Quantum integrable systems – 2023

Problems

1. Find the spectrum of the XYZ Heisenberg model for $N = 2$:

$$H = J_x \sigma_x^{(1)} \sigma_x^{(2)} + J_y \sigma_y^{(1)} \sigma_y^{(2)} + J_z \sigma_z^{(1)} \sigma_z^{(2)}.$$

   Consider the cases of the XXX ($J_x = J_y = J_z$) and XXZ ($J_x = J_y \neq J_z$) models and compare with the solution obtained through the Bethe ansatz.

2. Find the spectrum of the XXX model on 3 sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^{3} \left(\vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \qquad \vec{\sigma}^{(4)} \equiv \vec{\sigma}^{(1)}.$$

   Find multiplicity of the energy levels.

3. Consider the XXX model on N sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^{N} \left(\vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \qquad \vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}.$$

   Let $\vec{S}$ be the vector of the total spin: $\vec{S} = \sum_j \vec{\sigma}^{(j)}$. Let $|\Psi_m\rangle$ be Bethe states with m magnons (i.e. $S_z |\Psi_m\rangle = (N - 2m) |\Psi_m\rangle$). Prove that $S_+ |\Psi_m\rangle = 0$ at $m = 1, 2$, where $|\Psi_m\rangle$ are Bethe states with m magnons (such that the corresponding Bethe roots $\lambda_i < \infty$).

4. Find the Yang function for the XXX spin chain with spin $\frac{1}{2}$.

5. Find eigenstates and the energy spectrum of one and two Bose particles with the Hamiltonian

$$\hat{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le N} \delta(x_j - x_k) \qquad (N = 1, 2)$$

   on the segment $[0, L]$ with impenetrable walls (i.e. such that the wave function vanishes if at least one particle is at the endpoints of the segment).

6. For the system of three identical Bose particles with the Hamiltonian

$$\hat{H}_3 = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le 3} \delta(x_j - x_k)$$

   a) construct common eigenstates of the Hamiltonian and the total momentum

$$\hat{P} = -i \sum_{j=1}^{3} \frac{\partial}{\partial x_j}$$

b) impose periodic boundary conditions on the segment $[0, L]$ and obtain the system of Bethe equations.

7. Consider the Bethe equations for the model of the Bose gas with N particles on the segment $[0, L]$ with periodic boundary conditions:

$$L\lambda_j + 2\sum_{k=1}^{N} \operatorname{arctg}\frac{\lambda_j - \lambda_k}{c} = 2\pi n_j$$

Prove that in the state which is characterized by integer or half-integer numbers n_j the total momentum of the system is $P = \frac{2\pi}{L}\sum_{j=1}^{N} n_j$.

8. Prove that $|++ + \ldots +\rangle$ is an eigenvector for the transfer matrix of the 6-vertex model and find the corresponding eigenvalue.

9. Consider the 6-vertex model on thee $N \times N$ lattice with the R-matrix

$$\mathsf{R} = \mathsf{R}(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

where $a = \sinh(u + \eta)$, $b = \sinh u$, $c = \sinh \eta$ and the quantum monodromy matrix

$$\mathcal{T}(u) = \mathsf{R}_{10}(u)\mathsf{R}_{20}(u)\ldots\mathsf{R}_{N0}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Find the scalar product $\langle\Omega|\, C(v)B(u)\,|\Omega\rangle$. Here $|\Omega\rangle = |++ + \ldots +\rangle$ is the generating vector.

10. Let $\mathsf{R}_{0j}(u) = \mathbf{1}u + \mathsf{P}_{0j}$ be the quantum R-matrix of the XXX model (P_{0j} is the permutation operator). Consider the transfer matrix of the XXX model

$$\mathsf{T}(u) = \mathrm{tr}_0\Big(\mathsf{R}_{01}(u)\mathsf{R}_{02}(u)\ldots\mathsf{R}_{0N}(u)\Big).$$

Find $\partial_u^2 \log \mathsf{T}(u)\Big|_{u=0}$.

11. Find the commutation relations between the operators $A(u)$, $B(u)$, $C(u)$, $D(u)$ which are encoded in the intertwining relation $\mathsf{R}_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathsf{R}_{12}(u-v)$ with the 6-vertex R-matrix.

12. Prove that the quantity

$$\det\nolimits_q \mathcal{T}(u) = A(u+\eta)D(u) - B(u+\eta)C(u)$$

is a central element of the algebra generated by elements of the quantum monodromy matrix, i.e., that $\det_q \mathcal{T}(u')$ (the quantum determinant) commutes with $A(u)$, $B(u)$, $C(u)$, $D(u)$ for all u, u'.

13. Let $\mathsf{R}(u) = \sum_{a=0}^{3} W_a(u+\eta)\sigma_a \otimes \sigma_a$ be the R-matrix of the 8-vertex model (here $W_a(u) = \theta_{a+1}(u)/\theta_{a+1}(\eta)$) and $\mathsf{L}(u) = \sum_{a=0}^{3} W_a(u)\sigma_a \otimes S_a$ be the L-operator. Find the commutation relations between the operators S_a which are equivalent to the intertwining relation $\mathsf{R}_{12}(u-v)\mathsf{L}_1(u)\mathsf{L}_2(v) = \mathsf{L}_2(v)\mathsf{L}_1(u)\mathsf{R}_{12}(u-v)$ (these are the commutation relations of the Sklyanin algebra).

14. The Sklyanin algebra is the quadratic algebra with the generators S_a, $a = 0, \ldots, 3$ and the six commutation relations

$$[S_0, S_\alpha]_- = iJ_{\beta\gamma}[S_\beta, S_\gamma]_+, \quad [S_\alpha, S_\beta]_- = i[S_0, S_\gamma],$$

where $[\,,\,]_\mp$ is the commutator (anticommutator) and $(\alpha\beta\gamma)$ means any cyclic permutation of the indices (123). The structure constants $J_{\alpha\beta}$ are

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma},$$

where J_α, $\alpha = 1, 2, 3$ are free parameters. Using the commutation relations of the Sklyanin algebra, prove that

$$\Omega_1 = S_0^2 + S_1^2 + S_2^2 + S_3^2, \quad \Omega_2 = J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2$$

are central elements of the Sklyanin algebra, i.e., $[\Omega_1, \mathsf{S}_a] = [\Omega_2, \mathsf{S}_a] = 0$ for all $a = 0, \ldots, 3$. (Hint: by transforming $\mathsf{S}_\alpha \mathsf{S}_0 \mathsf{S}_\alpha$ and $\mathsf{S}_0 \mathsf{S}_\alpha \mathsf{S}_0$ in two ways obtain linear relations between cubic monomials in the generators.)