Integrable systems of particles and nonlinear equations. Lecture 7

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Spin systems

The CM and RS systems admit generalizations to the case when the particles possess some internal degrees of freedom (spins). These models are known in rational, trigonometric and elliptic versions. We will consider the elliptic version as the most general one.

Spin generalization of the CM system

The spin generalization of the CM system was suggested by Gibbons and Hermsen in 1984 [16].

The Hamiltonian and equations of motion. Let any i-th particle be equipped with some internal degrees of freedom, which are described by an $n \times n$ matrix S_i associated with the particle. The matrices S_i are called spin matrices. Their matrix elements are denoted as Их матричные $S_i^{\alpha\beta}$, where the Greek indices take values $1, \dots, n$. The Poisson brackets are

$$\{S_i^{\alpha\beta}, S_j^{\mu\nu}\} = \delta_{ij}\Big(\delta_{\alpha\nu}S_i^{\mu\beta} - \delta_{\beta\mu}S_i^{\alpha\nu}\Big).$$

The Hamiltonian has the form

$$H = \sum_i p_i^2 - g^2 \sum_{i\neq j} \mathrm{tr}(S_i S_j)\wp(x_i - x_j),$$

hence the equations of motion are

$$\dot S_i = 2g^2 \sum_{j\neq i} [S_i, S_j]\wp(x_i - x_j),$$

$$\ddot x_i = 4g^2 \sum_{j\neq i} \mathrm{tr}(S_i S_j)\wp'(x_i - x_j).$$

It follows from the first equation that $\mathrm{tr} S_i$ are integrals of motion. We fix them by imposing the conditions

$$\mathrm{tr} S_i = 1.$$

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In what follows we will concentrate on the case when S_i are matrices of rank 1; then the system is integrable. In this case, instead of the spin matrices one can introduce canonical nariables a_i^α, b_i^β with the brackets

$$\{a_i^\alpha, b_j^\beta\} = \delta_{ij}\delta_{\alpha\beta}, \quad \{a_i^\alpha, a_j^\beta\} = \{b_i^\alpha, b_j^\beta\} = 0,$$

in terms of which the spin matrix (of rank 1) is expressed as

$$S_i^{\alpha\beta} = a_i^\alpha b_i^\beta,$$

and the Hamiltonian acquires the form

$$H = \sum_i p_i^2 - g^2 \sum_{i\neq j} b_i^\mu a_j^\mu b_j^\nu a_i^\nu \wp(x_i - x_j).$$

Here and below, the summation over repeated Greek indices from 1 to n is assumed, if it is not stated otherwise. The Hamiltonian equations of motion

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{a}_i^\alpha = \frac{\partial H}{\partial b_i^\alpha}, \quad \dot{b}_i^\alpha = -\frac{\partial H}{\partial a_i^\alpha}$$

are equivalent to the equations

$$\dot{a}_i^\alpha = -2g^2 \sum_{j\neq i} a_j^\alpha b_j^\nu a_i^\nu \wp(x_i - x_j),$$

$$\dot{b}_i^\alpha = 2g^2 \sum_{j\neq i} b_j^\alpha b_i^\nu a_j^\nu \wp(x_i - x_j),$$

$$\ddot{x}_i = 4g^2 \sum_{j\neq i} b_i^\mu a_j^\mu b_j^\nu a_i^\nu \wp'(x_i - x_j).$$

The dynamical variables a_i^α, b_i^β are subject to the constraints

$$a_i^\nu b_i^\nu = 1.$$

The Lax representation and equations of motion. The spin CM system has the Lax representation with the matrices

$$L_{ij}(\lambda) = -p_i\delta_{ij} - g(1-\delta_{ij}) b_i^\nu a_j^\nu \Phi(x_i - x_j, \lambda),$$

$$M_{ij}(\lambda) = -2g(1-\delta_{ij}) b_i^\nu a_j^\nu \Phi'(x_i - x_j, \lambda),$$

depending on the spectral parameter λ.

Problem. Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion.

As usual, the Lax equation implies that the spectral invariants of the Lax matrix, for example $\mathrm{tr} L^k$, are integrals of motion. In particular, $\mathrm{tr} L^2 = H + \mathrm{const}$. The higher Hamiltonians H_m are linear combinations of traces of powers of the Lax matrix.

