Integrable systems of particles and nonlinear equations. Lecture 10

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The KP equation of type B and its singular solutions

The KP hierarchy is related to the infinite dimensional algebra of type A. There exist other versions of the hierarchy that are related to Lie algebras of types B,C,D. In some sense they are subhierarhies of the KP hierarchy. Here we will discuss, without going into details, the KP equation of type B (BKP) and its elliptic solutions. Dynamics of their poles is given by a new integrable system which, comparing with the CM system, looks very differently. This system first appeared in the paper [19].

The BKP equation

The BKP equation is the first member of an infinite hierarchy (the BKP hierarchy) with the independent variables $t_1, t_3, t_5, t_7, \ldots$ which are numbered by odd natural numbers. Set $t_1 = x$. What is called the BKP equation is in fact a system of two equations for two unknown functions u, w:

$$\left\{ \begin{array}{l} 3w' = 10u_{t_3} + 20u''' + 120uu' \\ \\ w_{t_3} - 6u_{t_5} = w''' - 6u''''' - 60uu''' - 60u'u'' + 6uw' - 6wu', \end{array} \right.$$

where prime means differentiating with respect to x. The dependent variable w can be excluded, so one can represent this system as a single equation for $U = \int^x u dx$.

The BKP equation has a commutation representation in the form of the Zakharov-Shabat equation

$$\partial_{t_5} B_3 - \partial_{t_3} B_5 + [B_3, B_5] = 0$$

for the differential operators

$$B_3 = \partial_x^3 + 6u\partial_x, \qquad B_5 = \partial_x^5 + 10u\partial_x^3 + 10u'\partial_x^2 + w\partial_x.$$

In its turn, the Zakharov-Shabat equation is the compatibility condition for the linear problems

$$\partial_{t_3}\psi = B_3\psi, \qquad \partial_{t_5}\psi = B_5\psi$$

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for the wave function ψ depending on a spectral parameter z.

As in the case of the KP hierarchy, one can introduce the tau-function $\tau = \tau(x, t_3, t_5, \ldots)$ for the BKP hierarchy, too. The substitutions

$$u = \partial_x^2 \log \tau, \qquad w = \frac{10}{3}\,\partial_{t_3}\partial_x \log \tau + \frac{20}{3}\,\partial_x^4 \log \tau + 20(\partial_x^2 \log \tau)^2$$

convert the first equation into identity while the second one becomes bilinear.

Problem. Show that the second equation from the BKP system in terms of the tau-function becomes bilinear and acquires the form

$$\Big(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5\Big)\tau \cdot \tau = 0,$$

where D_i are Hirota operators that act according to the rule

$$P(D_1, D_3, D_5, \ldots)\tau \cdot \tau = P(\partial_{y_1}, \partial_{y_3}, \partial_{y_5}, \ldots)\tau(x + y_1, t_3 + y_3, \ldots)\tau(x - y_1, t_3 - y_3, \ldots)\Big|_{y_i = 0}$$

for any polynomial $P(D_1, D_3, D_5, \ldots)$.

The wave function is expressed through the tau-function as follows:

$$\psi = \exp\Big(\sum_{k=1,3,5,\ldots} t_k z^k\Big)\,\frac{\tau\Big(t_1 - 2z^{-1}, t_3 - \frac{2}{3}\,z^{-3}, t_5 - \frac{2}{5}\,z^{-5}, \ldots\Big)}{\tau(t_1, t_3, t_5, \ldots)}.$$

This formula is similar to the corresponding formula in the theory of the KP hierarchy.

Equations of motion for poles of elliptic solutions to the BKP equation

Equations of motion for poles of elliptic solutions. Our goal is to study double-periodic (= elliptic) solutions regarded as functions of the variable x. For such solutions, the tau-function is an "elliptic polynomial" of x, i.e., a function of the form

$$\tau = Ae^{cx^2/2}\prod_{i=1}^{N}\sigma(x - x_i)$$

with some constants A, c. Accordingly, $u = \partial_x^2 \log \tau$ is an elliptic function with double poles at the points x_i:

$$u = c - \sum_{i=1}^{N}\wp(x - x_i).$$

The poles depend on the times t_3, t_5. We will show that the dependence on $t_3 = t$ is described by the following Newtonian equations of motion:

$$\ddot{x}_i + 6\sum_{j\neq i}(\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{[ijk]}\wp(x_i - x_j)\wp'(x_i - x_k) = 0,$$

where the last sum goes over all distinct indices i, j, k. The characteristic property of these equations is that there is not only 2-particle interaction but also the 3-particle one.

