Integrable systems of particles and nonlinear equations. Lecture 6

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Deformed Ruijsenaars-Schneider systems

The elliptic RS systems admit a further deformation preserving integrability. (see [11, 12]). However, the Hamiltonian structure of these deformed systems is not known, and they are probably not Hamiltonian. We will discuss them on the level of Newtonian equations of motion. Like the RS systems, they exist in rational, trigonometric and elliptic versions. We will consider the most general elliptic version.

Equations of motion

The equations of motion of the deformed RS system have the form

$$\ddot{x}_i + \sum_{j\neq i}^{N} \dot{x}_i \dot{x}_j \Big(\zeta(x_{ij}+\eta) + \zeta(x_{ij}-\eta) - 2\zeta(x_{ij})\Big) + g(U_i^- - U_i^+) = 0,$$

where

$$U_i^{\pm} = \prod_{j\neq i}^{N} U^{\pm}(x_{ij}), \qquad U^{\pm}(x_{ij}) = \frac{\sigma(x_{ij}\pm 2\eta)\sigma(x_{ij}\mp\eta)}{\sigma(x_{ij}\pm\eta)\sigma(x_{ij})}$$

and g is the deformation parameter. At $g=0$ they become the equations of motion of the elliptic RS system. Clearly, any $g\neq 0$ can be put equal to an arbitrary nonzero number by rescaling the time variable as $t\to g^{-1/2}t$. In what follows we set $g=\sigma(2\eta)$ without loss of generality.

We will show that the equations of motion of the deformed RS system can be obtained by restriction of the Hamiltonian flow with the Hamiltonian $H_+ - \sigma^2(\eta)H_-$ of the RS system with an even number of particles $N=2N_0$ to the half-dimensional subspace $\mathcal{P}\subset\mathcal{F}$ of the $4N_0$-dimensional phase space $\mathcal{F}$ that corresponds to configurations in which the $2N_0$ particles stick together in pairs forming N_0 pairs with the distance between particles in each pair equal to η. Such configurations are immediately destroyed by flows with the Hamiltonians H_+, H_- but are preserved by the flow with the Hamiltonian $H_+ - \sigma^2(\eta)H_-$, and the corresponding dynamics can be restricted to the subspace $\mathcal{P}$. It is this restriction

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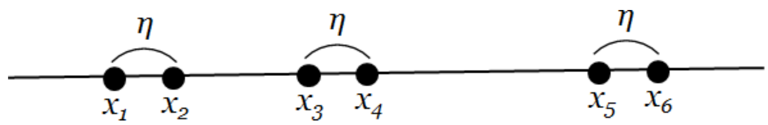

Рис. 1: Pairs of particles in the RS model ($N = 6$, $N_0 = 3$).

that gives the equations of motion written above, where one should substitute N by N_0, and x_i ($i = 1, \dots, N_0$) is the coordinate of the i-th pair which moves as a whole thing with the fixed distance between particles. In fact the subspace $\mathcal{P}$ is Lagrangian; the meaning of this fact for the theory of the deformed RS system is to be clarified.

So, let us show that the restriction of the RS dynamics of $N = 2N_0$ particles to the subspace $\mathcal{P}$ in which the particles stick together in N_0 pairs such that

$$x_{2i} - x_{2i-1} = \eta, \qquad i = 1, \dots, N_0$$

leads to the equations of motion of the deformed RS system for coordinates of the pairs. It is natural to introduce the variables

$$X_i = x_{2i-1}, \qquad i = 1, \dots, N_0,$$

which are coordinates of pairs. As we will see below, such structure is preserved by the flow with the Hamiltonian $\bar{H} = H_+ - \sigma^2(\eta)H_-$. Therefore, to define the dynamical system, one should consider the evolution in the time t that corresponds to the flow with the Hamiltonian $\bar{H} = H_+ - \sigma^2(\eta)H_-$ fixing the time variables that correspond to the other flows.

