Integrable systems of particles and nonlinear equations. Lecture 1

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The CM system with rational potential

In 1971, F.Calogero introduced a system of interacting quantum particles on the line with pairwise interaction potential that is proportional to the inverse square of distance between the particles. Later the classical version of this system was considered by Moser who revealed a remarkable algebraic structure of the system. At present, this system is usually called the Calogero-Moser (CM) system of particles.

The Hamiltonian and equations of motion. The state of the system is characterized by coordinates and momenta of the particles x_i, p_i, $i = 1, \ldots, N$ with the canonical Poisson brackets $\{x_i, x_j\} = \{p_i, p_j\} = 0$, $\{x_i, p_j\} = \delta_{ij}$. All particles are regarded as identical. Let us set their mass to be equal to $\frac{1}{2}$. The Hamiltonian has the form

$$H = \sum_{i=1}^{N} p_i^2 - g^2 \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}.$$

The parameter g^2 is a coupling constant. If g is real, than the particles attract each other, if it is purely imaginary, the interaction is repulsive. Without loss of generality one can put $g = 1$ that can be achieved by the transformation $x_i \to g x_i$. The Hamiltonian equations of motion are obtained by the standard rule:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = 2p_i,$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -4g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^3},$$

where dot means the time derivative. From this we can find the Newtonian equations of motion:

$$\ddot{x}_i = -8g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^3}.$$

It immediately follows that the center of masses $\bar{x} = \dfrac{1}{N} \sum_i x_i$ moves with a constant velocity.

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The Lax representation. The key to the proof of integrability of the CM system is representation of the equations of motion as a condition of commutativity of some matrices (the Lax representation). It was found by Moser in 1975. Consider two matrices L, M of size $N \times N$ with matrix elements

$$L_{ij} = -\delta_{ij} p_i - \frac{g(1-\delta_{ij})}{x_i - x_j},$$

$$M_{ij} = -2g\delta_{ij} \sum_{k \neq i} \frac{1}{(x_i - x_k)^2} + \frac{2g(1-\delta_{ij})}{(x_i - x_j)^2}.$$

The matrix L is called the Lax matrix, while the pair of matrices L, M is called the Lax pair. Let us show that the equations of motionof the CM system are equivalent to the matrix equation

$$\dot{L} + [L, M] = 0,$$

which is called the Lax equation. Note that the matrix Lax equation is an overdetermined system of equations as soon as it contains N^2 relations (according to the number of matrix elements), while there are only N equations of motion. As we shall see below, our matrices L, M are such that the non-diagonal elements of the left-hand side of the Lax equation vanish identically, and one is left with just N relations which imply vanishing the diagonal elements. These relations are just the equations of motion.

Let us note here that the Lax representation is not unique. As we will see later, there exist an one-parametric family of Lax pairs leading to the same equations of motion.

For convenience of the calculations, we introduce matrices A, B with zeros on the main diagonal and matrix elements

$$A_{ij} = \frac{1-\delta_{ij}}{x_i - x_j}, \quad B_{ij} = \frac{1-\delta_{ij}}{(x_i - x_j)^2},$$

and also diagonal matrices X, D with diagonal elements

$$X_{ii} = x_i, \qquad D_{ii} = \sum_{k \neq i} \frac{1}{(x_i - x_k)^2},$$

then

$$L = -\frac{1}{2}\dot{X} - gA, \quad M = 2gB - 2gD.$$

Clearly, $\dot{A} = [B, \dot{X}]$. The calculation of the left-hand side of the Lax equation yields:

$$\dot{L} + [L, M] = -\frac{1}{2}\ddot{X} + 4g^2([A, D] - [A, B]).$$

Problem. Prove the identity

$$[A, D] - [A, B] = D',$$

where D' is the diagonal matrix with the elements

$$D'_{ij} = -2\delta_{ij} \sum_{k \neq i} \frac{1}{(x_i - x_k)^3}.$$

Therefore, the Lax equation is reduced to equality of two diagonal matrices:

$$\ddot{X} = 4D',$$

which contains all the N Newtonian equations of motion.

