Integrable systems of particles and nonlinear equations. Lecture 2

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Rational CM system in quadratic potential

The CM particles can be put in an external field with quadratic potential, and the integrability is preserved. The Hamiltonian has the form

$$H = \sum_{i=1}^{N} p_i^2 - g^2 \sum_{i\neq j} \frac{1}{(x_i - x_j)^2} + \omega^2 \sum_{i=1}^{N} x_i^2,$$

which leads to the equations of motion:

$$\ddot{x}_i = -8g^2 \sum_{j\neq i} \frac{1}{(x_i - x_j)^3} - 4\omega^2 x_i.$$

They imply that the center of masses moves as the harmonic oscillator with frequency 2ω.

Commutation representation. Keeping the notation from the previous section, we introduce the matrices

$$L^{\pm} = L \pm i\omega X.$$

Problem. Show that the equations of motion are equivalent to the matrix equations

$$\dot{L}^{\pm} + [L^{\pm}, M] \pm 2i\omega L^{\pm} = 0$$

with the matrix M from the previous section.

Consider the matrices

$$L_1 = L^+ L^- = L^2 + \omega^2 X^2 - ig\omega(E - I),$$

$$L_2 = L^- L^+ = L^2 + \omega^2 X^2 + ig\omega(E - I).$$

Problem. Show that the matrices L_1, L_2 satisfy the Lax equations Лакса:

$$\dot{L}_\alpha + [L_\alpha, M] = 0, \quad \alpha = 1, 2.$$

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The same Lax equation holds for the matrix

$$\mathcal{L} = \frac{1}{2}(L_1 + L_2) = L^2 + \omega^2 X^2,$$

and we can immediately find a set of integrals of motion:

$$I_k = \operatorname{tr} \mathcal{L}^k.$$

At $k = 1$ we obtain the Hamiltonian of the system. It can be shown that all these integrals are in involution.

Problem. Find I_2 in the explicit form.

Note that the commutation representation can be written in terms of the matrices L, M и X. From the equations $\dot{L}^{\pm} + [L^{\pm}, M] \pm 2i\omega L^{\pm} = 0$ we find:

$$\dot{L} + [L, M] \pm i\omega\Big(\dot{X} + [X, M] + 2L\Big) = 2\omega^2 X.$$

The imaginary part must vanish, i.e.,

$$\dot{X} + [X, M] + 2L = 0.$$

Therefore, we arrive at the commutation representation in the form

$$\dot{L} + [L, M] = 2\omega^2 X.$$

The projection method. At $\omega \neq 0$ the free motion in the space of matrices should be substituted by harmonic oscillations with frequency 2ω and project onto eigenvalues. To wit, we will show that the eigenvalues of the matrix

$$X_0 \cos(2\omega t) - L_0 \frac{\sin(2\omega t)}{\omega}$$

move as CM particles in the quadratic potential. Obviously, at $\omega = 0$ this reduces to the statement of the previous section. Let V be the diagonalizing matrix:

$$X_0 \cos(2\omega t) - L_0 \frac{\sin(2\omega t)}{\omega} = V X V^{-1},$$

where X is a diagonal matrix. Differentiating this with respect to time, we obtain:

$$2\omega \sin(2\omega t) X_0 + \cos(2\omega t) L_0 = 2 V L V^{-1},$$

where we have put $L = -\frac{1}{2}(\dot{X} + [X, M])$ and $M = -\dot{V} V^{-1}$. Differentiating once again, we will have:

$$\dot{L} + [L, M] = 2\omega^2 X,$$

which is the commutation representation of the equations of motion.

Next, we should show that the matrix L is of the required form. The equation $\dot{L} + [L, M] = 2\omega^2 X$ is equivalent to the Lax equation for $\mathcal{L} = L^2 + \omega^2 X^2$. This we can write

$$V(L^2 + \omega^2 X^2)V^{-1} = L_0^2 + \omega^2 X_0^2.$$

Plugging here the time dependence of the matrix VXV^{-1}, we obtain, after some simple transformations:

$$VLV^{-1} = \cos(2\omega t)L_0 + \omega\sin(2\omega t)X_0.$$

This relation allows one to find the commutator $[L, X]$, which turns out to be time-independent and equal to

$$[L, X] = V^{-1}[L_0, X_0]V = gV^{-1}(I - E)V = g(I - E),$$

as before (the matrix V is normalized in the same way as before). It then follows that the matrix L has the same form as before.

