)^2}$$

or

$$\partial_{t_1} X\, \partial_{\bar t_1} X = \frac{1}{4\gamma^2}\, W^{-2} U_+ U_-,$$

where $U_\pm$ are diagonal matrices of the form

$$(U_\pm)_{ij} = \delta_{ij}\, \frac{\prod_k (w_i - q^{\pm 1} w_k)}{\prod_{l\neq i}(w_i-w_l)}.$$

We also need the matrix inverse to the Cauchy matrix

$$C_{ij} = \frac{1}{w_i - qw_j}.$$

It has the form

$$C^{-1}_{ij} = \frac{1}{qw_i - w_j}\, \frac{\prod_k (qw_i - w_k)(w_j - qw_k)}{q^{N-1}\prod_{l\neq j}(w_j-w_l)\prod_{l'\neq i}(w_i - w_{l'})}$$

or

$$C^{-1} = -qU_- C^{\mathrm{T}} U_+.$$

Now we can write: $L = 2\gamma q^{1/2}\partial_{t_1} X\, W^{1/2} C W^{1/2}$ и находим:

$$\begin{aligned} L^{-1} &= \frac{q^{-1/2}}{2\gamma} W^{-1/2} C^{-1} W^{-1/2} (\partial_{t_1} X)^{-1} \\ &= -2\gamma q^{1/2} W^{-1/2} U_- C^T W^{-1/2} \partial_{\bar t_1} X\, W^2 U_-^{-1} \\ &= -2\gamma q^{1/2} W^{-1} U_- \Big(\partial_{\bar t_1} X\, W^{1/2} C W^{1/2}\Big)^{\mathrm{T}} (W^{-1}U_-)^{-1} \\ &= -W^{-1} U_-\, \bar L^{\mathrm{T}} (W^{-1}U_-)^{-1}. \end{aligned}$$

We see that the matrix $-\bar L^{\mathrm{T}}$ is connected to the L^{-1} by a similarity transformation. Using the fact that $E_i \bar L = -\eta^{-1}\partial \bar L/\partial p_i$, we can rewrite the equations of motion

$$\partial_{\bar t_m} x_i = (-1)^m\, \frac{\sinh(m\gamma\eta)}{\gamma}\, \mathrm{tr}\,(E_i \bar L^m)$$

as

$$\partial_{\bar t_m} x_i = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\, \frac{\partial}{\partial p_i}\, \mathrm{tr}\, L^{-m} = \frac{\partial \bar H_m}{\partial p_i},$$

which is just the first half of the Hamiltonian equations for higher flows in the negative times.

After the calculations for the positive times, there are no problems to derive the second part of the Hamiltonian equations. Note that

$$\partial_{\bar t_m} x_i = m\eta\kappa_m \,\mathrm{tr}\,(E_i L^{-m}).$$

In the full analogy with the above calculations, we obtain:

$$\partial_{\bar t_m} p_i = m\kappa_m \,\mathrm{tr}\,(A^{(i)} L^{-m-1})$$

with the same matrix $A^{(i)}$. By virtue of the relation

$$A^{(i)} = -\frac{\partial L}{\partial x_i} - [C^{(i)}, L]$$

we have:

$$\partial_{\bar t_m} p_i = -m\kappa_m \,\mathrm{tr}\Big(\frac{\partial L}{\partial x_i}\, L^{-m-1}\Big) = m\kappa_m \,\mathrm{tr}\Big(\frac{\partial L^{-1}}{\partial x_i}\, (L^{-1})^{m-1}\Big) = \kappa_m \frac{\partial}{\partial x_i}\,\mathrm{tr}\, L^{-m}.$$

This gives the Hamiltonian equations

$$\partial_{\bar t_m} p_i = -\frac{\partial \bar H_m}{\partial x_i}.$$

In particular,

$$\bar H_1 = \frac{\sinh^2(\gamma\eta)}{\gamma^2\eta^2} \sum_i e^{-\eta p_i} \prod_{k\neq i} \frac{\sinh(\gamma(x_{ik}-\eta))}{\sinh(\gamma x_{ik})}.$$

