Integrable systems of particles and nonlinear equations. Lecture 3

A. Zabrodin*

The CM system with trigonometric potential

The interaction potential of the CM system admits a deformation such that it becomes periodic in the complex plane with one real or purely imaginary period. We will call such systems trigonometric (hyperbolic) CM systems (sometimes they are called Calogero-Sutherland systems). This deformation preserves integrability. All statements about the rational CM systems (except the self-duality) made in the previous section have their direct analogs for the trigonometric systems, although their formulations and proofs may be a little bit more complicated.

The Hamiltonian and equations of motion. The Hamiltonian has the form

$$H = \sum_i p_i^2 - g^2 \sum_{i \neq j} \frac{\gamma^2}{\sinh^2(\gamma(x_i - x_j))},$$

where g^2 is the coupling constant, and γ is a parameter characterizing period of the potential which is equal to $\pi i/\gamma$. At real γ we have a hyperbolic system, at purely imaginary γ the system is trigonometric. In what follows we will mostly ignore this difference and will call the system trigonometric in the both cases. In the limit $\gamma \to 0$ we come back to the rational CM system

The equations of motion are as follows:

$$\dot{x}_i = 2p_i,$$

$$\dot{p}_i = -4g^2\gamma^3 \sum_{j \neq i} \frac{\cosh(\gamma(x_i - x_j))}{\sinh^3(\gamma(x_i - x_j))},$$

or, in the Newtonian form,

$$\ddot{x}_i = -8g^2\gamma^3 \sum_{j \neq i} \frac{\cosh(\gamma(x_i - x_j))}{\sinh^3(\gamma(x_i - x_j))}.$$

*e-mail: zabrodin@itep.ru

The Lax representation and integrals of motion. The Lax matrix for the trigonometric CM system has the form

$$L_{ij} = -p_i\delta_{ij} - \frac{g\gamma(1-\delta_{ij})}{\sinh(\gamma(x_i - x_j))}.$$

It is convenient to pass to the new variables

$$w_i = e^{2\gamma x_i}$$

and along with the diagonal matrix $X = \mathrm{diag}(x_1, \ldots, x_N)$ introduce the diagonal matrix $W = e^{2\gamma X} = \mathrm{diag}(w_1, \ldots, w_N)$. Introduce also the matrices A, B with zeros on the main diagonal and with the matrix elements

$$A_{ik} = 2\gamma(1-\delta_{ik})\frac{w_i^{1/2}w_k^{1/2}}{w_i - w_k},$$

$$B_{ik} = 4\gamma^2(1-\delta_{ik})\frac{w_i^{3/2}w_k^{1/2}}{(w_i - w_k)^2},$$

as well as the diagonal matrix

$$D_{ik} = 4\gamma^2\delta_{ik}\sum_{l\neq i}\frac{w_i w_l}{(w_i - w_l)^2}.$$

We denote them by the same letters as the corresponding matrices in the previous section, since they are their direct analogs and in the limit $\gamma \to 0$ tend to them. The Lax pair is as follows:

$$L = -\tfrac{1}{2}\dot{X} - gA,$$

$$M = \gamma\dot{X} + 2gB - 2gD.$$

Problem. Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion.

The equation that characterizes the Lax matrix is

$$[W, L] = 2g\gamma\Big(W - W^{1/2}EW^{1/2}\Big)$$

or

$$W^{1/2}LW^{-1/2} - W^{-1/2}LW^{1/2} = 2g\gamma(I - E).$$

In the limit $\gamma \to 0$ it becomes the equation $[X, L] = g(I - E)$, which was discussed in the previous section.

As before, the Lax equation implies that the time evolution of the Lax matrix is an isospectral transformation, and $H_k = \mathrm{tr}L^k$ are integrals of motion.

Problem. Verify that $H_2 = H$ and find H_3 in the explicit form.

Problem. Using the method of the previous section, prove that the integrals H_k are in involution.

Linearization in the space of matrices. The linearization in the space of matrices has an analog for the trigonometric system: $w_i(t) = e^{2\gamma x_i(t)}$, where $x_i(t)$ are the coordinates of particles of the trigonometric CM system are eigenvalues of the matrix

$$e^{-4\gamma t L_0} e^{2\gamma X_0}.$$

Problem. Prove this statement.

Problem. Try to find the corresponding matrix for the higher flows.

The trigonometric CM system and the rational system in the external field. In 1997, Nekrasov obtained a remarkable result that establishes a correspondence between the dynamics of the trigonometric CM system and the dynamics of the rational CM system in the external quadratic potential.

