# Integrable systems of particles and nonlinear equations. Lecture 15

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## Field generalization of the CM and RS systems

The CM and RS system admit a so-called field generalization, when the system of $N$ particles with coordinates $x_i$ depending on time $t$ turns into a 2D field theory with $N$ “fields” $x_i$, which depend not only on time $t$ but also on an extra space variable $x$: $x_i = x_i(x, t)$. Such field systems can be obtained from analysis of more general elliptic solutions to nonlinear equations, when we consider solutions that are elliptic functions of a variable which is not necessarily the space variable.

### Elliptic families of solutions to the KP hierarchy and field generalization of the CM system

**Derivation of the equations of motion.** In previous sections we studied solutions of the KP hierarchy that are elliptic functions of the space variable $x = t_1$. Here, following the work [36], we are going to consider more general elliptic solutions that are elliptic functions of some linear combination of higher times: $u = \sum\limits_k \beta_k t_k$. In [36] such solutions were called elliptic families. They form a special subclass of general algebraic-geometrical solutions constructed from a complex algebraic curve and some additional data on it. It can be shown that tau-function for such solutions, as a function of $u$, has the form

$$\tau(\mathbf{t}) = f(\mathbf{t}) e^{\gamma_1 u + \gamma_2 u^2} \prod_{i=1}^{N} \sigma(u - u_i(\mathbf{t})),$$

where $\gamma_1, \gamma_2$ are constants, and $f$ is some function of the times $\mathbf{t}$. In what follows we restrict ourselves by considering the dependence on $x = t_1$ and $t = t_2$ and will write $u_i = u_i(x, t)$. (These $u_i$ will be the fields of the field CM system depending on $x, t$.) Accordingly, for the KP solution we have:

$$U = 2\partial_x^2 \log \tau = 2\partial_x^2 \sum_i \log \sigma(u - u_i)$$

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$$= -2\sum_i \Big(u_{i,x}^2 \wp(u-u_i) + u_{i,xx}\zeta(u-u_i)\Big) + c(x,t),$$

where $c(x,t)$ is some so far unknown function of $x,t$. Since $U$ should be an elliptic function of $u$, there is the relation

$$\sum_i u_{i,xx} = 0.$$

As before, our main technical tool is the first linear problem for the KP hierarchy:

$$\partial_t \psi - \partial_x^2 \psi - U\psi = 0.$$

We will find solutions among double-Bloch functions of $u$ with $N$ poles at $u = u_i$:

$$\psi = \sum_{i=1}^{N} c_i \Phi(u-u_i, \lambda).$$

Finding the derivatives, we have:

$$\begin{aligned}
\partial_t \psi &= \sum_i c_{i,t}\Phi(u-u_i,\lambda) - \sum_i c_i u_{i,t}\Phi'(u-u_i,\lambda), \\
\partial_x^2 \psi &= \sum_i c_{i,xx}\Phi(u-u_i,\lambda) - 2\sum_i c_{i,x}u_{i,x}\Phi'(u-u_i,\lambda) \\
&\qquad + \sum_i c_i u_{i,x}^2 \Phi''(u-u_i,\lambda) - \sum_i c_i u_{i,xx}\Phi'(u-u_i,\lambda).
\end{aligned}$$

Substituting then into the linear equation for $\psi$ together with the explicit form of $U$, we get in the left-hand side an expression that has poles at $u = u_i$ up to the third order. The highest poles cancel identically. Cancellation of the second and first order poles gives an overdetermined system for the coefficients $c_i$. Representing them as a column vector $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$, we can write this system as

$$\left\{\begin{array}{l}
\mathbf{c}_x = L\mathbf{c}, \\[2ex]
\mathbf{c}_t = M\mathbf{c}, \quad M = L^2 + L_x + A,
\end{array}\right.$$

where matrices $L$, $A$ have the following matrix elements:

$$L_{ij} = \xi_i \delta_{ij} + (1-\delta_{ij})u_{i,x}\Phi(u_i-u_j,\lambda), \qquad \xi_i = \frac{u_{i,t}+u_{i,xx}}{2u_{i,x}},$$

$$\begin{aligned}
A_{ij} = \delta_{ij}&\left(u_{i,x}^2\wp(\lambda) - 2\sum_{k\neq i}\Big(u_{k,x}^2\wp(u_i-u_k) + u_{k,xx}\zeta(u_i-u_k)\Big) + c(x,t)\right) \\
&-2(1-\delta_{ij})\Big(u_{i,x}^2\Phi'(u_i-u_j,\lambda) + u_{i,xx}\Phi(u_i-u_j,\lambda)\Big).
\end{aligned}$$