In order to simplify the formulas and to get rid of inessential multipliers and coefficients, here we redefine the momenta and take the Hamiltonians in the form

$$H_\pm = \sum_i e^{\pm p_i} \prod_{j \neq i} \frac{\sigma(x_{ij} \pm \eta)}{\sigma(x_{ij})}.$$

It is easy to see that these Hamiltonians lead to the same Newtonian equations of motion. For further convenience, we also change normalization of the integrals of motion:

$$J_{\pm k} = \frac{\sigma(|k|\eta)}{\sigma^k(\eta)} \sum_{\mathcal{I} \subset \{1,\dots,N\}, |\mathcal{I}|=k} \exp\Big(\pm \sum_{i \in \mathcal{I}} p_i\Big) \prod_{i \in \mathcal{I}, j \notin \mathcal{I}} \frac{\sigma(x_{ij} \pm \eta)}{\sigma(x_{ij})},$$

then $\bar{H} = J_1 - J_{-1}$.

For the velocities $\dot{x}_i = \partial \bar{H}/\partial p_i$ we have:

$$\dot{x}_{2i-1} = e^{p_{2i-1}} \prod_{j=1, \neq 2i-1}^{2N_0} \frac{\sigma(x_{2i-1,j} + \eta)}{\sigma(x_{2i-1,j})} + \sigma^2(\eta) e^{-p_{2i-1}} \prod_{j=1, \neq 2i-1}^{2N_0} \frac{\sigma(x_{2i-1,j} - \eta)}{\sigma(x_{2i-1,j})},$$

$$\dot{x}_{2i} = e^{p_{2i}} \prod_{j=1, \neq 2i}^{2N_0} \frac{\sigma(x_{2i,j} + \eta)}{\sigma(x_{2i,j})} + \sigma^2(\eta) e^{-p_{2i}} \prod_{j=1, \neq 2i}^{2N_0} \frac{\sigma(x_{2i,j} - \eta)}{\sigma(x_{2i,j})}.$$

Imposing the constraint $x_{2i}-x_{2i-1}=\eta$, one sees that the first term in the right-hand side of the first equation, as well as the second term in the second one, vanish since there appears a zero multiplier in the product. Then in terms of the coordinates of the pairs X_i these equations are:

$$\dot{x}_{2i-1}=\sigma(\eta)\sigma(2\eta)e^{-p_{2i-1}}\prod_{j=1,\neq i}^{N_0}\frac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij})},$$

$$\dot{x}_{2i}=\frac{\sigma(2\eta)}{\sigma(\eta)}\,e^{p_{2i}}\prod_{j=1,\neq i}^{N_0}\frac{\sigma(X_{ij}+2\eta)}{\sigma(X_{ij})}.$$

Hence, if to put

$$p_{2i-1}=\alpha_i+P_i,\quad p_{2i}=\alpha_i-P_i,\quad i=1,\ldots,N_0,$$

where

$$\alpha_i=\log\sigma(\eta)+\frac{1}{2}\sum_{j\neq i}^{N_0}\log\frac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij}+2\eta)}$$

with arbitrary P_i, then $\dot{x}_{2i-1}=\dot{x}_{2i}$ for all i, so the distance between particles in each pair is preserved by the evolution. Therefore, under the $\bar{H}$-flow each pair moves as a whole thing, and

$$\dot{X}_i=\sigma(2\eta)e^{-P_i}\prod_{j\neq i}^{N_0}\frac{(\sigma(X_{ij}-2\eta)\sigma(X_{ij}+2\eta))^{1/2}}{\sigma(X_{ij})}.$$

We have passed from the initial $4N_0$-dimensional phase space $\mathcal{F}$ with coordinates $(\{x_i\}_N,\{p_i\}_N)$ to the $2N_0$-dimensional subspace $\mathcal{P}\subset\mathcal{F}$ of pairs that is defined by imposing the constraints

$$\begin{cases} x_{2i}-x_{2i-1}=\eta, \quad x_{2i-1}=X_i,\\[2ex] p_{2i-1}+p_{2i}=2\log\sigma(\eta)+\displaystyle\sum_{j\neq i}\log\frac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij}+2\eta)}.\end{cases}$$

The coordinates in $\mathcal{P}$ are $(\{X_i\}_{N_0},\{P_i\}_{N_0})$.