Elliptic solutions. Consider solutions that are elliptic functions of x and find the dynamics of their poles as functions of t_1. For elliptic solutions we have the linear problem

$$\partial_{t_1}\psi(x) = \psi(x+\eta) + v(x)\psi(x),$$

where the function $v(x)$ has the form

$$v(x) = \sum_i \dot x_i \Big(\zeta(x-x_i) - \zeta(x-x_i+\eta)\Big)$$

(the dot means the t_1-derivative). It is a double-periodic function of x, so it is natural to search for solutions among double-Bloch functions. As before, we represent the solution as a sum of the elementary double-Bloch functions

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}\, e^{-\zeta(\lambda)x}.$$

The expression has the form

$$\psi = k^{x/\eta} e^{t_1 k} \sum_i c_i \Phi(x-x_i,\lambda),$$

where k is the second spectral parameter, which is going to be connected with λ by equation of the spectral curve. Substituting this ansatz into the linear problem, we obtain the conditions of cancellation of the poles at $x = x_i$ and $x = x_i - \eta$ in the form

$$\left\{\begin{array}{l} kc_i + \dot{c}_i = \dot{x}_i \sum\limits_{j \neq i} c_j \Phi(x_i - x_j) + c_i \sum\limits_{j \neq i} \dot{x}_j \zeta(x_i - x_j) - c_i \sum\limits_{j} \dot{x}_j \zeta(x_i - x_j + \eta) \\[2ex] kc_i - \dot{x}_i \sum\limits_{j} c_j \Phi(x_i - x_j - \eta) = 0, \end{array}\right.$$

where we omit the second argument of the function Φ for brevity. These conditions can be written in the matrix form as a system of linear equations for the vector $\mathbf{c} = (c_1, \dots, c_N)^{\mathrm{T}}$:

$$\left\{\begin{array}{l} L(\lambda)\mathbf{c} = k\mathbf{c} \\[2ex] \dot{\mathbf{c}} = M(\lambda)\mathbf{c}, \end{array}\right.$$

where the $N \times N$ matrices L, M have the form

$$L_{ij}(\lambda) = \dot{x}_i \Phi(x_i - x_j - \eta, \lambda),$$

$$M_{ij}(\lambda) = \delta_{ij}\Big(\sum_{l \neq i} \dot{x}_l \zeta(x_i - x_l) - \sum_{l} \dot{x}_l \zeta(x_i - x_l + \eta)\Big) + (1 - \delta_{ij})\dot{x}_i \Phi(x_i - x_j, \lambda) - \dot{x}_i \Phi(x_i - x_j - \eta, \lambda).$$

We recognize the Lax pair for the elliptic RS system. The compatibility condition of the linear system is the Lax equation $\dot{L} + [L, M] = 0$. The last term in the expression for the matrix M can be ignored because it is proportional to the matrix L and thus cancels in the Lax equation.

The correspondence of elliptic solutions to the 2D Toda chain and the elliptic RS system on the level of hierarchies was established in the work [23]. The result is as follows. The dynamics in all the times $\mathbf{t}$, $\bar{\mathbf{t}}$ is Hamiltonian, and the corresponding Hamiltonians are the higher RS Hamiltonians. Their generating function is $\lambda(z)$, and the spectral parameters λ, z are connected by the equation of the spectral curve

$$\det_{N \times N}\Big(ze^{\eta\zeta(\lambda)}I - L(\lambda)\Big) = 0.$$

The spectral curve is conserved for the dynamics in all times. A point on it is a pair $P = (z, \lambda)$, where z, λ are connected by the equation. There are two distinguished points on the spectral curve: $P_\infty = (\infty, 0)$ and $P_0 = (0, N\eta)$. The Hamiltonians that generate dynamics in positive times $\mathbf{t}$ are defined as coefficients of the expansion of the function $\lambda(z)$ in inverse powers of z near the point P_∞, while the Hamiltonians that generate dynamics in negative times $\bar{\mathbf{t}}$ are coefficients in the expansion of $\lambda(z)$ in positive powers of z near the point P_0.

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