It is easy to see that the compatibility condition for this system has the form of the Zakharov-Shabat equation:

$$L_t - M_x + [L, M] = 0.$$

We need the following identities for the function $\Phi(u, \lambda)$:

$$\Phi(x, \lambda)\Phi(y, \lambda) = \Phi(x + y, \lambda)\Big(\zeta(x) + \zeta(y) - \zeta(x + y + \lambda) + \zeta(\lambda)\Big),$$

$$\Phi(x - y, \lambda)\Phi(y - z, \lambda) = -\Phi'(x - z, \lambda) + \Phi(x - z, \lambda)\eta(x, y, z),$$

where

$$\eta(x, y, z) = \zeta(x - y) + \zeta(y - z) + \zeta(z - x),$$

and

$$\Phi''(x, \lambda) = \Phi(x, \lambda)(2\wp(x) + \wp(\lambda)).$$

Note that $\eta(x, y, z)$ is an anti-symmetric elliptic function in all the three arguments.

**Problem.** Prove these identities.

**Problem.** Using these identities, prove that matrix elements of the matrix $M$ have the following form:

$$M_{ii} = u_{i,x}\Big(\sum_k u_{k,x}\Big)\wp(\lambda) + m_i,$$

$$M_{ij} = -u_{i,x}\Big(\sum_k u_{k,x}\Big)\Phi'(u_i - u_j, \lambda) + m_{ij}\Phi(u_i - u_j, \lambda), \quad i \neq j,$$

where

$$m_i = \xi_i^2 + \xi_{i,x} + c(x, y) - \sum_{k\neq i}\Big((2u_{k,x}^2 + u_{i,x}u_{k,x})\wp(u_i - u_k) + 2u_{k,xx}\zeta(u_i - u_k)\Big),$$

$$m_{ij} = (\xi_i + \xi_j)u_{i,x} - u_{i,xx} + u_{i,x}\sum_{k\neq i,j} u_{k,x}\eta(u_i, u_k, u_j).$$

Consider first off-diagonal matrix elements in the left-hand side of the Zakharov-Shabat equation. A direct calculation, using the above identities, leads to the following expressions for them as functions of $\lambda$:

$$\Big(L_t - M_x + [L, M]\Big)_{ij} = \Big[r_2\wp(\lambda) + r_1\Big(\zeta(u_i - u_j + \lambda) - \zeta(u_i - u_j) - \zeta(\lambda)\Big) + r_0\Big]\Phi(u_i - u_j, \lambda),$$

where $i \neq j$, и $r_0, r_1, r_2$ are some coefficients which depend only on $u_l$, $l = 1, \ldots, N$ and their derivatives with respect to $x, t$. It is easy to see that $r_2 = 0$.

**Problem.** Prove that $r_1 = r_0 = 0$.

Let us make some remarks concerning this problem. The proof that $r_1 = 0$ is a relatively direct but heavy calculation. The proof that $r_0 = 0$ is more difficult. It requires a rather complicated identity for elliptic functions of several variables which can be proved by the standard method: one should consider $r_0$ as elliptic function of $u_i$ and check that it does not have poles at $u_i = u_l$, $l \neq i$, and then find it at $u_i = u_j$.