Restricting the second set of the Hamiltonian equations $\dot{p}_i=-\partial\bar{H}/\partial x_i$ to the subspace $\mathcal{P}$, we have:

$$\begin{aligned}\dot{p}_{2i-1} &= \sigma(\eta)\sigma(2\eta)e^{-\alpha_i-P_i}\prod_{k=1,\neq i}^{N_0}\frac{\sigma(X_{ik}-2\eta)}{\sigma(X_{ik})}\left[\sum_{j=1,\neq i}^{N_0}\Big(\zeta(X_{ij}-2\eta)-\zeta(X_{ij})\Big)+\zeta(\eta)-\zeta(2\eta)\right]\\ &+\sigma(\eta)\sigma(2\eta)\sum_{l=1,\neq i}^{N_0}e^{-\alpha_l-P_l}\prod_{k=1,\neq l}^{N_0}\frac{\sigma(X_{lk}-2\eta)}{\sigma(X_{lk})}\Big(\zeta(X_{il}+\eta)-\zeta(X_{il})\Big)\\ &-\frac{\sigma(2\eta)}{\sigma(\eta)}\sum_{l=1}^{N_0}e^{\alpha_l-P_l}\prod_{k=1,\neq l}^{n}\frac{\sigma(X_{lk}+2\eta)}{\sigma(X_{lk})}\Big(\zeta(X_{il}-2\eta)-\zeta(X_{il}-\eta)\Big)\\ &+\sigma^{-1}(\eta)e^{\alpha_i+P_i}\prod_{k=1,\neq i}^{N_0}\frac{\sigma(X_{ik}+\eta)}{\sigma(X_{ik}-\eta)}-\sigma(\eta)e^{-\alpha_i+P_i}\prod_{k=1,\neq i}^{N_0}\frac{\sigma(X_{ik}-\eta)}{\sigma(X_{ik}+\eta)}.\end{aligned}$$

Taking the time derivative of $\dot{X}_i$, we get:

$$\ddot{X}_i = -\sigma(2\eta)\dot{P}_i e^{-P_i} \prod_{j\neq i}^{N_0} \frac{(\sigma(X_{ij}-2\eta)\sigma(X_{ij}+2\eta))^{1/2}}{\sigma(X_{ij})}$$

$$+\frac{1}{2}\sum_{j\neq i}^{N_0} \dot{X}_i(\dot{X}_i-\dot{X}_j)\Big(\zeta(X_{ij}-2\eta)+\zeta(X_{ij}+2\eta)-2\zeta(X_{ij})\Big).$$

Here one should substitute $\dot{P}_i = -\dot{\alpha}_i + \dot{p}_{2i-1}$ from the equation above:

$$\dot{P}_i = -\dot{\alpha}_i + \dot{X}_i\left[\sum_{j\neq i}^{N_0}\Big(\zeta(X_{ij}-2\eta)-\zeta(X_{ij})\Big)+\zeta(\eta)-\zeta(2\eta)\right]$$

$$+\sum_{l\neq i}^{N_0}\dot{X}_l\Big(\zeta(X_{il}+\eta)-\zeta(X_{il})\Big)-\sum_{l=1}^{N_0}\dot{X}_l\Big(\zeta(X_{il}-2\eta)-\zeta(X_{il}-\eta)\Big)$$

$$+e^{P_i}\prod_{k\neq i}^{N_0}\frac{\sigma^{1/2}(X_{ik}-2\eta)\sigma(X_{ik}+\eta)}{\sigma^{1/2}(X_{ik}+2\eta)\sigma(X_{ik}-\eta)} - e^{P_i}\prod_{k\neq i}^{N_0}\frac{\sigma^{1/2}(X_{ik}+2\eta)\sigma(X_{ik}-\eta)}{\sigma^{1/2}(X_{ik}-2\eta)\sigma(X_{ik}+\eta)}.$$

Plugging here $\dot{\alpha}_i$ from

$$\alpha_i = \log\sigma(\eta) + \frac{1}{2}\sum_{j\neq i}^{N_0}\log\frac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij}+2\eta)},$$

we finally obtain:

$$\ddot{X}_i = -\sum_{j\neq i}^{N_0}\dot{X}_i\dot{X}_j\Big(\zeta(X_{ij}+\eta)+\zeta(X_{ij}-\eta)-2\zeta(X_{ij})\Big)+\sigma(2\eta)\Big(U_i^+ - U_i^-\Big),$$

where

$$U_i^{\pm} = \prod_{j\neq i}^{N_0}\frac{\sigma(X_{ij}\pm 2\eta)\sigma(X_{ij}\mp\eta)}{\sigma(X_{ij}\pm\eta)\sigma(X_{ij})}.$$

We have thus obtained the equations of motion of the deformed RS system (with $g = \sigma(2\eta)$, $N = N_0$).