Integrals of motion. The Lax representation implies existence of an infinite set of integrals of motion (N of which are independent). Consider, for example, the quantities

$$H_k = \mathrm{tr}\, L^k, \qquad k = 1, 2, 3, \ldots.$$

By virtue of the Lax equation and the cyclic property of the trace, we have:

$$\dot{H}_k = k\mathrm{tr}(\dot{L}L^{k-1}) = k\mathrm{tr}([M, L]L^{k-1}) = k\mathrm{tr}(MLL^{k-1} - LML^{k-1}) = 0.$$

It is easy to see that the Lax representation means that in the process of the time evolution the Lax matrix undergoes an isospectral transformation: $L(t) = UL(0)U^{-1}$, where the matrix U is such that $M = \dot{U}U^{-1}$. Hence all eigenvalues of the Lax matrix are conserved quantities. The first N integrals of motion $H_k = \mathrm{tr}\, L^k$, $k = 1, \ldots, N$ are functionally independent. Here are the first few of them:

$$H_1 = -\sum_i p_i,$$

$$H_2 = H = \sum_i p_i^2 - g^2 \sum_{i \neq j} \frac{1}{(x_i - x_j)^2},$$

$$H_3 = -\sum_i p_i^3 + 3g^2 \sum_{i \neq j} \frac{p_i}{(x_i - x_j)^2},$$

$$H_4 = \sum_i p_i^4 - 2g^2 \sum_{i \neq j} \frac{p_i p_j + 2p_i^2}{(x_i - x_j)^2} - +g^4 \sum_{i \neq j} \frac{1}{(x_i - x_j)^4} + 2g^4 \sum_{[ijk]} \frac{1}{(x_i - x_j)^2 (x_j - x_k)^2},$$

where the last sum goes over all distinct indices i, j, k. One sees that $-H_1 = P$ is the total momentum of the system and $H_2 = H$ is the Hamiltonian. The rest integrals of motion are sometimes called the higher Hamiltonians. Like H_2, they define certain Hamiltonian flows in the phase space. By t_k we will denote the time variable corresponding to the flow with the Hamiltonian H_k. For example, the flow with the Hamiltonian H_3 is defined by the equations

$$\partial_{t_3} x_i = \frac{\partial H_3}{\partial p_i} = -3p_i^2 + 3g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2},$$

$$\partial_{t_3} p_i = -\frac{\partial H_3}{\partial x_i} = 6g^2 \sum_{j \neq i} \frac{p_i + p_j}{(x_i - x_j)^3}.$$

Below we will prove that the H_k's are in involution, so all these flows are compatible, the higher Hamiltonians are integrals of motion for each other and the CM system is a completely integrable Hamiltonian system.

Involutivity of the integrals of motion. Following the method suggested in [?], we will prove that all eigenvalues of the Lax matrix (and thus all the higher Hamiltonians H_k) are in involution.

Suppose that the Lax matrix is diagonalizable, which is the case of general position. Let λ, μ be two its distinct eigenvalues, $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$, $\mathbf{b} = (b_1, \ldots, b_N)^{\mathrm{T}}$ be the corresponding right eigenvectors (column-vectors), and $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_N)$, $\tilde{\mathbf{b}} = (\tilde{b}_1, \ldots, \tilde{b}_N)$ be the left eigenvectors (row-vectors):

$$L\mathbf{c} = \lambda\mathbf{c}, \quad L\mathbf{b} = \mu\mathbf{b},$$

$$\tilde{\mathbf{c}}L = \lambda\tilde{\mathbf{c}}, \quad \tilde{\mathbf{b}}L = \mu\tilde{\mathbf{b}}.$$

Let us normalize them by the conditions

$$(\tilde{\mathbf{c}}, \mathbf{c}) = \sum_i \tilde{c}_i c_i = 1, \quad (\tilde{\mathbf{b}}, \mathbf{b}) = \sum_i \tilde{b}_i b_i = 1.$$