We have shown that $\Big(L_t - M_x + [L, M]\Big)_{ij} = 0$ at $i \neq j$. Vanishing of diagonal elements is equivalent to equations of motion for $u_i(x, t)$. The diagonal elements are given by

$$\Big(L_t - M_x + [L, M]\Big)_{ii} = \xi_{i,t} - m_{i,x} - u_{i,xx}\Big(\sum_l u_{l,x}\Big)\wp(\lambda)$$

$$+\sum_{k\neq i}\sum_{l} u_{i,x}u_{k,x}u_{l,x}\Big(\Phi'(u_i-u_k,\lambda)\Phi(u_k-u_i,\lambda)-\Phi'(u_k-u_i,\lambda)\Phi(u_i-u_k,\lambda)\Big)$$

$$+\sum_{k\neq i}(u_{i,x}m_{ki}-u_{k,x}m_{ik})\Phi(u_i-u_k,\lambda)\Phi(u_k-u_i,\lambda).$$

One should substitute here the expressions for $\xi_i$, $m_i$, $m_{ik}$ and use the identities

$$\Phi'(x,\lambda)\Phi(-x,\lambda)-\Phi'(-x,\lambda)\Phi(x,\lambda)=-\wp'(x),$$

$$\Phi(x,\lambda)\Phi(-x,\lambda)=\wp(\lambda)-\wp(x).$$

After some algebra we obtain:

$$u_{i,tt}=2u_{i,x}c_x(x,t)+\Big(\frac{u_{i,t}^2-u_{i,xx}^2}{u_{i,x}}+u_{i,xxx}\Big)_x$$

$$+4u_{i,x}\sum_{k\neq i}\Big[u_{k,x}^3\wp'(u_i-u_k)-3u_{k,x}u_{k,xx}\wp(u_i-u_k)-u_{k,xxx}\zeta(u_i-u_k)\Big].$$

To find the function $c(x,t)$, we use the following arguments. As it follows from the geberal theory of the KP hierarchy, the wave function is proportional to the ratio $\dfrac{\tau(\mathbf{t}-[z^{-1}])}{\tau(\mathbf{t})}$. Our tau-function is proportional to the product $\prod\limits_i \sigma(u-u_i(\mathbf{t}))$. Therefore, the condition that the wave function is a double-Bloch one with time independent Bloch multipliers implies that the sum $\sum\limits_i u_i(\mathbf{t})$ has to have a linear dependence on time. In particular, we have:

$$\sum_i u_{i,tt}=0, \quad \sum_i u_{i,x}=h=\text{const},$$

and the function $c(x,t)$ can be fixed by summation of the equations over all $i$ from 1 to $N$. Then the left-hand side vanishes and we can express $c_x(x,t)$ through $u_l$'s and their $x$-derivatives. It turns out that it is possible to integrate over $x$ once, so the result has the form

$$c(x,t)=\frac{1}{h}V(x,t)+\sum_i\frac{u_{i,xx}^2-u_{i,t}^2}{2hu_{i,x}},$$

where

$$V(x,t)=\sum_{i\neq k}\Big[u_{i,x}u_{k,x}(u_{i,x}+u_{k,x})\wp(u_i-u_k)+(u_{i,x}u_{k,xx}-u_{k,x}u_{i,xx})\zeta(u_i-u_k)\Big].$$

The function $c(x,t)$ is defined up to adding an arbitrary function of $t$, but this does not influence the equations of motion. So, we have obtained the equations of motion of the field elliptic CM system. They are reduced to the standard CM system after the substitution $u_i(x,t)=x+x_i(t)$.

**The Hamiltonian formulation.** Let us introduce canonical field variables $p_i(x)$, $u_i(x)$ with the Poisson brackets

$$\{p_i(x),p_j(x')\}=\{u_i(x),u_j(x')\}=0, \quad \{u_i(x),p_j(x')\}=\delta_{ij}\delta(x-x').$$

The field CM system is Hamiltonian with the Hamiltonian

$$\mathcal{H} = \int H dx,$$

where the Hamiltonian density has the form

$$H = -\sum_i p_i^2 u_{i,x} + \sum_i \frac{u_{i,xx}^2}{4u_{i,x}} + \frac{1}{2} V + \frac{1}{h}\Big(\sum_i p_i u_{i,x}\Big)^2 .$$

