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Lectures on Bethe ansatz and quantum integrable systems

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Abstract

This is a course on quantum integrable systems solved by Bethe ansatz based on lectures given for students of Moscow Institute of Physics and Technology, Higher School of Economics and Skolkovo Institute of Science and Technology. We start from the coordinate Bethe ansatz for the spin chains and the one-dimensional Bose gas with point-like interaction, proceed to the algebraic Bethe ansatz for the 6-vertex and 8-vertex models and end with a discussion of such advanced issues as scalar products of Bethe vectors, the master T-operator and quantum-classical duality.

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1 Introductory remarks

The history of quantum integrable systems began in 1931 when Bethe managed to find exact eigenfunctions of the Hamiltonian of the Heisenberg spin chain using a special ansatz which is now famous and is named after him (Bethe ansatz). In one or another form, this method turned out to be applicable to many other integrable models of solid state physics and quantum field theory. From the mathematical point of view, the Bethe method is related to representation theory of quantum algebras (q-deformations of universal envelopping algebras of Lie algebras).

Although many different generalizations and variants of the Bethe method have been proposed over the years, the secret of its amazing effectiveness and universality has not been revealed until now.

These lectures contain a presentation of the following matters:

- Coordinate Bethe ansatz on the examples of the Heisenberg model and one-dimensional Bose-gas with pairwise point-like interaction between particles;
- Bethe ansatz in exactly solvable models of statistical mechanics on the lattice on the example of the 6-vertex model;
- Bethe equations and the Yang function, calculation of norms of Bethe vectors;
- Calculation of physical quantities in integrable models in thermodynamic limit;
- Quantum inverse scattering method and algebraic Bethe ansatz, quantum R-matrices, the Yang-Baxter equation;
- Generalized algebraic Bethe ansatz for the XYZ Heisenberg magnet and the 8-vertex model;
- Calculation of scalar products of Bethe vectors, determinant representation of the scalar products;
- The method of Baxter's Q-operators, functional relations for transfer matrices;
- Connection with integrable hierarchies of classical soliton equations: the master T-operator as tau-function;
- Connection with classical integrable systems of the Calogero-Moser type: quantum-classical duality.

As is seen from this list, the presentation starts from very classical matters which were repeatedly reviewed in the literature (such that coordinate Bethe ansatz) and ends with a discussion of rather recent achievements.

Here is also a list of topics related to the theory quantum integrable systems that are not covered in the lectures:

- Bethe ansatz for the sine-Gordon model and the massive Thirring model;

- Chiral Potts model;
- Hierarchical (nested) Bethe ansatz in its coordinate and algebraic versions;
- Thermodynamic Bethe ansatz;
- Integrable spin chains and vertex models connected with classical Lie algebras and superalgebras;
- Integrable systems with non-periodic boundary conditions, reflection equations;
- Exactly solvable models of the IRF-type (interaction round a face) on the lattice, the correspondence between vertex models and IRF models (the vertex-IRF correspondence);
- Calculation of correlation functions in quantum integrable systems and exactly solvable models of statistical physics on the lattice.

Knowledge of the basics of quantum mechanics and statistical physics is highly desirable but not absolutely necessary for understanding these lectures. Out of the physical context, the Bethe ansatz in its finite-dimensional version is simply a method to diagonalize large matrices of a special form, and in this sense it does not require any prior knowledge except the basics of linear algebra.

These notes are based on lectures given for students of Moscow Institute of Physics and Technology (MIPT), Higher School of Economics (HSE) and Skolkovo Institute of Science and Technology in 2013-2020. The lectures were intended for an audience unfamiliar with the subject and were aimed at an initial introduction to it. There are a number of exercises and problems in the text which are an essential part of the course. Exercises are very simple; problems are somewhat more difficult.

The literature on quantum integrable systems and Bethe ansatz is enormous. Our list of references contains, along with a very limited set of well-known monographs [1]–[4] and reviews [5]–[11], some recent original papers which have had a substantial impact on the content of the present lectures (sections 5, 6.3 and 6.5 are based on the papers [12], [13] and [14] respectively). As a rule, we do not give explicit references to the literature in the main text.