Let us consider the systems with repulsion, i.e., change in the previous formulas $g \to ig$ and $\gamma \to i\gamma$. In order not to mix the notation, denote the coordinates and momenta of particles in the trigonometric system as θ_i, $-\xi_i$ with the canonical Poisson brackets between them. The Hamiltonian and the Lax matrix of the trigonometric system acquire then the form

$$H = \sum_i \xi_i^2 + g^2 \sum_{i \neq j} \frac{\gamma^2}{\sin^2(\gamma(\theta_i - \theta_j))},$$

$$L_{jk} = \xi_j \delta_{jk} - \frac{ig\gamma(1 - \delta_{jk})}{\sin(\gamma(\theta_j - \theta_k))}.$$

Let us also change the notation for the Lax matrix of the rational system and introduce the matrix

$$P_{jk} = -p_j \delta_{jk} - \frac{ig(1 - \delta_{jk})}{x_j - x_k}$$

(previously it was L); in the case of repulsion it is Hermitian. Instead of $L^{\pm}$ introduce the matrices

$$Z = P + i\omega X, \quad Z^{\dagger} = P - i\omega X.$$

As it follows from the discussion in the previous section, the matrix Z satisfies the equation

$$\dot{Z} + [Z, M] + 2i\omega Z = 0$$

of the Lax type which implies that

$$Z(t) = e^{2i\omega t} V Z(0) V^{-1}$$

with some unitary matrix V.

Consider the decomposition of the matrix Z into a product of a unitary and a Hermitian matrices, which is analogous to the representation of a complex number z in the form $z = re^{i\varphi}$. We will write it in the symmetrized form

$$Z = U^{1/2} R^{1/2} U^{1/2}, \quad R^{\dagger} = R, \quad U^{\dagger} = U^{-1},$$

then

$$Z^{\dagger} Z = U^{-1/2} R U^{1/2}.$$

Let V be a unitary matrix that diagonalizes the matrix U, i.e.,

$$U = VWV^{-1}, \quad W = \mathrm{diag}(e^{2i\omega\theta_1}, \ldots, e^{2i\omega\theta_N}).$$

It is defined up to multiplication from the right by a diagonal matrix. We fix this freedom by imposing the condition $U\mathbf{e} = \mathbf{e}$.

Put

$$L = V^{-1}RV.$$

The commutation relation $[X, P] = ig(I - E)$ or $[Z, Z^\dagger] = -2\omega g(I - E)$ can be rewritten as

$$U^{1/2}RU^{-1/2} - U^{-1/2}RU^{1/2} = -2\omega g(I - E)$$

or

$$W^{1/2}LW^{-1/2} - W^{-1/2}LW^{1/2} = 2\omega g(I - E).$$

Comparing with the similar relation for the trigonometric CM system, we conclude that one should put $\omega = \gamma$, then L becomes its Lax matrix with the diagonal elements $\xi_i = (V^{-1}RV)_{ii}$. Therefore, the Hamiltonians $\mathrm{tr}(Z^\dagger Z)^k$ of the rational system turn over to the Hamiltonians $\mathrm{tr}L^k$ of the trigonometric system. In particular, at $k = 1$ we see that the dynamics of the system in the external field is equivalent to the free dynamics of particles of the trigonometric system with the Hamiltonian $\sum_i \xi_i$, i.e., the θ_j's move with a constant velocity which is the same for all particles. This also follows from the relation

$$Z(t) = e^{2i\omega t} V Z(0) V^{-1},$$

found earlier.

It remains to show that the transformation from (p_i, x_i) to (ξ_i, θ_i) is canonical. Again, it is easier to do this in the matrix form. We have:

$$\mathrm{tr}(dZ \wedge dZ^\dagger) = 2i\omega \mathrm{tr}(dP \wedge dX).$$

Problem. Using the identity

$$\mathrm{tr}(d\tilde{X} \wedge d\tilde{Y}) = \mathrm{tr}(dX \wedge dY) - d\mathrm{tr}([X, Y]U^{-1}dU)$$

from the previous section, show that

$$\sum_i d\xi_i \wedge d\theta_i = -\mathrm{tr}(dP \wedge dX) = \sum_i dp_i \wedge dx_i,$$

which just means that the transformation is canonical.

Список литературы

[1] F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971) 419—436.

[2] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16** (1975) 197–220.

[3] A. Perelomov, *Integrable systems of classical mechanics and Lie algebras*, Birkhäuser Basel, 1990.

[4] M.A. Olshanetsky, A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71** (1981) 313–400.

[5] Yu. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Springer Basel AG, 2003.

[6] N.I. Akhiezer, *Elements of the theory of elliptic functions*, "Nauka", Moscow, 1970.

[7] E.T. Whittaker, G.N. Watson, *A course of modern analysis*, Cambridge At the University Press, 1927 (Russian translation: Э.Т. Уиттекер, Дж.Н. Ватсон, *Курс современного анализа*, том II, Государственное издательство физико-математической литературы, Москва, 1963).

[8] T. Takebe, *Elliptic integrals and elliptic functions*, Springer, 2023.

[9] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986) 370–405.

[10] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987) 191–213.