Note also that if $x_i(t)$ is a solution of the equations of motion of the CM system with $\omega = 0$, the above formulas imply that

$$\tilde{x}_i(t) = x_i(\omega^{-1}\tan(2\omega t))\cos(2\omega t)$$

is a solution of the equations of motion of the CM system in the external field with $\omega \neq 0$.

The CM system with trigonometric potential

The interaction potential of the CM system admits a deformation such that it becomes periodic in the complex plane with one real or purely imaginary period. We call such systems trigonometric (or hyperbolic) CM systems (sometimes they are called the Calogero-Sutherland systems). This deformation preserves integrability. All statements of the previous section related to the rational CM system (except the self-duality) have direct analogs for the trigonometric systems; however, their formulations and proofs may be of a more complicated form.

The Hamiltonian and equations of motion. The Hamiltonian has the form

$$H = \sum_i p_i^2 - g^2 \sum_{i \neq j} \frac{\gamma^2}{\sinh^2(\gamma(x_i - x_j))},$$

where g^2 is the coupling constant and γ is a parameter characterizing the period of the potential, which is equal to $\pi i/\gamma$. At real γ we have a hyperbolic system, at purely imaginary γ the system is trigonometric. In what follows we will not pay attention to this difference, and will call the system trigonometric in the both cases. In the limit as $\gamma \to 0$ we come to the rational CM system.

The integrals of motion are:

$$\dot{x}_i = 2p_i,$$

$$\dot{p}_i = -4g^2\gamma^3 \sum_{j \neq i} \frac{\cosh(\gamma(x_i - x_j))}{\sinh^3(\gamma(x_i - x_j))},$$

or, in the Newtonian form,

$$\ddot{x}_i = -8g^2\gamma^3 \sum_{j \neq i} \frac{\cosh(\gamma(x_i - x_j))}{\sinh^3(\gamma(x_i - x_j))}.$$

Lax representation and integrals of motion. The Lax matrix of the trigonometric CM system has the form

$$L_{ij} = -p_i \delta_{ij} - \frac{g\gamma(1-\delta_{ij})}{\sinh(\gamma(x_i - x_j))}.$$

It is convenient to pass to the new variables

$$w_i = e^{2\gamma x_i}$$

and together with the diagonal matrix $X = \mathrm{diag}(x_1, \ldots, x_N)$ consider the diagonal matrix $W = \mathrm{diag}(w_1, \ldots, w_N)$. We also introduce the matrices A, B with matrix elements

$$A_{ik} = 2\gamma(1-\delta_{ik}) \frac{w_i^{1/2} w_k^{1/2}}{w_i - w_k},$$

$$B_{ik} = 4\gamma^2(1-\delta_{ik}) \frac{w_i^{3/2} w_k^{1/2}}{(w_i - w_k)^2},$$

and the diagonal matrix

$$D_{ik} = 4\gamma^2 \delta_{ik} \sum_{l \neq i} \frac{w_i w_l}{(w_i - w_l)^2}.$$

We keep the same notation for them as for the corresponding matrices of the previous section, since the former ones are direct analogs of the latter, and reduce to them as $\gamma \to 0$. Here is the Lax pair:

$$L = -\tfrac{1}{2}\dot{X} - gA,$$

$$M = \gamma \dot{X} + 2gB - 2gD.$$

Problem. Prove that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion.

The equation that characterizes the Lax matrix is

$$[W, L] = 2g\gamma\Big(W - W^{1/2} E W^{1/2}\Big)$$

or

$$W^{1/2} L W^{-1/2} - W^{-1/2} L W^{1/2} = 2g\gamma(I - E).$$

In the limit $\gamma \to 0$ it reduces to the equation $[X, L] = g(I - E)$, which was discussed in the previous section.

As before, the Lax equation implies that the evolution of the Lax matrix in time is an isospectral transformation, and the quantities $H_k = \mathrm{tr} L^k$ are integrals of motion.

Problem. Check that $H_2 = H$ and find H_3 in the explicit form.

Problem. Using the method of the previous section (suggested by A.Perelomov), prove that the integrals H_k are in involution.

Linearization in the space of matrices. The linearization in the space of matrices has an analog for the trigonomatric system: the quantities $w_i(t) = e^{2\gamma x_i(t)}$, where $x_i(t)$ are coordinates of the particles in the trigonometric CM system, are eigenvalues of the matrix

$$e^{-4\gamma t L_0} e^{2\gamma X_0}.$$

Problem. Prove this statement.