By differentiation of the eigenvalue equations with respect to p_i, x_i, we obtain the relations

$$\frac{\partial\lambda}{\partial p_i} = \Big(\tilde{\mathbf{c}}, \frac{\partial L}{\partial p_i}\,\mathbf{c}\Big), \qquad \frac{\partial\lambda}{\partial x_i} = \Big(\tilde{\mathbf{c}}, \frac{\partial L}{\partial x_i}\,\mathbf{c}\Big)$$

and similar relations for the derivatives of μ. Plugging here our matrix L expressed through the coordinates and momenta, we find:

$$\frac{\partial\lambda}{\partial p_i} = -\tilde{c}_i c_i,$$

$$\frac{\partial\lambda}{\partial x_i} = g\sum_{k\neq i}\frac{\tilde{c}_i c_k - \tilde{c}_k c_i}{(x_i - x_k)^2}.$$

Now we can find the Poisson bracket $\{\lambda, \mu\}$:

$$\{\lambda, \mu\} = \sum_i \Big(\frac{\partial\lambda}{\partial p_i}\frac{\partial\mu}{\partial x_i} - \frac{\partial\lambda}{\partial x_i}\frac{\partial\mu}{\partial p_i}\Big)$$

$$= g\sum_i \left[-\tilde{c}_i c_i \sum_{k\neq i}\frac{\tilde{b}_i b_k - \tilde{b}_k b_i}{(x_i - x_k)^2} + \tilde{b}_i b_i \sum_{k\neq i}\frac{\tilde{c}_i c_k - \tilde{c}_k c_i}{(x_i - x_k)^2}\right].$$

Multiplying the equation

$$\sum_k L_{ik} c_k = \lambda c_i$$

by b_i, and the equation

$$\sum_k L_{ik} b_k = \mu b_i$$

by c_i, and subtracting them from each other after that, we obtain the relation

$$c_i b_i = \frac{g}{\lambda - \mu}\sum_{k\neq i}\frac{b_k c_i - c_k b_i}{x_i - x_k},$$

and, similarly,

$$\tilde{c}_i\tilde{b}_i = -\frac{g}{\lambda-\mu}\sum_{k\neq i}\frac{\tilde{b}_k\tilde{c}_i - \tilde{c}_k\tilde{b}_i}{x_i - x_k}.$$

Substituting this into the above formula for $\{\lambda, \mu\}$, we get:

$$\{\lambda,\mu\} = \frac{g^2}{\lambda-\mu}\sum_i\left[\sum_{j\neq i}\frac{b_jc_i - c_jb_i}{x_i-x_j}\sum_{l\neq i}\frac{\tilde{c}_i\tilde{b}_l - \tilde{b}_i\tilde{c}_l}{(x_i-x_l)^2} + \sum_{j\neq i}\frac{\tilde{b}_j\tilde{c}_i - \tilde{c}_j\tilde{b}_i}{x_i-x_j}\sum_{l\neq i}\frac{c_ib_l - b_ic_l}{(x_i-x_l)^2}\right]$$

$$= \frac{g^2}{\lambda-\mu}\sum_i\sum_{j,l\neq i}\left[\frac{1}{(x_i-x_j)(x_i-x_l)^2} + \frac{1}{(x_i-x_l)(x_i-x_j)^2}\right](b_jc_i - c_jb_i)(\tilde{c}_i\tilde{b}_l - \tilde{b}_i\tilde{c}_l)$$

$$= \frac{g^2}{\lambda-\mu}\sum_i\sum_{j,l\neq i}\left[\frac{1}{(x_i-x_l)^2} - \frac{1}{(x_i-x_j)^2}\right]\frac{1}{x_l-x_j}(b_jc_i - c_jb_i)(\tilde{c}_i\tilde{b}_l - \tilde{b}_i\tilde{c}_l).$$

Rearranging the terms and re-denoting the summation indices, we arrive at the following expression:

$$\{\lambda,\mu\} = \frac{g^2}{\lambda-\mu}\sum_i\sum_{l\neq i}\frac{\tilde{c}_i\tilde{b}_l - \tilde{b}_i\tilde{c}_l}{(x_i-x_l)^2}\sum_{j\neq i}\frac{b_jc_i - c_jb_i}{x_l-x_j} - \frac{g^2}{\lambda-\mu}\sum_{j\neq i}\frac{b_jc_i - c_jb_i}{(x_i-x_j)^2}\sum_{l\neq i}\frac{\tilde{c}_i\tilde{b}_l - \tilde{b}_i\tilde{c}_l}{x_l-x_j}.$$