Here $V$ is the same as above:

$$V = \sum_{i \neq k} \Big[ u_{i,x} u_{k,x} (u_{i,x} + u_{k,x}) \wp(u_i - u_k) + (u_{i,x} u_{k,xx} - u_{k,x} u_{i,xx}) \zeta(u_i - u_k) \Big].$$

Let us show that the Hamiltonian equations

$$u_{i,t} = \frac{\delta \mathcal{H}}{\delta p_i}, \quad p_{i,t} = -\frac{\delta \mathcal{H}}{\delta u_i}$$

are equivalent to the equations of motion given above. Here $\delta\mathcal{H}/\delta p_i$, $\delta\mathcal{H}/\delta u_i$ are variational derivatives of the functional $\mathcal{H}$ with respect to the fields $p_i(x)$, $u_i(x)$. Since the Hamiltonian density depends on $p_i$, $u_i$, $u_{i,x}$ and $u_{i,xx}$, we have, according to the rules of variational calculus:

$$\frac{\delta \mathcal{H}}{\delta p_i} = \frac{\partial H}{\partial p_i}, \qquad \frac{\delta \mathcal{H}}{\delta u_i} = \frac{\partial H}{\partial u_i} - \frac{d}{dx} \frac{\partial H}{\partial u_{i,x}} + \frac{d^2}{dx^2} \frac{\partial H}{\partial u_{i,xx}} .$$

After the calculations, we obtain the Hamiltonian equations in the explicit form:

$$\begin{aligned} u_{i,t} &= -2p_i u_{i,x} + \frac{2}{h}\Big(\sum_k p_k u_{k,x}\Big) u_{i,x}, \\ p_{i,t} &= -2\sum_{k\neq i}\Big(u_{k,x}^3 \wp'(u_i - u_k) - 3u_{k,x} u_{k,xx} \wp(u_i - u_k) - u_{k,xxx} \zeta(u_i - u_k)\Big) \\ &\quad -2p_i p_{i,x} + \frac{1}{2u_{i,x}}\Big(\frac{u_{i,xx}^2}{u_{i,x}}\Big)_x - \frac{u_{i,xxxx}}{2u_{i,x}} + \frac{2}{h}\Big(p_i \sum_k p_k u_{k,x}\Big)_x . \end{aligned}$$

**Problem.** Find $u_{i,tt}$ and show that these equations are equivalent to the equations of motion given above.

This is a direct but rather long calculation.

## Elliptic families of solutions to the 2D Toda chain and field generalization of the RS system

The field generalization of the RS system was sugegsted in the work [37]. It is based on the same idea to consider more general elliptic solutions, now not of the KP hierarchy but the 2D Toda chain hierarchy. For simplicity, we fix all negative times and consider only dependence on the positive times. We assume that solutions are elliptic functions of some linear combination of the higher times which we denote by $u$.

**Derivation of equations of motion.** The starting point is again the linear problem

$$\partial_t \psi(x) = \psi(x+\eta) + b(x)\psi(x), \qquad b(x) = \partial_t \log \frac{\tau(x+\eta)}{\tau(x)}.$$

Like in the KP case, one can show that the tau-function for elliptic families has the general form

$$\tau = f(x,t) e^{\gamma_1 u + \gamma_2 u^2} \prod_{i=1}^{N} \sigma(u - u_i(x,t)),$$

where $\gamma_1, \gamma_2$ are constants and $f$ is some function of $x, t$. Then

$$b(x) = b(u,x,t) = \sum_i \Big( \dot{u}_i(x)\zeta(u - u_i(x)) - \dot{u}_i(x+\eta)\zeta(u - u_i(x+\eta)) \Big) + c(x,t)$$

with some function $c(x,t)$. As in the KP case, the sum $\sum_i u_i$ has a linear dependence on $x, t$. Denote

$$\sum_i \dot{u}_i = \beta = \text{const}.$$

Since $b(u,x,t)$ is an elliptic function of $u$, so we should search solutions among double-Bloch functions

$$\psi(x) = \sum_{i=1}^{N} c_i(x) \Phi(u - u_i(x), \lambda),$$

where we skip the dependence on $t$ for brevity. Below we will also often skip the second argument of $\Phi$.