It is not difficult to show that

$$G^{\alpha\beta} = \sum_i S_i^{\alpha\beta} = \sum_i a_i^\alpha b_i^\beta$$

are integrals of motion for all the flows ∂_{t_m} corresponding to the higher Hamiltonians: $\partial_{t_m} G^{\alpha\beta} = 0$. Indeed,

$$\partial_{t_m}\Bigl(\sum_i a_i^\alpha b_i^\beta\Bigr) = \sum_i \left(b_i^\beta \frac{\partial H_m}{\partial b_i^\alpha} - a_i^\alpha \frac{\partial H_m}{\partial a_i^\beta}\right),$$

and this is zero because H_m is a linear combination of traces $\mathrm{tr}\, L^j(\lambda)$, and

$$\begin{aligned}
&\sum_i \left(b_i^\beta \mathrm{tr}\Bigl(\frac{\partial L}{\partial b_i^\alpha}\, L^{j-1}\Bigr) - a_i^\alpha \,\mathrm{tr}\Bigl(\frac{\partial L}{\partial a_i^\beta}\, L^{j-1}\Bigr)\right) \\
&= \sum_i \sum_{l,k} \left(b_i^\beta \frac{\partial L_{lk}}{\partial b_i^\alpha}\, L_{kl}^{j-1} - a_i^\alpha \frac{\partial L_{lk}}{\partial a_i^\beta}\, L_{kl}^{j-1}\right) \\
&= \sum_i \sum_{l\neq k} (\delta_{ik} - \delta_{il}) b_l^\beta a_k^\alpha \Phi(x_l - x_k) L_{kl}^{j-1} = 0.
\end{aligned}$$

A simple lemma from linear algebra states that the eigenvalues ν_α of the $n\times n$ matrix G coincide with the non-zero eigenvalues of the $N\times N$ matrix F of rank n with matrix elements

$$F_{ij} = b_i^\nu a_j^\nu$$

(we assume that $n \leq N$). Indeed, consider the rectangular $N\times n$ matrices $\mathbf{A}_{i\alpha} = a_i^\alpha$ and $\mathbf{B}_{i\beta} = b_i^\beta$, then $G = \mathbf{A}^T\mathbf{B}$, $F = \mathbf{B}\mathbf{A}^T$, and a direct verification shows that traces of all powers of these matrices are the same: $\mathrm{tr}\, G^m = \mathrm{tr}\, F^m$ for all $m \geq 1$. This means that their non-zero eigenvalues coincide. Note that $\mathrm{tr}\, G = \sum_{\alpha=1}^{n} \nu_\alpha = N$.

Spin generalization of the RS system

The spin generalization of the RS system was suggested in the work by Krichever and the author in 1995 [18]. In that work, the spin model was obtained as a dynamical system for poles of elliptic solutions to the non-abelian two-dimensional Toda lattice. The Hamiltonian structure of this system is still unknown. That is why we start from the equations of motion from the very beginning.

The equations of motion. Let a_i^α, b_i^α be n-component vectors associated with i-th particle, as before. The equations of motion have the form

$$\begin{aligned}
\dot a_i^\alpha &= \sum_{j\neq i} a_j^\alpha b_j^\nu a_i^\nu \Bigl(\zeta(x_{ij}) - \zeta(x_{ij}+\eta)\Bigr), \\
\dot b_i^\alpha &= -\sum_{j\neq i} b_j^\alpha b_i^\nu a_j^\nu \Bigl(\zeta(x_{ij}) - \zeta(x_{ij}-\eta)\Bigr), \\
\ddot x_i &= -\sum_{j\neq i} b_i^\nu a_j^\nu b_j^\mu a_i^\mu \Bigl(\zeta(x_{ij}+\eta) + \zeta(x_{ij}-\eta) - 2\zeta(x_{ij})\Bigr).
\end{aligned}$$

It is clear from them that $\dot{x}_i - b_i^\nu a_i^\nu$ are integrals of motion. We will fix their zero values, imposing the constraints

$$\dot{x}_i = b_i^\nu a_i^\nu.$$

The Lax representation. The Lax pair for this system is as follows:

$$L_{ij}(\lambda) = b_i^\nu a_j^\nu \Phi(x_{ij} - \eta, \lambda),$$

$$M_{ij}(\lambda) = -\delta_{ij}(\zeta(\eta) + \tfrac{1}{2}\wp(\lambda)) + (1 - \delta_{ij}) b_i^\nu a_j^\nu \Phi(x_{ij}, \lambda).$$

Problem. Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion.

The derivation of this Lax pair will be given later in connection with the non-abelian Toda lattice and its elliptic solutions.

Traces of powers of the Lax matrix are integrals of motion. The spectral curve given by the equation $\det(kI - L(\lambda)) = 0$ is an integral of motion, too.

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