2 Coordinate Bethe ansatz

2.1 Bethe ansatz for the Heisenberg model

2.1.1 The notation and terminology

Pauli matrices. The Pauli matrices are the following 2×2 matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They from a convenient basis in the space of Hermitean 2×2 matrices with zero trace. They are also denoted as $\sigma_1, \sigma_2, \sigma_3$, and the unity matrix is denoted as σ_0. We will sometimes write 1 instead of the unity matrix. The main properties of the Pauli matrices are

1) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$,

2) $\sigma_x\sigma_y = i\sigma_z, \;\; \sigma_y\sigma_z = i\sigma_x, \;\; \sigma_z\sigma_x = i\sigma_y$,

3) $\sigma_j\sigma_k = -\sigma_k\sigma_j$ at $j \neq k$.

It is convenient to represent the Pauli matrices as a 3-vector with matrix components: $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. We will also often use the matrices

$$\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that $\sigma_+^2 = \sigma_-^2 = 0$, $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$ and $[\sigma_+, \sigma_-] = \sigma_z$.

The following equality in $\mathrm{End}\,(\mathbb{C}^2 \otimes \mathbb{C}^2)$ holds:

$$\sigma_0 \otimes \sigma_0 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z = 2\mathsf{P}_{12}, \tag{2.1}$$

where P_{12} is the permutation operator. It acts as follows: if u, v are any two vectors in $\mathbb{C}^2$, then $\mathsf{P}_{12}\,(u \otimes v) = v \otimes u$. Equivalently, if X, Y are any two operators in $\mathrm{End}\,(\mathbb{C}^2)$, then $\mathsf{P}_{12}\, X \otimes Y = Y \otimes X\, \mathsf{P}_{12}$. In the notation $\vec{\sigma}^{(1)} = \vec{\sigma} \otimes 1$, $\vec{\sigma}^{(2)} = 1 \otimes \vec{\sigma}$, the identity (2.1) can be also written in the form

$$\mathsf{P}_{12} = \frac{1}{2}\,(\mathsf{I} + \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}) \tag{2.2}$$

(here $\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$ is undestood as the scalar product of the “vectors” $\vec{\sigma}^{(1)}$ and $\vec{\sigma}^{(2)}$, i.e., sum of products of the components, and I is the unity matrix of size 4×4).

Another convenient basis in the space of 2×2 matrices consists of the “matrix units” e_{ab} $(a, b = 1, 2)$. The ab matrix element of the matrix e_{ab} is equal to 1, and all other matrix elements are 0. The permutation operator can be written as

$$\mathsf{P}_{12} = \sum_{a,b=1}^{2} e_{ab}^{(1)} e_{ba}^{(2)}.$$

In the case when there are more than two tensor factors, it is convenient to introduce the operators $\vec{\sigma}^{(j)} \in \mathrm{End}\,(\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{N})$ according to the rule

$$\vec{\sigma}^{(j)} = \underbrace{1 \otimes \ldots \otimes 1}_{j-1} \otimes \vec{\sigma} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{N-j},$$

and similarly for $e_{ab}^{(j)}$. Obviously, if $j \neq k$, then the components of the operator vector $\vec{\sigma}^{(j)}$ commute with components of $\vec{\sigma}^{(k)}$ because they act nontrivially in different spaces.

The space of states. Consider the linear space

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{N}.$$

We call it the *space of states.* (A state is any vector from $\mathcal{H}$. If two vectors are proportional, they define the same state.)

In quantum mechanics this is the space of states of a system of N fixed atoms having magnetic degrees of freedom which we for brevity call *spins.* Other degrees of freedom of the atom are not taken into account in this simplified picture. The space of states of each atom is two-dimensional (spin $\frac{1}{2}$).

Let us choose the basis in $\mathbb{C}^2$ as follows:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the state $|+\rangle$ the z-projection of spin is $+1$ (up-looking arrow). Similarly, in the state $|-\rangle$ the z-projection of spin is -1 (down-looking arrow). For brevity, we say that in the first (second) case spin looks up (down). All other states of spin are linear combinations of these two with complex coefficients. In such states the z-projection does not have a definite value. The Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ are operators of projections of spin on the axis x, y, z (more precisely, in physics the operator of spin $\frac{1}{2}$ is not $\vec{\sigma}$ but $\frac{1}{2}\vec{\sigma}$, and possible values of spin projections are $\pm\frac{1}{2}$). Since they do not commute, only projection to one axis can have a definite value ($+1$ or -1). Clearly, we have: $\sigma_z |+\rangle = |+\rangle$, $\sigma_z |-\rangle = -|-\rangle$, $\sigma_+ |+\rangle = \sigma_- |-\rangle = 0$, $\sigma_+ |-\rangle = |+\rangle$, $\sigma_- |+\rangle = |-\rangle$.