Problem. Try to find the corresponding matrix for the higher flows.

The trigonometric CM system versus the rational one in an external field. In 1997, Nekrasov obtained a remarkable result which establishes a correspondence between the trigonometric CM dynamics and the dynamics of the rational CM system in an external quadratic potential.

Let us consider the systems with repulsive potential, i.e., change $g \to ig$ and $\gamma \to i\gamma$. To avoid mixing of the notation, we will denote the coordinates and momenta of the trigonometric system by θ_i, $-\xi_i$, with the canonical Poisson brackets. The Hamiltonian and the Lax matrix of the trigonometric system have the form

$$H = \sum_i \xi_i^2 + g^2 \sum_{i \neq j} \frac{\gamma^2}{\sin^2(\gamma(\theta_i - \theta_j))},$$

$$L_{jk} = \xi_j \delta_{jk} - \frac{ig\gamma(1-\delta_{jk})}{\sin(\gamma(\theta_j - \theta_k))}.$$

Also, we change the notation for the Lax matrix of the rational system and introduce the matrix

$$P_{jk} = -p_j \delta_{jk} - \frac{ig(1-\delta_{jk})}{x_j - x_k}$$

(it is just the former matrix$L$); in the case of repulsion it is Hermitian. Instead of the matrices $L^{\pm}$ we consider the matrices

$$Z = P + i\omega X, \quad Z^{\dagger} = P - i\omega X.$$

As it follows from the previous section, the matrix Z satisfies the equation of the Lax type:

$$\dot{Z} + [Z, M] + 2i\omega Z = 0,$$

which implies that

$$Z(t) = e^{2i\omega t} V Z(0) V^{-1}$$

with some unitary matrix V.

Let us consider decomposition of the matrix Z into a product of a unitary and a Hermitian matrix, which is analogous to representation of a complex number z in the form $z = re^{i\varphi}$. We will write such a decompositon in the symmetrized form:

$$Z = U^{1/2} R^{1/2} U^{1/2}, \quad R^{\dagger} = R, \quad U^{\dagger} = U^{-1},$$

then

$$Z^{\dagger} Z = U^{-1/2} R U^{1/2}.$$

Let V be the unitary matrix diagonalizing the matrix U, i.e.,

$$U = VWV^{-1}, \quad W = \mathrm{diag}(e^{2i\omega\theta_1}, \dots, e^{2i\omega\theta_N}).$$

It is defined up to multiplication by a diagonal matrix from the right. We fix this freedom by the condition $U\mathbf{e} = \mathbf{e}$.

Set

$$L = V^{-1}RV.$$

The commutation relation $[X, P] = ig(I - E)$ or $[Z, Z^\dagger] = -2\omega g(I - E)$ is then rewritten in the form

$$U^{1/2}RU^{-1/2} - U^{-1/2}RU^{1/2} = -2\omega g(I - E)$$

or

$$W^{1/2}LW^{-1/2} - W^{-1/2}LW^{1/2} = 2\omega g(I - E).$$

Comparing this with the similar relation for the trigonometric CM system, we conclude that there should be $\omega = \gamma$, and then L becomes its Lax matrix with diagonal elements $\xi_i = (V^{-1}RV)_{ii}$. Therefore, the Hamiltonians $\mathrm{tr}(Z^\dagger Z)^k$ of the rational system transform into the Hamiltonians $\mathrm{tr}L^k$ of the trigonometric system. In particular, at $k = 1$ we have that the dynamics of the CM particles in the external field is equivalent to the free dynamics of the trigonometric system with the Hamiltonian $\sum_i \xi_i$, i.e. all the θ_j's move with the same constant velocity. This follows from the relation

$$Z(t) = e^{2i\omega t} V Z(0) V^{-1}$$

established earlier.

It remains to prove that the transformation from (p_i, x_i) to (ξ_i, θ_i) is canonical. We have:

$$\mathrm{tr}(dZ \wedge dZ^\dagger) = 2i\omega \mathrm{tr}(dP \wedge dX).$$

Problem. Using the identity

$$\mathrm{tr}(d\tilde{X} \wedge d\tilde{Y}) = \mathrm{tr}(dX \wedge dY) - d\mathrm{tr}([X, Y]U^{-1}dU)$$

from the problem of the previous section, show that

$$\sum_i d\xi_i \wedge d\theta_i = -\mathrm{tr}(dP \wedge dX) = \sum_i dp_i \wedge dx_i,$$

which just means that the transformation is canonical.

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