Also, there is dependence not only on coordinates but on velocities, too. A Hamiltonian structure of this dynamical system, if any, is not known.

According to Krichever's method, we should address the linear problem $\partial_t \psi = B_3 \psi$ for the function ψ:

$$\partial_t \psi = \partial_x^3 \psi + 6u\partial_x \psi.$$

Since the function u is double-periodic, it is natural to seek for solutions among double-Bloch functions. The pole ansatz for the wave function is similar to the one used in the KP case:

$$\psi = e^{xz+tz^3} \sum_{i=1}^{N} c_i \Phi(x - x_i, \lambda).$$

As usual, we will often omit the second argument of the function Φ: $\Phi(x) = \Phi(x, \lambda)$. We also need derivatives of this function with respect to x: $\Phi'(x) = \partial_x \Phi(x, \lambda)$, $\Phi''(x) = \partial_x^2 \Phi(x, \lambda)$, etc.

Note that the constant c in the formula for u can be eliminated by a simple transformation $x \to x - 6ct$, $t \to t$ (or $\partial_x \to \partial_x$, $\partial_t \to \partial_t + 6c\partial_x$ for the vector fields). So, we put $c = 0$ from the very beginning.

Plugging the pole ansatz into the linear problem with $u = -\sum\limits_i \wp(x - x_i)$, we come to the relation

$$\sum_i \dot{c}_i \Phi(x-x_i) - \sum_i c_i \dot{x}_i \Phi'(x-x_i) = 3z^2 \sum_i c_i \Phi'(x-x_i) + 3z \sum_i c_i \Phi''(x-x_i) + \sum_i c_i \Phi'''(x-x_i)$$

$$-6z\Big(\sum_k \wp(x - x_k)\Big)\Big(\sum_i c_i \Phi(x - x_i)\Big) - 6\Big(\sum_k \wp(x - x_k)\Big)\Big(\sum_i c_i \Phi'(x - x_i)\Big).$$

It is enough to cancel all the poles at x_i's (up to the fourth order). It is easy to see that the poles of the fourth and third order cancel identically. A direct calculation gives the following conditions for cancellation of the second and first order poles:

$$c_i \dot{x}_i = -(3z^2 - 3\wp(\lambda))c_i - 6z \sum_{k \neq i} c_k \Phi(x_i - x_k) - 6 \sum_{k \neq i} c_k \Phi'(x_i - x_k) + 6c_i \sum_{k \neq i} \wp(x_i - x_k),$$

$$\dot{c}_i = 3z\wp(\lambda)c_i + 2\wp'(\lambda)c_i - 6z \sum_{k \neq i} c_k \Phi'(x_i - x_k) - 6zc_i \sum_{k \neq i} \wp(x_i - x_k)$$

$$- 6 \sum_{k \neq i} c_k \Phi''(x_i - x_k) + 6c_i \sum_{k \neq i} \wp'(x_i - x_k).$$

As before, these conditions can be rewritten in the matrix form:

$$\left\{ \begin{array}{l} L\mathbf{c} = (3z^2 - 3\wp(\lambda))\mathbf{c} \\ \\ \dot{\mathbf{c}} = M\mathbf{c}, \end{array} \right.$$

where

$$L = -\dot{X} - 6zA - 6B + 6D,$$

$$M = (3z\wp(\lambda) + 2\wp'(\lambda))I - 6zB - 6zD - 6C + 6D',$$

and the matrices A, B, C, D, D', I are as follows:

$$A_{ik} = (1-\delta_{ik})\Phi(x_i - x_k),$$

$$B_{ik} = (1-\delta_{ik})\Phi'(x_i - x_k),$$

$$C_{ik} = (1-\delta_{ik})\Phi''(x_i - x_k),$$

$$D_{ik} = \delta_{ik}\sum_{j\neq i}\wp(x_i - x_j),$$

$$D'_{ik} = \delta_{ik}\sum_{j\neq i}\wp'(x_i - x_j).$$

The matrices A, B, C are off-diagonal while the matrices D, D' are diagonal. The equations of the spectral curve has the form