Commutation representation

The equations of motion of the deformed RS system do not admit any representation of the Lax type. However, they admit a more general commutation representation which is known as the Manakov's triple. It allows one to find integrals of motion. Here we present only the result; the derivation will be given later in the section devoted to the dynamics of poles of elliptic solutions to the Toda lattice of type B.

In this more general case the matrices L and M depend on two spectral parameters, λ and k (they will be connected by equation of the spectral curve). Consider the matrices

$$L_{ij}(k,\lambda) = \dot{x}_i \Phi(x_{ij}-\eta,\lambda) + k^{-1} g U_i^- \Phi(x_{ij}-2\eta,\lambda),$$

$$M_{ij}(k,\lambda) = \dot{x}_i(1-\delta_{ij})\Phi(x_{ij},\lambda) + k^{-1} g U_i^+ \Phi(x_{ij}-\eta,\lambda) + \delta_{ij}\Big(\sum_l \dot{x}_l \zeta(x_{il}+\eta) - \sum_{l\neq i} \dot{x}_l \zeta(x_{il})\Big),$$

$$R_{ij}(k,\lambda) = g k^{-1}(U_i^- - U_i^+)\Phi(x_{ij}-\eta,\lambda).$$

Here and below $g=\sigma(2\eta)$. The statement is that the matrix equation

$$\dot{L} + [L,M] = R(L-kI)$$

is equivalent to the equations of motion of the deformed RS system. The matrix equation can be written in a little bit different way in terms of the matrices $\mathcal{L}(k,\lambda) = L(k,\lambda) - kI$ and

$$M^{\pm}_{ij}(k,\lambda) = \dot{x}_i(1-\delta_{ij})\Phi(x_{ij},\lambda) + k^{-1} g U_i^{\pm} \Phi(x_{ij}-\eta,\lambda) + \delta_{ij}\Big(\sum_l \dot{x}_l \zeta(x_{il}+\eta) - \sum_{l\neq i} \dot{x}_l \zeta(x_{il})\Big),$$

then

$$\dot{\mathcal{L}} = M^- \mathcal{L} - \mathcal{L} M^+,$$

and the matrices $\mathcal{L}$, M^+, M^- form the Manakov's triple.

The calculation necessary for the proof is non-trivial and requires some explanations. To this end, introduce the matrices

$$A^0_{ij} = (1-\delta_{ij})\Phi(x_{ij}), \quad A_{ij} = \Phi(x_{ij}-\eta), \quad B_{ij} = \Phi(x_{ij}-2\eta)$$

and diagonal matrices

$$\dot{X}_{ij} = \delta_{ij}\dot{x}_i, \quad U^{\pm}_{ij} = \delta_{ij} U_i^{\pm},$$

$$Z^{\pm}_{ij} = \delta_{ij}\Big(\sum_l \dot{x}_l \zeta(x_{il}\pm\eta) - \sum_{l\neq i} \dot{x}_l \zeta(x_{il})\Big),$$

$$D^-_{ij} = \delta_{ij}\sum_{l\neq i}\Big(\zeta(x_{il}-2\eta) + \zeta(x_{il}+\eta) - \zeta(x_{il}-\eta) - \zeta(x_{il})\Big),$$

$$S^-_{ij} = \delta_{ij}\sum_{l\neq i}\dot{x}_l\Big(\zeta(x_{il}-2\eta) + \zeta(x_{il}+\eta) - \zeta(x_{il}-\eta) - \zeta(x_{il})\Big).$$