Multiply the equation

$$-p_ic_i - g\sum_{l\neq i}\frac{1}{x_i-x_l}\,c_l = \lambda c_i$$

by b_j the equation

$$-p_ib_i - g\sum_{l\neq i}\frac{1}{x_i-x_l}\,b_l = \mu b_i$$

by c_j, and subtract them from each other. As a result, we get:

$$g\sum_{l\neq i}\frac{c_lb_j - b_lc_j}{x_i-x_l} = \mu b_ic_j - \lambda c_ib_j + p_i(b_ic_j - c_ib_j).$$

In a similar way, we obtain the relation

$$g\sum_{l\neq i}\frac{\tilde{c}_l\tilde{b}_j - \tilde{b}_l\tilde{c}_j}{x_l-x_i} = \mu\tilde{b}_i\tilde{c}_j - \lambda\tilde{c}_i\tilde{b}_j + p_i(\tilde{b}_i\tilde{c}_j - \tilde{c}_i\tilde{b}_j).$$

Plugging this into the formula for $\{\lambda, \mu\}$, we find:

$$\{\lambda,\mu\} = \frac{g\lambda}{\lambda-\mu}\sum_{j\neq i}\frac{\tilde{c}_ic_jb_i\tilde{b}_j - \tilde{c}_jc_ib_j\tilde{b}_i}{(x_i-x_j)^2} - \frac{g\mu}{\lambda-\mu}\sum_{j\neq i}\frac{c_i\tilde{c}_j\tilde{b}_ib_j - c_j\tilde{c}_ib_i\tilde{b}_j}{(x_i-x_j)^2}.$$

It is obvious that each of the two sums here vanishes. Indeed, the expression under the sum is antisymmetric with respect to permutations of the indices i, j. Therefore, we have proved that the Poisson brackets between the eigenvalues of the Lax matrix are zero, and so the integrals of motion H_k are in involution.

Linearization of the CM dynamics in the space of matrices (the projection method). The CM dynamics can be linearized in a space that is larger than the N-dimensional configuration space of the system. This larger space is the space of matrices of size N на N. It turns out that if a matrix from this space linearly depends on time, then the dynamics of its eigenvalues coincides with the dynamics of CM particles. This method of an implicit solving of the equations of motion is also called the projection method because the free dynamics in the space of matrices, being projected to a smaller space gives the dynamics of our system with a non-trivial interaction. The simplest example is the projection of a linear motion along a line in the plane on the radial direction (the distance between a point on the line and the origin). If the line does not contain the origin, the dependence of the radial distance on time is non-trivial.

More precisely, we will show that eigenvalues of the matrix $X_0 - 2tL_0$ (where $X_0 = \mathrm{diag}(x_1(0), \dots, x_N(0))$, $L_0 = L(0)$ is the Lax matrix at $t = 0$), which we denote as $x_i = x_i(t)$, move as particles in the CM system. At $t = 0$ the eigenvalues coincide with the initial values of the coordinates $x_i(0)$.

Let $V = V(t)$ be a matrix that diagonalizes the matrix $X_0 - 2tL_0$:

$$X_0 - 2tL_0 = VXV^{-1},$$

where $X = X(t) = \mathrm{diag}(x_1(t), \dots, x_N(t))$. Obviously, $V(0) = I$, where I is the unity matrix. The matrix V is defined up to multiplication from the right by a diagonal matrix. We fix this freedom by imposing the condition

$$V\mathbf{e} = \mathbf{e},$$

where $\mathbf{e}$ is the column vector such that all its components are equal to 1: $\mathbf{e} = (1, \dots, 1)^{\mathrm{T}}$.