$$\det\Bigl(L(z,\lambda) - (3z^2 - 3\wp(\lambda))I\Bigr) = 0.$$

Note that the matrices L, M depend not only on λ but also on z. The linear system for the vector $\mathbf{c}$ is overdetermined. Taking the t-derivative of the first equation, we find the compatibility condition in the form

$$\Bigl(\dot L + [L, M]\Bigr)\mathbf{c} = 0.$$

One can prove that the following matrix identity holds:

$$\dot L + [L, M] = -12D'\Bigl(L - (3z^2 - 3\wp(\lambda))I\Bigr) - \ddot X + 12D'(6D - \dot X) + 6\dot D - 6D''',$$

where $D'''_{ik} = \delta_{ik}\sum_{j\neq i}\wp'''(x_i - x_j)$. Let us present a short proof. Substituting the explicit form of the matrices L, M, we write:

$$\begin{aligned}
\dot L + [L, M] \;=\;& 36z^2\Bigl([A, B] + [A, D]\Bigr)\\
& - 6z\Bigl(\dot A - [\dot X, B]\Bigr)\\
& + 36z\Bigl([A, C] - [A, D'] + 2[B, D]\Bigr)\\
& - 6\Bigl(\dot B - [\dot X, C]\Bigr) - \ddot X + 6\dot D\\
& + 36\Bigl([B, C] - [B, D'] + [C, D]\Bigr).
\end{aligned}$$

First of all, notice that $\dot A_{ik} = (\dot x_i - \dot x_k)\Phi'(x_i - x_k)$, $\dot B_{ik} = (\dot x_i - \dot x_k)\Phi''(x_i - x_k)$, so $\dot A = [\dot X, B]$, $\dot B = [\dot X, C]$. The expression $[A, B] + [A, D]$ can be transformed using the identity

$$\Phi(x,\lambda)\Phi'(y,\lambda) - \Phi(y,\lambda)\Phi'(x,\lambda) = \Phi(x+y,\lambda)(\wp(x) - \wp(y)).$$

If $i \neq k$ we have:

$$\begin{aligned}
&\Bigl([A, B] + [A, D]\Bigr)_{ik}\\
&= \sum_{j\neq i,k}\Phi(x_i - x_j)\Phi'(x_j - x_k) - \sum_{j\neq i,k}\Phi'(x_i - x_j)\Phi(x_j - x_k)
\end{aligned}$$

$$+\Phi(x_i-x_k)\Big(\sum_{j\neq k}\wp(x_j-x_k)-\sum_{j\neq i}\wp(x_i-x_j)\Big)=0,$$

so $[A,B]+[A,D]$ is a diagonal matrix. In order to find it, take the limit of the identity given above as $y\to -x$:

$$\Phi(x,\lambda)\Phi'(-x,\lambda)-\Phi(-x,\lambda)\Phi'(x,\lambda)=\wp'(x),$$

hence

$$\Big([A,B]+[A,D]\Big)_{ii}$$
$$=\sum_{j\neq i}\Big(\Phi(x_i-x_j)\Phi'(x_j-x_i)-\Phi'(x_i-x_j)\Phi(x_j-x_i)\Big)=\sum_{j\neq i}\wp'(x_i-x_j)=D'_{ii},$$

so we obtain the matrix identity

$$[A,B]+[A,D]=D'.$$

Combining derivatives of the identity given above with respect to x and y, we obtain the identities

$$\Phi(x)\Phi''(y)-\Phi(y)\Phi''(x)=2\Phi'(x+y)(\wp(x)-\wp(y))+\Phi(x+y)(\wp'(x)-\wp'(y)),$$
$$\Phi'(x)\Phi''(y)-\Phi'(y)\Phi''(x)=\Phi''(x+y)(\wp(x)-\wp(y))+\Phi'(x+y)(\wp'(x)-\wp'(y)),$$

where the second argument of the function Φ is omitted. Their limits as $y\to -x$ give the identities

$$\Phi(x)\Phi''(-x)-\Phi(-x)\Phi''(x)=0,$$
$$\Phi'(x)\Phi''(-x)-\Phi'(-x)\Phi''(x)=-\frac{1}{6}\,\wp'''(x)-\wp(\lambda)\wp'(x).$$

Using these formulas, it is not difficult to prove the following matrix identities:

$$[A,C]=2[D,B]+D'A+AD',$$
$$[B,C]=[D,C]+D'B+BD'-\frac{1}{6}\,D'''-\wp(\lambda)D'.$$

Now it is easy to obtain our main identity for $\dot L+[L,M]$.