We will also need the matrices $A'_{ij} = \Phi'(x_{ij}-\eta,\lambda)$, $B'_{ij} = \Phi'(x_{ij}-2\eta,\lambda)$. In this notation

$$L = \dot{X}A + gk^{-1}U^- B,$$

$$M = \dot{X}A^0 + gk^{-1}U^+ A - Z^+.$$

Now we find:

$$\dot{L}+[L,M]=\Big(\ddot{X}+(Z^{+}+Z^{-})\dot{X}+g(U^{-}-U^{+})\Big)A+gk^{-1}(U^{-}-U^{+})A(L-kI)$$
$$+\,W_0+gk^{-1}W_1+g^2k^{-2}W_2,$$

where

$$W_0=\dot{X}^2A'-\dot{X}A'\dot{X}+\dot{X}A\dot{X}A^0-\dot{X}A^0\dot{X}A-\dot{X}AZ^{+}-\dot{X}Z^{-}A,$$

$$W_1=U^{-}\dot{X}DB-U^{-}(S^{-}-Z^{+})B+U^{-}\dot{X}B'-U^{-}B'\dot{X}+\dot{X}AU^{+}A$$
$$+\,U^{-}B\dot{X}A^0-\dot{X}A^0U^{-}B-U^{-}BZ^{+}-U^{-}A\dot{X}A,$$

$$W_2=U^{-}BU^{+}A-U^{+}AU^{-}B-(U^{-}-U^{+})AU^{-}B.$$

A direct calculation (which uses the identities from Section 1) shows that $W_0=0$. The calculation of W_2 yields:

$$\Big(U^{-}BU^{+}A-U^{+}AU^{-}B\Big)_{ij}$$
$$=\Phi(x_{ij}-3\eta)\sum_l\Big[U_i^{-}U_l^{+}\Big(\zeta(x_{il}-2\eta)+\zeta(x_{lj}-\eta)-\zeta(x_{ij}-3\eta+\lambda)+\zeta(\lambda)\Big)$$
$$-\,U_i^{+}U_l^{-}\Big(\zeta(x_{il}-\eta)+\zeta(x_{lj}-2\eta)-\zeta(x_{ij}-3\eta+\lambda)+\zeta(\lambda)\Big)\Big].$$

Using the fact that the sum of residues of the elliptic function

$$\Big(\zeta(x-x_j-\eta)-\zeta(x-x_i+2\eta)\Big)\prod_l\frac{\sigma(x-x_l+2\eta)\sigma(x-x_l-\eta)}{\sigma(x-x_l+\eta)\sigma(x-x_l)}$$

in the fundamental domain is zero, we conclude that

$$\Big(U^{-}BU^{+}A-U^{+}AU^{-}B\Big)_{ij}$$
$$=\Phi(x_{ij}-3\eta)(U_i^{-}-U_i^{+})\sum_l\Big[U_l^{-}\Big(\zeta(x_{il}-\eta)+\zeta(x_{lj}-2\eta)-\zeta(x_{ij}-3\eta+\lambda)+\zeta(\lambda)\Big),$$

so $W_2=0$. The calculation that shows that $W_1=0$ too, is the most difficult. One should use the fact that the sum of residues of the elliptic function

$$\Big(\zeta(x-x_j-\eta)-\zeta(x-x_i+\eta)\Big)\prod_l\frac{\sigma(x-x_l+2\eta)\sigma(x-x_l-\eta)}{\sigma(x-x_l+\eta)\sigma(x-x_l)}$$

in the fundamental domain is zero (this function has simple poles at $x=x_l$ and $x=x_l-\eta$, and a second order pole at $x=x_i-\eta$).

As a result, we obtain the matrix identity

$$\dot{L}+[L,M]=R(L-kI)+P,$$

where the matrix P is

$$P=\Big(\ddot{X}+(Z^{+}+Z^{-})\dot{X}+g(U^{-}-U^{+})\Big)A,$$

and

$$R=gk^{-1}(U^{-}-U^{+})A.$$

The equations of motion are equivalent to the condition $P=0$.