Substitution to the linear problem gives:

$$\sum_{i=1}^{N} \dot{c}_i(x)\Phi(u - u_i(x)) - \sum_{i=1}^{N} c_i(x)\dot{u}_i(x)\Phi'(u - u_i(x)) - \sum_{i=1}^{N} c_i(x+\eta)\Phi(u - u_i(x+\eta))$$

$$- \sum_{i=1}^{N} \Big( \dot{u}_i(x)\zeta(u - u_i(x)) - \dot{u}_i(x+\eta)\zeta(u - u_i(x+\eta)) \Big) \sum_{j=1}^{N} c_j(x)\Phi(u - u_j(x))$$

$$-c(x,t) \sum_{i=1}^{N} c_i(x)\Phi(u - u_i(x)) = 0.$$

Cancellation of poles leads to the following system:

$$c_i(x+\eta) = \dot{u}_i(x+\eta) \sum_{j=1}^{N} c_j(x) \Phi(u_i(x+\eta) - u_j(x)),$$

$$\dot{c}_i(x) = \dot{u}_i(x) \sum_{j \neq i}^{N} c_j(x)\Phi(u_i(x) - u_j(x)) + c_i(x) \sum_{j \neq i}^{N} \dot{u}_j(x)\zeta(u_i(x) - u_j(x))$$

$$-c_i(x) \sum_{j=1}^{N} \dot{u}_j(x+\eta)\zeta(u_i(x) - u_j(x+\eta)) + c_i(x)c(x,t).$$

They can be represented in matrix form:

$$c_i(x+\eta) = \sum_{j=1}^{N} L_{ij}(x)c_j(x),$$

$$\dot{c}_i(x) = \sum_{j=1}^{N} M_{ij}(x)c_j(x),$$

where the mattices $L$ and $M$ are

$$L_{ij}(x,\lambda) = \dot{u}_i(x+\eta)\Phi(u_i(x+\eta)-u_j(x),\lambda),$$

$$M_{ij}(x,\lambda) = (1-\delta_{ij})\dot{u}_i(x)\Phi(u_i(x)-u_j(x),\lambda)$$
$$+\delta_{ij}\Big(\sum_{k\neq i}\dot{u}_k(x)\zeta(u_i(x)-u_k(x)) - \sum_k \dot{u}_k(x+\eta)\zeta(u_i(x)-u_k(x+\eta)) + c(x,t)\Big).$$

The compatibility condition is the semi-discrete Zakharov-Shabat equation

$$R(x) =: \dot{L}(x) + L(x)M(x) - M(x+\eta)L(x) = 0.$$

The matrices $L$, $M$ depend on the spectral parameter $\lambda$, and this equation should hold for all $\lambda$.

It is useful to introduce the matrices

$$A^+_{ij}(x) = \Phi(u_i(x+\eta)-u_j(x)),$$

$$A^0_{ij}(x) = (1-\delta_{ij})\Phi(u_i(x)-u_j(x))$$

and diagonal matrices

$$\Lambda_{ij}(x) = \delta_{ij}u_i(x),$$

$$D^0_{ij}(x) = \delta_{ij}\sum_{k\neq i}\dot{u}_k(x)\zeta(u_i(x)-u_k(x)),$$

$$D^\pm_{ij}(x) = \delta_{ij}\sum_k \dot{u}_k(x\pm\eta)\zeta(u_i(x)-u_k(x\pm\eta)).$$

In this notation

$$L(x) = \dot{\Lambda}(x+\eta)A^+(x), \qquad M(x) = \dot{\Lambda}(x)A^0(x) + D^0(x) - D^+(x) + c(x,t)I.$$

We have:

$$R(x) = \ddot{\Lambda}(x+\eta)A^+(x) + \dot{\Lambda}(x+\eta)\Big(S(x) + A^+(x)(D^0(x)-D^+(x))$$
$$-(D^0(x+\eta)-D^+(x+\eta))A^+(x) + (c(x,t)-c(x+\eta,t))A^+(x)\Big),$$

where

$$S(x) = \dot{A}^+(x) + A^+(x)\dot{\Lambda}(x)A^0(x) - A^0(x+\eta)\dot{\Lambda}(x+\eta)A^+(x).$$

The expression for $S(x)$ can be transformed with the help of already familiar identities for the function $\Phi$.