It follows from this identity that the compatibility condition for the linear system for the vector $\mathbf{c}$ requires vanishing of the diagonal matrix

$$-\ddot X+12D'(6D-\dot X)+6\dot D-6D'''.$$

This gives the equations of motion for the poles x_i. Writing the diagonal elements explicitly, we have:

$$\ddot x_i+6\sum_{j\neq i}(\dot x_i+\dot x_j)\wp'(x_i-x_j)-72\sum_{j\neq i}\sum_{k\neq i}\wp(x_i-x_j)\wp'(x_i-x_k)+6\sum_{j\neq i}\wp'''(x_i-x_j)=0.$$

Taking into account the identity $\wp'''(x)=12\wp(x)\wp'(x)$, we obtain the equations of motion in the form

$$\ddot x_i+6\sum_{j\neq i}(\dot x_i+\dot x_j)\wp'(x_i-x_j)-72\sum_{[ijk]}\wp(x_i-x_j)\wp'(x_i-x_k)=0,$$

where the last sum goes over all distinct indices i,j,k. In the rational limit these equations are

$$\ddot x_i-12\sum_{j\neq i}\frac{\dot x_i+\dot x_j}{(x_i-x_j)^3}+144\sum_{[ijk]}\frac{1}{(x_i-x_j)^2(x_i-x_k)^3}=0.$$

The Manakov representation and integrals of motion. The equations obtained above do not have a Lax representation. However, the following commutation representation hols:

$$\dot{L} + [L, M] = -12D'(L - \Lambda I),$$

where $\Lambda = 3(z^2 - \wp(\lambda))$. A commutation representation of this form is known as the Manakov triple. The time evolution of the Lax matrix $L \to L(t)$ is not isospectral; nevertheless, $\det(L(z, \lambda) - \Lambda I)$ is an integral of motion. Indeed, from the Manakov representation it follow that

$$\frac{d}{dt} \log\det(L - \Lambda I) = \frac{d}{dt} \operatorname{tr}\log(L - \Lambda I)$$
$$= \operatorname{tr}\Big(\dot{L}(L - \Lambda I)^{-1}\Big) = -12 \operatorname{tr} D' = 0$$

since the matrix D' is traceless: the function $\wp'$ is an odd function, and, therefore,

$$\operatorname{tr} D' = \sum_{i \neq j} \wp'(x_i - x_j) = 0.$$

The expression

$$R(z, \lambda) = \det\Big(3(z^2 - \wp(\lambda))I - L(z, \lambda)\Big)$$

is a polynomial in z of degree $2N$. The coefficients are integrals of motion (not all of them are independent and some of them may be trivial). Making the similarity transformation $L \to G^{-1}LG$ with the diagonal matrix $G_{ij} = \delta_{ij} e^{-\zeta(\lambda)x_i}$, we conclude that the coefficients $R_k(\lambda)$ of the polynomial

$$R(z, \lambda) = \sum_{k=0}^{2N} R_k(\lambda) z^k$$

are elliptic functions of λ with a pole (of a high order for large k) at $\lambda = 0$. They obey the property $R_k(-\lambda) = (-1)^k R_k(\lambda)$ and can be represented as linear combinations of the $\wp$-function and its derivatives.

Example ($N = 2$):

$$\det_{2\times 2}\Big(3(z^2 - \wp(\lambda))I - L\Big) = 9z^4 + 3z^2\Big(\dot{x}_1 + \dot{x}_2 - 18\wp(\lambda)\Big) - 36z\wp'(\lambda) - 3\wp(\lambda)(\dot{x}_1 + \dot{x}_2)$$
$$+ \dot{x}_1\dot{x}_2 - 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2) - 27\wp^2(\lambda) + 9g_2,$$

where g_2 is the coefficient in the expansion of the $\wp$-function around $x = 0$: $\wp(x) = x^{-2} + \frac{1}{20} g_2 x^2 + \frac{1}{28} g_3 x^4 + O(x^6)$. Therefore, at $N = 2$ there are two equations of motion: $I_1 = \dot{x}_1 + \dot{x}_2$ и $I_2 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2)$.