Integrals of motion

The equation of the spectral curve is

$$\det \mathcal{L}(k,\lambda) = \det\Big(kI - L(k,\lambda)\Big) = 0.$$

The time evolution $L \to L(t)$ of our "Lax matrix" is not isospectral. Nevertheless, the characteristic polynomial $\det(kI - L(k,\lambda))$ (which is in fact a Laurent polynomial in k) is an integral of motion, so the spectral curve does not depend on time. This fact follows from the Manakov's representation. Indeed,

$$\begin{aligned}\frac{d}{dt}\log\det(L-kI) &= \frac{d}{dt}\operatorname{tr}\log(L-kI)\\ &= \operatorname{tr}\Big(\dot{L}(L-kI)^{-1}\Big) = \operatorname{tr}R = 0\end{aligned}$$

because the matrix R is traceless:

$$\operatorname{tr} R = gk^{-1}\Phi(-\eta,\lambda)\sum_i (U_i^- - U_i^+) = 0.$$

This follows from the fact that $\sum_i (U_i^- - U_i^+)$ is proportional to the sum of residues of the elliptic function

$$F(x) = \prod_j \frac{\sigma(x-x_j-2\eta)\sigma(x-x_j+\eta)}{\sigma(x-x_j-\eta)\sigma(x-x_j)}$$

in the fundamental domain. The characteristic polynomial $\det(kI-L(k,\lambda))$ is the generating function of integrals of motion.

In order to study properties of the spectral curve, it is convenient to pass to the gauge-equivalent Lax matrix $\tilde{L} = e^{-\eta\zeta(\lambda)}G^{-1}LG$, where G is the diagonal matrix with matrix elements $G_{ii} = e^{-\zeta(\lambda)x_i}$, and to the spectral parameter $z = ke^{-\eta\zeta(\lambda)}$. Then the equation of the spectral curve acquires the form

$$\det\Big(zI - \tilde{L}(z,\lambda)\Big) = 0,$$

where

$$\tilde{L}_{ij}(z,\lambda) = \dot{x}_i\phi(x_{ij}-\eta,\lambda) - gz^{-1}U_i^-\phi(x_{ij}-2\eta,\lambda), \quad \phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}.$$

Denote

$$Q(z,\lambda) = \frac{\det\Big(zI - \tilde{L}(z,\lambda)\Big)}{\sigma(2N\eta-\lambda)}.$$

It is a generating function of integrals of motion. Calculation of the determinant yields:

$$\begin{aligned}Q(z,\lambda) &= \frac{z^N}{\sigma(2N\eta-\lambda)} - \frac{z^{-N}}{\sigma(\lambda)}\\ &+ \sum_{k=1}^{N} z^{N-k}\frac{\sigma(\lambda-k\eta)}{\sigma(\lambda)\sigma(2N\eta-\lambda)\sigma(k\eta)}\,\mathsf{J}_k - \sum_{k=1}^{N-1} z^{k-N}\frac{\sigma(\lambda-2N\eta+k\eta)}{\sigma(\lambda)\sigma(2N\eta-\lambda)\sigma(k\eta)}\,\mathsf{J}_k,\end{aligned}$$

where J_k are integrals of motion of the deformed RS system. They can be found in an explicit form (see below). From the equation of the spectral curve $Q(z, \lambda) = 0$ it is seen that the curve has a holomorphic involution $\iota : (z, \lambda) \mapsto (z^{-1}, 2N\eta - \lambda)$ with two fixed points $(\pm 1, N\eta)$.

The calculation of the determinant requires some comments. The calculation is rather long but direct. It uses the formula for determinant of sum of two matrices and the formula for determinant of the elliptic Cauchy matrix. First of all, the determinant $\det(I + M)$ is equal to sum of all diagonal minors of the matrix M of all sizes, including the "empty minor", which should be put equal to 1. After that we encounter the determinants of the form $\det(A_{\mathcal{J}} + B_{\mathcal{J}})$, where $A_{\mathcal{J}}$, $B_{\mathcal{J}}$ are diagonal minors of the matrices $\dot{X}_i \phi(X_{ij} - \eta, u)$, $\sigma(2\eta) z^{-1} U_i^- \phi(X_{ij} - 2\eta, u)$ of size $n \le N$ with rows and columns indexed by indices from the set $\mathcal{J} = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, N\}$ $(j_1 < j_2 < \ldots < j_n \le N)$. The formula for determinant of sum of two matrices states that