Let us show that $L = V^{-1}L_0V$ is the Lax matrix at the time t. To this end, we note that the matrix L_0 id characterized by the commutation relation

$$[X_0, L_0] = g(I - E),$$

where $E = \mathbf{e}\,\mathbf{e}^{\mathrm{T}}$ is the matrix of rank 1 with all the matrix elements being equal to 1, and . $X_0 = X(0)$. Let us check that the matrices $L = V^{-1}L_0V$ and X at the time t satisfy the same commutation relation. Indeed,

$$[X, L] = [V^{-1}X_0V - 2tV^{-1}L_0V,\, V^{-1}L_0V] = V^{-1}[X_0, L_0]V = g(I - V^{-1}\mathbf{e}\,\mathbf{e}^{\mathrm{T}}V).$$

By virtue of the normalization condition we have $V^{-1}\mathbf{e} = \mathbf{e}$. Note that the diagonal elements of the matrix in the left-hand side are equal to 0, hence $\mathbf{e}^{\mathrm{T}}V = \mathbf{e}^{\mathrm{T}}$, and so the right-hand side equals $g(I - E)$, i.e., we have

$$[X, L] = g(I - E).$$

This commutation relation allows one to find the non-diagonal elements of the matrix L:

$$L_{ij} = -\frac{g}{x_i - x_j}, \qquad i \neq j.$$

The next step is to differentiate the equality $X_0 - 2tL_0 = VXV^{-1}$ with respect to time. After a simple calculation this yields:

$$2L = -\dot{X} + [M, X],$$

where $M = -V^{-1}\dot{V}$. This relation allows one to fix the diagonal elements of the matrix L: $L_{ii} = -\frac{1}{2}\dot{x}_i$. Hence L is indeed the Lax matrix of the CM system at the time t. Substituting it into the above equality and written everything in matrix elements, we obtain:

$$2L_{ij} = M_{ij}(x_j - x_i), \qquad i \neq j,$$

hence the non-diagonal elements of the matrix M are:

$$M_{ij} = -\frac{2L_{ij}}{x_i - x_j} = \frac{2g}{(x_i - x_j)^2}, \qquad j \neq j.$$

Its diagonal elements are fixed by the normalization condition $V\mathbf{e} = \mathbf{e}$ which implies that $M\mathbf{e} = 0$, i.e.,

$$M_{ii} = -\sum_{j\neq i} M_{ij} = -2g \sum_{j\neq i} \frac{1}{(x_i - x_j)^2}.$$

So, we have shown that the matrices L and M introduced above form the Lax pair for the CM system. The time derivative of the relation $L = V^{-1}L_0V$ gives the Lax equation $\dot{L} + [L, M] = 0$, and the equations of motion for $x_i(t)$ follow from it.

Our next goal is to extend the above statement about the linearization to the higher Hamiltonian flows. We will show that the eigenvalues of the matrix $X_0 - kt_kL_0^{k-1}$ depending on the time t_k move as the CM particles in the flow with the Hamiltonian H_k.

First of all, we should derive the Hamiltonian equations of motion. We have:

$$\frac{\partial x_i}{\partial t_k} = \frac{\partial}{\partial p_i}\,\mathrm{tr}L^k = k\,\mathrm{tr}\Big(\frac{\partial L}{\partial p_i}L^{k-1}\Big) = -k(L^{k-1})_{ii},$$

$$\frac{\partial p_i}{\partial t_k} = -\frac{\partial}{\partial x_i}\,\mathrm{tr}L^k = -k\,\mathrm{tr}\Big(\frac{\partial L}{\partial x_i}L^{k-1}\Big).$$

Problem. Prove the identity

$$2\frac{\partial L}{\partial x_i} = [E_i, M],$$

where E_i is the matrix whose matrix elements are all equal to 0 except the ii-element on the diagonal which is equal to 1.

This identity allows one to write the equations of motion in the form

$$\frac{\partial x_i}{\partial t_k} = -k(L^{k-1})_{ii},$$

$$\frac{\partial p_i}{\partial t_k} = -\frac{k}{2}[M, L^{k-1}]_{ii}.$$

They imply the relation

$$\partial_{t_k} L_{ab} = \frac{k}{2}[M, L^{k-1}]_{aa}\delta_{ab} - kg(1 - \delta_{ab})\frac{(L^{k-1})_{aa} - (L^{k-1})_{bb}}{(x_a - x_b)^2},$$

which we will need later.