Problem. Prove that

$$I_1 = \sum_i \dot{x}_i,$$

$$I_2 = \frac{1}{2}\sum_i \dot{x}_i^2 + 6\sum_{i \neq j} \dot{x}_i \wp(x_{ij}) - 18\sum_{[ijk]} \wp(x_{ij})\wp(x_{ik})$$

are integrals of motion. In the formula for I_2, the last sum goes over all distinct indices i, j, k from 1 to N.

The proof is based on the following identities:

$$\sum_{i=1}^{n}\partial_{x_i}\prod_{k=1,\neq i}^{n}\wp(x_i-x_k)=0, \quad n=2,3,\dots$$

(one needs them for $n=3$ and $n=4$). They can be proved in the standard way. The left-hand side is an elliptic function of x_1. Expanding it near possible poles at $x_1=x_k$, $k=2,\dots,n$, one can see that it is a regular function, and, therefore, does not depend on x_1. Since there is a symmetry under permutations of the arguments, this function is equal to a constant that does not depend on all the x_i's. Putting $x_k=kx$, we can see that it is equal to zero.

The spectral curve is defined by the equation

$$R(z,\lambda)=\det\Big(3(z^2-\wp(\lambda))I-L(z,\lambda)\Big)=0.$$

It is easy to see that $L(-z,-\lambda)=L^{\mathrm{T}}(z,\lambda)$, so the spectral curve admits a holomorphic involution $\iota:(z,\lambda)\mapsto(-z,-\lambda)$. If values of the integrals of motion are fixed, the equation $R(z,\lambda)=0$ defines an algebraic curve Γ, which is a covering (with $2N$ sheets) of the elliptic curve $\mathcal{E}$, realized as a factor of complex plane over the lattice with periods $2\omega_1$, $2\omega_2$. It can be proved that genus of the curve Γ is equal to $2N$.

CM particles sticking together

In the work [20] it was shown that the equations of motion for poles of elliptic solutions to the BKP equation derived above coincide with the dynamics of CM particles that stick together in pairs, under the Hamiltonian flow with the Hamiltonian H_3. The standard flow with the Hamiltonian H_2 breaks such configuration immediately; but, as we shall see, it is invariant under the flow with the Hamiltonian

$$H_3=-\sum_i p_i^3+3\sum_{i\neq j}p_i\wp(x_i-x_j).$$

Consider the configuration of particles in the CM system with even number of particles $N=2n$ in which the particles merge in pairs, so that their coordinated satisfy the conditions $x_{2i-1}=x_{2i}$, $i=1,\dots,n$. However, we can not literally put $x_{2i-1}=x_{2i}$ since this leads to a very singular expressions that do not make any sense. To avoid this difficulty, we put $x_{2i}-x_{2i-1}=O(\varepsilon)$, and consider the limit $\varepsilon\to 0$ in the very end. Namely, put

$$x_{2i}-x_{2i-1}=\varepsilon\delta_i, \quad \varepsilon\to 0, \quad i=1,\dots,n,$$

where δ_i in general can depend on time. Define the submanifold $\mathcal{B}_n(\varepsilon)=\mathcal{B}_n(\varepsilon,\{\delta_i\})$ in the $2N=4n$-dimensional phase space of the CM system, imposing, together with $x_{2i}-x_{2i-1}=\varepsilon\delta_i$, the following conditions on the momenta:

$$p_{2i-1}=\frac{1}{\varepsilon\delta_i}+\alpha_i\varepsilon+\beta_i\varepsilon^2,$$

$$p_{2i}=-\frac{1}{\varepsilon\delta_i}-\alpha_i\varepsilon+\beta_i\varepsilon^2,$$

where

$$\alpha_i = \alpha_i(\varepsilon) = \alpha_{i,0} + \alpha_{i,1}\varepsilon + \alpha_{i,2}\varepsilon^2 + \dots, \quad \beta_i = \beta_i(\varepsilon) = \beta_{i,0} + \beta_{i,1}\varepsilon + \beta_{i,2}\varepsilon^2 + \dots$$

(these are some series in ε). Given δ_i, the quantities β_i will be fixed (we will see this later), so the submanifold $\mathcal{B}_n(\varepsilon)$ with coordinates x_{2i-1}, α_i has dimension $2n$.