**Problem.** Show that

$$S_{ij}(x) = \Phi(u_i(x+\eta) - u_j(x))\Big(D^-_{ii}(x+\eta) + D^+_{jj}(x) - D^0_{jj}(x) - D^0_{ii}(x+\eta)\Big).$$

Collecting everything together, we obtain the matrix identity

$$R(x) = \Big(\ddot{\Lambda}(x+\eta)\dot{\Lambda}^{-1}(x+\eta) + D^-(x+\eta) + D^+(x+\eta) - 2D^0(x+\eta)$$

$$+(c(x,t) - c(x+\eta,t))I\Big)L(x),$$

which implies that the compatibility condition is equivalent to vanishing of all elements of the diagonal matrix that stands in front of $L(x)$:

$$\ddot{\Lambda}(x+\eta)\dot{\Lambda}^{-1}(x+\eta) + D^-(x+\eta) + D^+(x+\eta) - 2D^0(x+\eta) + (c(x,t) - c(x+\eta,t))I = 0.$$

Hence we obtain the equations of motion:

$$\ddot{u}_i(x) + \sum_k \Big(\dot{u}_i(x)\dot{u}_k(x-\eta)\zeta(u_i(x) - u_k(x-\eta)) + \dot{u}_i(x)\dot{u}_k(x+\eta)\zeta(u_i(x) - u_k(x+\eta))\Big)$$

$$-2\sum_{k\neq i} \dot{u}_i(x)\dot{u}_k(x)\zeta(u_i(x) - u_k(x)) + (c(x-\eta,t) - c(x,t))\dot{u}_i(x) = 0.$$

They resemble the equqations of motion for the RS system. The latter can be reproduced by the substitution $u_i = x_i + x$ (with $c(x,t) = c(x+\eta,t)$). Summing over $i$ from 1 to $N$, we find the function $c(x,t)$:

$$c(x,t) = \frac{1}{\beta}\sum_{i,k=1}^{N} \dot{u}_i(x)\dot{u}_k(x+\eta)\zeta(u_i(x) - u_k(x+\eta))$$

(up to adding a $t$ dependent and $\eta$-periodic function of $x$, which does not influence the equations of motion).

The lattice version of the equations of motion is obtained after the substitution $u_i^k = u_i(k\eta + x_0)$:

$$\ddot{u}_i^k + \sum_j \Big(\dot{u}_i^k\dot{u}_j^{k-1}\zeta(u_i^k - u_j^{k-1}) + \dot{u}_i^k\dot{u}_j^{k+1}\zeta(u_i^k - u_j^{k+1})\Big)$$

$$-2\sum_{j\neq i} \dot{u}_i^k\dot{u}_j^k\zeta(u_i^k - u_j^k) + (c^{k-1}(t) - c^k(t))\dot{u}_i^k = 0$$

with the function $c^k(t)$ given by

$$c^k(t) = \frac{1}{\beta}\sum_{i,j=1} \dot{u}_i^k\dot{u}_j^{k+1}\zeta(u_i^k - u_j^{k+1}).$$

**The Hamiltonian formulation.** The field extension of the RS system is an infinite dimensional Hamiltonian system. It is more convenient to consider its lattice version.We introduce canonical variables $u_i^k$, $p_i^k$ with the Poisson brackets

$$\{p_i^k, p_j^l\} = \{u_i^k, u_j^l\} = 0, \quad \{u_i^k, p_j^l\} = \delta_{ij}\delta_{kl}.$$