$$\det(A_{\mathcal{J}} + B_{\mathcal{J}}) = \sum_{\mathcal{I} \subseteq \mathcal{J}} \det A^{(B)}_{\mathcal{J} \setminus \mathcal{I}},$$

where the sum goes over all subsets $\mathcal{I}$ of the set $\mathcal{J}$ and $A^{(B)}_{\mathcal{J} \setminus \mathcal{I}}$ is the matrix $A_{\mathcal{J}}$, in which the rows numbered by indices from the set $\mathcal{I}$ are changed to the corresponding rows of the matrix $B_{\mathcal{J}}$. Each matrix $A^{(B)}_{\mathcal{J} \setminus \mathcal{I}}$ is an elliptic Cauchy matrix multiplied by a diagonal matrix, so its determinant is known. To see this, put $x_j = X_j$ and

$$y_j = X_j - \eta \quad \text{if } j \in \mathcal{J} \setminus \mathcal{I},$$

$$y_j = X_j - 2\eta \quad \text{if } j \in \mathcal{I}$$

in the elliptic Cauchy matrix. The determinant is then represented as a Laurent polynomial in z with coefficients that are written as sums over the sets $\mathcal{I}$, $\mathcal{I}' \subseteq \{1, \ldots, N\}$ such that $\mathcal{I} \cap \mathcal{I}' = \emptyset$.

Problem. Do this calculation with all details and find the explicit form of the coefficients J_k.

The explicit form of the integrals of motion is as follows:

$$\mathsf{J}_{\pm n} = \sum_{m=0}^{[k/2]} J_{\pm n, m},$$

where $J_{\pm n,m}$ is given by

$$J_{\pm n,m} = \frac{\sigma(n\eta)}{\sigma^{n-2m}(\eta)} \sum_{\substack{\mathcal{I}, \mathcal{I}', \mathcal{I} \cap \mathcal{I}' = \emptyset \\ |\mathcal{I}| = m, |\mathcal{I}'| = n-2m}} \Big(\prod_{j \in \mathcal{I}'} \dot{X}_j \Big) \Big(\prod_{\substack{i,j \in \mathcal{I}' \\ i<j}} V(X_{ij}) \Big) \Big(\prod_{i \in \mathcal{I}} \prod_{\ell \in \mathcal{N} \setminus (\mathcal{I} \cup \mathcal{I}')} U^{\pm}(X_{i\ell}) \Big).$$

Here

$$V(X_{ij}) = \frac{\sigma^2(X_{ij})}{\sigma(X_{ij} + \eta)\, \sigma(X_{ij} - \eta)},$$

$$U^{\pm}(X_{ij}) = \frac{\sigma(X_{ij} \pm 2\eta)\, \sigma(X_{ij} \mp \eta)}{\sigma(X_{ij} \pm \eta)\, \sigma(X_{ij})}.$$

Here are the first two integrals:

$$\mathsf{J}_1 = \sum_{i=1} \dot{x}_i,$$

$$\mathsf{J}_2 = \frac{\sigma(2\eta)}{2\sigma^2(\eta)} \left[\sum_{i\neq j} \dot{x}_i \dot{x}_j V(x_{ij}) + \sigma^2(\eta) \sum_i \Big(\prod_{\ell\neq i} U^+(x_{i\ell}) + \prod_{\ell\neq i} U^-(x_{i\ell}) \Big) \right],$$

One can prove that J_1 is an integral of motion directly summing all the equations of motion. One can prove that $J_{n,m} = J_{-n,m}$. This follows from the identity

$$\sum_{\mathcal{I}\subset\mathcal{N}'} \prod_{i\in\mathcal{I}} \prod_{\ell\in\mathcal{N}'\setminus\mathcal{I}} U^+(X_{i\ell}) = \sum_{\mathcal{I}\subset\mathcal{N}'} \prod_{i\in\mathcal{I}} \prod_{\ell\in\mathcal{N}'\setminus\mathcal{I}} U^-(X_{i\ell}),$$

where $\mathcal{N}'$ is any subset of the set $\mathcal{N} = \{1, \ldots, N\}$. The already familiar for us identity $\sum\limits_i (U_i^- - U_i^+) = 0$ is a very particular case of the general identity given here.

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