Let us show that the submanifold $\mathcal{B}_n(\varepsilon)$ is invariant under the flow with the Hamiltonian H_3. The Hamiltonian equations of motion are

$$\begin{cases} \dot{x}_i = -3p_i^2 + 3\sum_{j\neq i} \wp(x_i - x_j), \\ \dot{p}_i = -3\sum_{j\neq i} (p_i + p_j)\wp'(x_i - x_j). \end{cases}$$

We have:

$$\dot{x}_{2i-1} = -6\frac{\alpha_i}{\delta_i} + 6\sum_{j\neq i} \wp(x_{2i-1} - x_{2j-1}) + O(\varepsilon),$$

and for compatibility we should require

$$\dot{x}_{2i} - \dot{x}_{2i-1} = \varepsilon\dot{\delta}_i,$$

meaning that the time evolution does not generate non-vanishing as $\varepsilon \to 0$ terms (i.e., the pairs of stuck together particles are not destroyed). From the equation for $\dot{x}_i$ we have:

$$\begin{aligned} \partial_t(x_{2i} - x_{2i-1}) \;=\; & 3(p_{2i-1}^2 - p_{2i}^2) \\ & +3\sum_{j\neq i}\Big(\wp(x_{2i}-x_{2j-1}) + \wp(x_{2i}-x_{2j})\Big) \\ & -3\sum_{j\neq i}\Big(\wp(x_{2i-1}-x_{2j-1}) + \wp(x_{2i-1}-x_{2j})\Big). \end{aligned}$$

Taking into account that $p_{2i-1}^2 - p_{2i}^2 = 4\beta_i\delta_i^{-1}\varepsilon + 4\alpha_i\beta_i\varepsilon^3$, we obtain, expanding the right-hand side as a series in ε, that $\dot{x}_{2i} - \dot{x}_{2i-1} = \varepsilon\dot{\delta}_i$ is satisfied and

$$\dot{\delta}_i = 12\beta_{i,0}\delta_i^{-1} + 6\delta_i\sum_{j\neq i}\wp'(x_{2i-1} - x_{2j-1}) \quad \text{in order } \varepsilon,$$

$$4\beta_{i,1} = \delta_i^2\sum_{j\neq i}(\delta_j - \delta_i)\wp''(x_{2i-1} - x_{2j-1}) \quad \text{in order } \varepsilon^2,$$

$$24\alpha_i\beta_{i,0}\delta_i + 24\beta_{i,2} + \delta_i^2\sum_{j\neq i}(2\delta_i^2 + 3\delta_j^2 - 3\delta_i\delta_j)\wp'''(x_{2i-1} - x_{2j-1}) = 0 \quad \text{in order } \varepsilon^3.$$

Next, we write the Hamiltonian equations from the second half:

$$\dot{p}_{2i-1} = -3(p_{2i-1} + p_{2i})\wp'(x_{2i-1} - x_{2i})$$

$$-3\sum_{j\neq i}\Big((p_{2i-1} + p_{2j-1})\wp'(x_{2i-1} - x_{2j-1}) + (p_{2i-1} + p_{2j})\wp'(x_{2i-1} - x_{2j})\Big).$$

Expanding the right-hans side in powers of ε in the orders ε^{-1}, ε^0 we get the same equations (this means consistency of the procedure), while in the next order we obtain the equation

$$\dot{\alpha}_i = -6\alpha_i \sum_{j\neq i} \wp'(x_{2i-1}-x_{2j-1}) - \frac{3}{2}\sum_{j\neq i}(\delta_i^{-1}-\delta_j^{-1})\delta_j^2\wp'''(x_{2i-1}-x_{2j-1}) - 12\delta_i^{-3}\beta_{i,2}.$$