On the lattice with $n$ sites and periodic boundary conditions, the Hamiltonian has the form

$$\mathcal{H} = \frac{\beta}{\eta} \sum_{k=1}^{n} \log H_k,$$

where

$$H_k = \sum_i e^{\eta p_i^k} \frac{\prod\limits_{j=1} \sigma(u_i^k - u_j^{k-1})}{\prod\limits_{j\neq i} \sigma(u_i^k - u_j^k)} .$$

The first group of Hamiltonian equations is

$$\dot{u}_i^k = \frac{\partial \mathcal{H}}{\partial p_i^k} = \frac{\beta}{H_k} \eta e^{\eta p_i^k} \frac{\prod\limits_{j=1} \sigma(u_i^k - u_j^{k-1})}{\prod\limits_{j\neq i} \sigma(u_i^k - u_j^k)} .$$

Taking time derivative of the both sides (after taking logarithms), we obtain:

$$\eta \dot{p}_i^k = \frac{\ddot{u}_i^k}{\dot{u}_i^k} - \sum_{j=1} (\dot{u}_i^k - \dot{u}_j^{k-1}) \zeta(u_i^k - u_j^{k-1}) + \sum_{j\neq i} (\dot{u}_i^k - \dot{u}_j^k) \zeta(u_i^k - u_j^k) + \partial_t \log H_k .$$

To write down the second group of Hamiltonian equations,

$$\dot{p}_i^k(x) = -\frac{\partial \mathcal{H}}{\partial \lambda_i^k},$$

we consider the variation

$$\eta \delta \mathcal{H} = \beta \sum_{k=1}^{n} \frac{\delta H_k}{H_k}$$

$$= \sum_{k=1}^{n} \sum_{i,l=1}^{N} \dot{u}_i^k \zeta(u_i^k - u_l^{k-1})(\delta u_i^k - \delta u_l^{k-1}) - \sum_{k=1}^{n} \sum_{i\neq l}^{N} \dot{u}_i^k \zeta(u_i^k - u_l^k)(\delta u_i^k - \delta u_l^k).$$

Changing the summation indices as $k \to k+1$ and $i \leftrightarrow l$ where necessary, we have:

$$\eta \delta \mathcal{H} = \sum_{k=1}^{n} \sum_{i,l=1}^{N} \dot{u}_i^k \zeta(u_i^k - u_l^{k-1}) \delta u_i^k + \sum_{k=1}^{n} \sum_{i,l=1}^{N} \dot{u}_l^{k+1} \zeta(u_i^k - u_l^{k+1}) \delta u_i^k$$

$$- \sum_{k=1}^{n} \sum_{i\neq l}^{N} (\dot{u}_i^k + \dot{u}_l^k) \zeta(u_i^k - u_l^k) \delta u_i^k.$$

Hence we see that

$$\eta \dot{p}_i^k = -\sum_{l=1}^{N} \dot{u}_i^k \zeta(u_i^k - u_l^{k-1}) - \sum_{l=1}^{N} \dot{u}_l^{k+1} \zeta(u_i^k - u_l^{k+1}) + \sum_{l\neq i}^{N} (\dot{u}_i^k + \dot{u}_l^k) \zeta(u_i^k - u_l^k).$$

Comparing this with the expression for $\dot p_i^k$ obtained by differentiating of the equations from the first group, we find:

$$\frac{\ddot u_i^k}{\dot u_i^k}+\sum_{l=1}^{N}\dot u_l^{k+1}\zeta(u_i^k-u_l^{k+1})+\sum_{l=1}^{N}\dot u_l^{k-1}\zeta(u_i^k-u_l^{k-1})-2\sum_{l\neq i}^{N}\dot u_l^k\zeta(u_i^k-u_l^k)+\partial_t\log H_k=0.$$

**Problem.** Show that

$$\partial_t\log H_k=c^{k-1}(t)-c^k(t),$$

where

$$c^k(t)=\frac{1}{\beta}\sum_{i,j=1}\dot u_i^k\dot u_j^{k+1}\zeta(u_i^k-u_j^{k+1}).$$

So, we have reproduced the equations of motion for the field RS system.

