Problems for the course "Integrable systems of particles and nonlinear equations" – 2025

1. Let the matrices of Lax pair be of the form

$$L_{ij} = -\delta_{ij} p_i - \frac{g(1-\delta_{ij})}{x_i - x_j},$$

$$M_{ij} = -2g\delta_{ij} \sum_{k\neq i} \frac{1}{(x_i - x_k)^2} + \frac{2g(1-\delta_{ij})}{(x_i - x_j)^2}.$$

Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion of the rational Calogero-Moser (CM) system.

2. Put $L^{\pm} = L \pm i\omega X$, where $X = \mathrm{diag}\,(x_1, \ldots, x_N)$. Show that matrix equations

$$\dot{L}^{\pm} + [L^{\pm}, M] \pm 2i\omega L^{\pm} = 0,$$

where L, M are matrices from problem 1, are equivalent to the equations of motion of the rational CM system in an external quadratic potential.

3. Let $L(\lambda) = -\delta_{ij} p_i - g(1-\delta_{ij})\Phi(x_i - x_j, \lambda)$ ($\Phi(x, \lambda)$ is the Lamé-Hermite function) be the Lax matrix of the elliptic CM system.

a) Find $\mathrm{tr}\, L^3(\lambda)$,

b) Find trigonometric degeneration of this Lax matrix at $\omega_2 = \infty$.

4. Let

$$H_{\pm} = \sum_{i=1}^{N} e^{\pm \eta p_i} \prod_{j\neq i} \frac{(x_i - x_j + \eta)^{1/2}(x_i - x_j - \eta)^{1/2}}{x_i - x_j}$$

be the Hamiltonians of the rational Ruijsenaars-Schneider (RS) system, where x_i, p_i are canonical variables. Prove that $\{H_+, H_-\} = 0$.

5. The Lax pair for rational RS system has the form

$$L_{ij} = \frac{\dot{x}_i}{x_i - x_j - \eta},$$

$$M_{ij} = \delta_{ij}\Big(\sum_{k\neq i} \frac{\dot{x}_k}{x_i - x_k} - \sum_{k} \frac{\dot{x}_k}{x_i - x_k + \eta}\Big) + \frac{(1-\delta_{ij})\dot{x}_i}{x_i - x_j}.$$

Obtain equations of motion which follow from the Lax equation $\dot{L} + [L, M] = 0$.

6. The Hamiltonian of the elliptic RS system has the form

$$H = \sum_{i} e^{\sigma(\eta) p_i} \prod_{j\neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)},$$

where p_i, x_j are canonical variables with the standard Poisson brackets. Obtain Newtonian equations of motion.

7. The Lax pair for elliptic spin CM system has the form

$$L_{ij}(\lambda) = -p_i \delta_{ij} - g(1-\delta_{ij}) b_i^\nu a_j^\nu \Phi(x_i - x_j, \lambda),$$

$$M_{ij}(\lambda) = -2g(1-\delta_{ij}) b_i^\nu a_j^\nu \Phi'(x_i - x_j, \lambda),$$

where $\Phi'(x,\lambda) = \partial_x \Phi(x,\lambda)$ and summation over repeated Greek indices is assumed. Obtain equations of motion that follow from the Lax equation $\dot{L} + [L, M] = 0$.

8. Prove that the substitution $u = 2\partial_x^2 \log \tau$ converts the KP equation

$$3u_{yy} = \Big(4u_t - 6uu_x - u_{xxx}\Big)_x$$

to a bilinear differential equation for the tau-function $\tau(x,y,t)$.

9. Show that the tau-function of the one-soliton solution to the KP equation,

$$\tau(\mathbf{t}) = 1 + \alpha e^{\xi(\mathbf{t},p) - \xi(\mathbf{t},q)},$$

where $\xi(\mathbf{t}, z) = \sum\limits_{k\geq 1} t_k z^k$, and p, q, α are parameters, satisfy the bilinear integral functional relation

$$\oint_{C_\infty} e^{\xi(\mathbf{t}-\mathbf{t}', z)} \tau(\mathbf{t} - [z^{-1}]) \tau(\mathbf{t}' + [z^{-1}]) dz = 0$$

for all $\mathbf{t}, \mathbf{t}'$; $\mathbf{t} \pm [z^{-1}] = \{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots\}$.

10. a) As a corollary of the functional relation from problem 9, obtain the following 3-term bilinear equation for the tau-function:

$$(a-d)(b-c)\tau\Big(\mathbf{t} + [b^{-1}] + [c^{-1}]\Big)\tau\Big(\mathbf{t} + [a^{-1}] + [d^{-1}]\Big)$$

$$+ (b-d)(c-a)\tau\Big(\mathbf{t} + [a^{-1}] + [c^{-1}]\Big)\tau\Big(\mathbf{t} + [b^{-1}] + [d^{-1}]\Big)$$

$$+ (c-d)(a-b)\tau\Big(\mathbf{t} + [a^{-1}] + [b^{-1}]\Big)\tau\Big(\mathbf{t} + [c^{-1}] + [d^{-1}]\Big) = 0,$$

for all $a, b, c, d \in \mathbb{C}$.

b) Prove that the equation above and the simpler equation obtained as its particular case at $d = \infty$ are equivalent.

c) Expanding the equation from item b) of this problem in powers of a^{-1}, b^{-1}, c^{-1} as $a, b, c \to \infty$, obtain the KP equation for the tau-function in bilinear form from problem 8.