Compatibility with the time evolution requires that

$$\dot{p}_{2i-1} + \dot{p}_{2i} = O(\varepsilon^2).$$

From the Hamiltonian equations from the second group we have:

$$\begin{aligned}
-(\dot{p}_{2i-1} + \dot{p}_{2i}) \;\; = \;\; & 3p_{2i-1}\sum_{j\neq i}\Big(\wp'(x_{2i-1}-x_{2j-1}) + \wp'(x_{2i-1}-x_{2j})\Big) \\
& +3p_{2i}\sum_{j\neq i}\Big(\wp'(x_{2i}-x_{2j-1}) + \wp'(x_{2i}-x_{2j})\Big) \\
& +3\sum_{j\neq i} p_{2j-1}\wp'(x_{2i-1}-x_{2j-1}) + 3\sum_{j\neq i} p_{2j}\wp'(x_{2i-1}-x_{2j}) \\
& +3\sum_{j\neq i} p_{2j-1}\wp'(x_{2i}-x_{2j-1}) + 3\sum_{j\neq i} p_{2j}\wp'(x_{2i}-x_{2j}).
\end{aligned}$$

On the first glance, the right-hand side is of order $O(1)$. However, an accurate expansion in powers of ε shows that the terms or orders $O(1)$ and $O(\varepsilon)$ cancel, and the right-hand side has the order $O(\varepsilon^2)$, as required.

Let us restrict the time evolution to the submanifold $\mathcal{B}_n = \lim_{\varepsilon\to 0}\mathcal{B}_n(\varepsilon)$. This will allow us to obtain equations of motion for the pairs with coordinates x_{2i-1}, i.e., the equations that connect $\ddot{x}_{2i-1}$, $\dot{x}_{2i-1}$ and x_{2i-1}. From

$$\dot{x}_{2i-1} = -6\frac{\alpha_i}{\delta_i} + 6\sum_{j\neq i}\wp(x_{2i-1}-x_{2j-1}) + O(\varepsilon)$$

we have:

$$\ddot{x}_{2i-1} = -6\delta_i^{-1}\dot{\alpha}_i + 6\alpha_i\delta_i^{-2}\dot{\delta}_i + 6\sum_{j\neq i}(\dot{x}_{2i-1}-\dot{x}_{2j-1})\wp'(x_{2i-1}-x_{2j-1}).$$

Substituting $\dot{\alpha}_i$, we see that all δ_i's cancel, and in the limit $\varepsilon \to 0$ we obtain the following equations of motion:

$$\ddot{x}_i + 6\sum_{j\neq i}(\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{j\neq i}\sum_{k\neq i}\wp(x_i-x_j)\wp'(x_i-x_k) + 6\sum_{j\neq i}\wp'''(x_i-x_j) = 0$$

(here i, j, k are odd numbers from 1 to $2n-1$). The identity $\wp'''(x) = 12\wp(x)\wp'(x)$ allows us to represent them in the form

$$\ddot{x}_i + 6\sum_{j\neq i}(\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{[ijk]}\wp(x_i-x_j)\wp'(x_i-x_k) = 0.$$

These are exactly the equations of motion for poles of elliptic solutions to the BKP equation.

The calculations can be considerably simplified if one puts $\delta_i = 1$ from the very beginning. This fixes the "gauge freedom" in the definition of $\mathcal{B}_n(\varepsilon)$. The coefficients $\beta_{i,a}$ become functions of x_{2j-1}, α_j. The equations given above imply that the restriction of the CM dynamics with the Hamiltonian H_3 to $\mathcal{B}_n$ can be represented as a system of first order equations:

$$\left\{\begin{array}{l} \dot{x}_{2i-1} = -6\alpha_i + 6\sum\limits_{j\neq i} \wp(x_{2i-1} - x_{2j-1}), \\ \\ \dot{\alpha}_i = -12\alpha_i \sum\limits_{j\neq i} \wp'(x_{2i-1} - x_{2j-1}) + \sum\limits_{j\neq i} \wp'''(x_{2i-1} - x_{2j-1}). \end{array}\right.$$

Excluding α_i, one arrives at the equations of motion given above.

Of course the reader can notice the analogy with the procedure of obtaining the equations of motion of the deformed RS system, when the particles merged in pairs keeping the fixed distance η between the particles in each pair. This analogy is not accidental. In fact, the equations of motion for poles of elliptic solutions to the BKP equation can be obtained from the ones for the deformed RS system in the limit $\eta \to 0$.

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