**The limit $\eta\to 0$.** Let us show that the limit $\eta\to 0$ gives the field CM system. The limit is not simple. The easiest way is to find it for the Hamiltonian density. It is convenient to come back to the difference version of the field RS system with the continuous space variable $x$ and the Hamiltonian density

$$H(x)=\frac{\beta}{\eta}\log\left(\sum_{i=1}^{N}e^{\eta p_i(x)}\sigma(u_i(x)-u_i(x-\eta))\prod_{l\neq i}^{N}\frac{\sigma(u_i(x)-u_l(x-\eta))}{\sigma(u_i(x)-u_l(x))}\right).$$

The canonical Poisson brackets are

$$\{p_i(x),p_j(y)\}=\{u_i(x),u_j(y)\}=0,\quad \{u_i(x),p_j(y)\}=\delta_{ij}\delta(x-y).$$

The Hamiltonian is the integral $\int H(x)dx$. All calculations are similar to the ones done for the lattice version; although the derivatives with respect to the canonical variables $u_i^k$, $p_i^k$ become variations derivatives with respect to $u_i(x)$, $p_i(x)$. We are interested in expansion of the Hamiltonian density in $\eta$ as $\eta\to 0$. We have:

$$H(x)=\frac{\beta}{\eta}\log\Bigg[\eta\sum_{i=1}^{N}\Big(1+\eta p_i+\frac12\eta^2p_i^2+O(\eta^3)\Big)\big(u_i'-\frac12\eta u_i''+\frac16\eta^2u_i'''+O(\eta^3)\big)$$

$$\times\exp\Big(\sum_{j\neq i}^{N}\Big(\eta u_j'\zeta(u_i-u_j)-\frac12\eta^2u_j''\zeta(u_i-u_j)-\frac12\eta^2u_j'^2\wp(u_i-u_j)+O(\eta^3)\Big)\Big)\Bigg],$$

where prime means $x$-derivative.

Let us make the canonical transformation

$$p_i\longrightarrow\tilde p_i=p_i-\frac{u_i''}{2u_i'}+\sum_{j\neq i}^{N}u_j'\zeta(u_i-u_j),\quad u_i\longrightarrow u_i.$$

**Problem.** Check that this transformation is indeed canonical.

For the proof that $\{\tilde p_i(x),\tilde p_j(y)\}=0$, one should use of the identities

$$f(x)\delta''(x-y)-f(y)\delta''(y-x)=-\Big(f'(x)\delta'(x-y)-f'(y)\delta'(y-x)\Big),$$

$$f(x)\delta'(x-y)+f(y)\delta'(y-x)=-f'(x)\delta(x-y).$$

Note that in the limit $\eta\to 0$ the evolution in $x$ in the 2D Toda chain becomes evolution in $t$, so

$$\lim_{\eta\to 0}\sum_{i=1}^{N}u_i'(x)=\beta,$$

and we get the expansion in the form

$$H(x)=\text{const}\,-(1+O(\eta))H_1(x)-\frac{\eta}{2}\,H_2(x)+O(\eta^2),$$

where

$$H_1(x)=\sum_{i=1}^{N}\tilde p_i u_i'$$

is the density of the first Hamiltonian (an analog of the total momentum), while

$$H_2(x)=-\sum_{i=1}^{N}\tilde p_i^2 u_i'+\frac{1}{4}\sum_{i=1}^{N}\frac{u_i''^2}{u_i'}-\frac{1}{3}\sum_{i=1}^{N}u_i'''+\frac{1}{\beta}\Big(\sum_i \tilde p_i u_i'\Big)^2$$

$$-\frac{1}{2}\sum_{i\neq j}^{N}(u_i''u_j'-u_j''u_i')\zeta(u_i-u_j)+\frac{1}{2}\sum_{i\neq j}^{N}(u_i'u_j'^{\,2}+u_j'u_i'^{\,2})\wp(u_i-u_j).$$

This is the Hamiltonian density of the field CM system (up to adding a full derivative term, which does not influence the equations of motion).

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