Let V be a matrix that diagonalizes the matrix $X_0 - kt_k L_0^{k-1}$:

$$X_0 - kt_k L_0^{k-1} = VXV^{-1},$$

where $X = \mathrm{diag}(x_1, \ldots, x_N)$. To simplify the notation, we denote it by the letter V, the same as for $k = 2$ (see above), but one should keep in mind that it may depend on k. This matrix is defined up to multiplication from the right by a diagonal matrix. We fix this freedom by imposing the normalization condition $V\mathbf{e} = \mathbf{e}$, as above. Introduce matrices the $L = V^{-1}L_0V$, $M_k = -V^{-1}\partial_{t_k}V$. Note that the normalization condition implies that $M_k\mathbf{e} = 0$. As above, the differentiation with respect to t_k leads to the relations

$$\partial_{t_k} L = [M_k, L],$$

$$\partial_{t_k} X = [M_k, X] - kL^{k-1}.$$

The first one is the Lax equation for the higher (k-th) flow. The second one allows to find the matrix M_k. For its non-diagonal elements we immediately obtain:

$$(M_k)_{ab} = -\frac{k(L^{k-1})_{ab}}{x_a - x_b}, \quad a \neq b.$$

The diagonal elements can be found from the condition $M_k\mathbf{e} = 0$, which states that

$$(M_k)_{aa} = -\sum_{j\neq a}(M_k)_{aj},$$

so

$$(M_k)_{ab} = -k(1-\delta_{ab})\frac{(L^{k-1})_{ab}}{x_a - x_b} + k\delta_{ab}\sum_{j\neq a}\frac{(L^{k-1})_{aj}}{x_a - x_j}.$$

At last, we will show that the Lax equation with this matrix M_k coincides with the equation for $\partial_{t_k} L_{ab}$ obtained above using the Hamiltonian equations of motion for the k-th flow. We write:

$$\partial_{t_k} L_{ab} = \sum_j (M_k)_{aj}L_{jb} - \sum_j L_{aj}(M_k)_{jb}$$

$$= \sum_{j\neq a}(M_k)_{aj}L_{jb} - \sum_{j\neq b} L_{aj}(M_k)_{jb} + L_{ab}((M_k)_{aa} - (M_k)_{bb}).$$

At $b = a$ this gives $\partial_{t_k} L_{aa} = \frac{1}{2}\, k[M, L^{k-1}]_{aa}$, as it must be due to the equations of motion. Consider now the more difficult case $a \neq b$. Substituting the explicit form of the matrices L, M_k, we obtain:

$$\partial_{t_k} L_{ab} = kg\sum_{j\neq a,b}\frac{(L^{k-1})_{aj}}{(x_a - x_j)(x_j - x_b)} - kg\sum_{j\neq a,b}\frac{(L^{k-1})_{jb}}{(x_a - x_j)(x_j - x_b)}$$

$$-kg\sum_{j\neq a}\frac{(L^{k-1})_{aj}}{(x_a - x_b)(x_a - x_j)} + kg\sum_{j\neq b}\frac{(L^{k-1})_{bj}}{(x_a - x_b)(x_a - x_j)}$$

$$- k(L^{k-1})_{ab}\frac{p_a - p_b}{x_a - x_b}.$$

To transform the first two sums, we use the identity

$$\frac{1}{(x_a - x_j)(x_j - x_b)} = \frac{1}{(x_a - x_b)(x_a - x_j)} + \frac{1}{(x_a - x_b)(x_j - x_b)}.$$

After a rearranging the terms, we come to the following expression:

$$\partial_{t_k} L_{ab} = \frac{kg((L^{k-1})_{bb} - (L^{k-1})_{aa}}{(x_a - x_b)^2} + \frac{k}{x_a - x_b} Y,$$

where

$$Y = g \sum_{j \neq b} \frac{(L^{k-1})_{aj} - (L^{k-1})_{bj}}{x_j - x_b} - g \sum_{j \neq b} \frac{(L^{k-1})_{jb}}{x_j - x_b} + g \sum_{j \neq a} \frac{(L^{k-1})_{jb}}{x_j - x_a} - (p_a - p_b)(L^{k-1})_{ab}.$$

Problem. Prove that $Y = 0$.

Hint: express everything through matrix elements of the Lax matrix.

The rest expression

$$\partial_{t_k} L_{ab} = \frac{kg((L^{k-1})_{bb} - (L^{k-1})_{aa}}{(x_a - x_b)^2}$$

at $a \neq b$ is just the one that was obtained before from the equations of motion. So, we have achieved the linearization of the higher Hamiltonian flows for the CM system and obtained the Lax representation for them.

Self-duality. The rational CM system has an interesting property called self-duality. In general, the duality transformation sends eigenvalues of the Lax matrix (the action-type variables) to coordinates of the dual system, that us why this transformation is also called the action-coordinate duality.

Let U be a matrix that diagonalizes the Lax matrix, i.e.,

$$L = U \tilde{L} U^{-1}, \quad \tilde{L} = \text{diag}(\lambda_1, \ldots, \lambda_N).$$

It is defined up to multiplication from the right by a diagonal matrix. We fix this freedom by imposing the condition

$$U\mathbf{e} = \mathbf{e}.$$

Put

$$\tilde{X} = U^{-1} X U,$$

where $\tilde{X}$ is in general a non-diagonal matrix. It is easy to see that the relation

$$[X, L] = g(I - E)$$

becomes $[\tilde{X}, \tilde{L}] = g(I - E)$, from which it follows that

$$\tilde{X}_{ij} = \xi_i \delta_{ij} + \frac{g(1 - \delta_{ij})}{\lambda_i - \lambda_j}$$

with some ξ_i. The variables (ξ_i, λ_i) are called dual variables, and the transformation

$$(p_i, x_i) \to (\xi_i, \lambda_i)$$

is called the duality transformation. We see that the matrix $\tilde{X}$ in the dual variables has the same form as the Lax matrix in the original variables. The variables λ_i are usually referred to as dual coordinates, and the ξ_i's are referred to as dual momenta. Below we shall see that the duality is a canonical transformation[1], so the Hamiltonians $\tilde{H}_k = \mathrm{tr}\tilde{X}^k$ define dynamics in the dual variables which, because $\tilde{X}$ has the same form as the Lax matrix of the original system, is the same (in the dual variables) as the CM dynamics. This just means the self-duality of the system.

Let us prove that the duality transformation is canonical. The standard symplectic form on the phase space of the CM system is

$$\Omega = \sum_i dp_i \wedge dx_i = \mathrm{tr}(dX \wedge dL).$$

Consider the form

$$\tilde{\Omega} = \sum_i d\xi_i \wedge d\lambda_i = \mathrm{tr}(d\tilde{X} \wedge d\tilde{L}).$$

We will prove that $\tilde{\Omega} = \Omega$, which just means that the transformation is canonical. The proof is based on a simple technical lemma which we formulate as a problem.

Problem. Let X, Y be quadratic matrices of the same size, and put $\tilde{X} = UXU^{-1}$, $\tilde{Y} = UYU^{-1}$. Then

$$\mathrm{tr}(d\tilde{X} \wedge d\tilde{Y}) = \mathrm{tr}(dX \wedge dY) - d\mathrm{tr}([X, Y]U^{-1}dU).$$

The proof consists in a direct calculation which we omit. In our case $\tilde{X} = U^{-1}XU$, $\tilde{L} = U^{-1}LU$ (in the formulation of the problem above one should change $U \to U^{-1}$), and the identity yields:

$$\begin{aligned}\tilde{\Omega} = \mathrm{tr}(d\tilde{X} \wedge d\tilde{L}) &= \mathrm{tr}(dX \wedge dL) - d\mathrm{tr}([X, L]UdU^{-1}) \\ &= \Omega - g\, d\mathrm{tr}((I - E)UdU^{-1}) \\ &= \Omega - g\, d\mathrm{tr}(UdU^{-1}) + \Omega g\, d\mathrm{tr}(\mathbf{e}\mathbf{e}^{\mathrm{T}}UdU^{-1}).\end{aligned}$$

The second and the third terms in the right-hand side are identically zero. Indeed, the second term is equal to $g\mathrm{tr}(dUU^{-1} \wedge dUU^{-1}) = 0$, and the third one vanishes by virtue of the condition $U\mathbf{e} = \mathbf{e}$. Therefore, $\tilde{\Omega} = \Omega$ and so the duality transformation is canonical.

Список литературы

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