The basis vectors in $\mathcal{H}$ are tensor products of the basis vectors in each tensor factor. For example,

$$|+\rangle \otimes |+\rangle \otimes |-\rangle \otimes |+\rangle \otimes |-\rangle \otimes |-\rangle \otimes \; \ldots \; \otimes |-\rangle \otimes |+\rangle,$$

which we will also write as

$$|+\rangle_1 |+\rangle_2 |-\rangle_3 |+\rangle_4 |-\rangle_5 |-\rangle_6 \; \ldots \; |-\rangle_{N-1} |+\rangle_N$$

or simply $|++-+-- \; \ldots \; -+\rangle$. This is the state in which the first spin has z-projection $+1$, the second $+1$, the third -1 and so on. There are 2^N such vectors, i.e., $\dim \mathcal{H} = 2^N$.

Let us note that in the basis $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$ the permutation operator P_{12} is represented as

$$\mathsf{P}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, we introduce in $\mathcal{H}$ the scalar product (which allows one to express physical quantities such as correlation functions). In each space $\mathbb{C}^2$ we define the scalar product in the natural way by the formula $\langle \epsilon | \, \epsilon' \rangle = \delta_{\epsilon,\epsilon'}$, where $\epsilon, \epsilon' = \pm$ and extend it to their tensor product by the rule

$$\langle \epsilon_1 \epsilon_2 \ldots \epsilon_n | \, \epsilon'_1 \epsilon'_2 \ldots \epsilon'_n \rangle = \prod_{i=1}^{N} \delta_{\epsilon_i, \epsilon'_i}.$$

Let us discuss this in some more details. The dual vectors (covectors or row-vectors) have the form $\langle +| = (1, 0)$, $\langle -| = (0, 1)$. The scalar product of vectors $|\Phi\rangle$ and $|\Psi\rangle$ is written in the form $(|\Phi\rangle, |\Psi\rangle) = \langle \Phi|\, \Psi\rangle$. For any operator $\mathbf{O}$ we have $\langle \Phi|\, \mathbf{O}\, |\Psi\rangle = \langle \Psi|\, \mathbf{O}^\dagger\, |\Phi\rangle$, where $\mathbf{O}^\dagger$ is the conjugated (transposed) operator. The operator $\mathbf{O}$ acts to dual vectors (to the left) according to the rule $\langle \Phi|\, \mathbf{O} = \left(\mathbf{O}^\dagger\, |\Phi\rangle\right)^\dagger$ for any vector $|\Phi\rangle$ (note that in our notation $(|\Phi\rangle)^\dagger = \langle\Phi|$). For example, $\langle +|\, \sigma_+ = \langle -|$, $\langle -|\, \sigma_- = \langle +|$, $\langle +|\, \sigma_- = \langle -|\, \sigma_+ = 0$.

The operator of total spin and decomposition of the space of states. The operator

$$\vec{S} = \sum_{j=1}^{N} \vec{\sigma}^{(j)}$$

is called, for the clear reason, the operator of total spin of the system of atoms. By analogy, one can also introduce $S_\pm = \frac{1}{2}(S_x \pm i S_y)$. The commutation relations of these operators are obviously the same as for the Pauli matrices, i.e., $[S_z, S_\pm] = \pm 2 S_\pm$, $[S_+, S_-] = S_z$, but S_x^2, S_y^2, S_z^2, are no longer equal to the unity operator and $S_\pm^2 \neq 0$.

It is easy to see that all basis vectors in $\mathcal{H}$ in which m spins look up and the other $N - m$ look down (independently of their order) are eigenvectors for the operator S_z with eigenvalue $N - 2m$. Accordingly, the space $\mathcal{H}$ can be represented as a direct sum of subspaces $\mathcal{H}(m)$ for $m = 0, 1, 2, \ldots, N$, on which the z-projection of the total spin is equal to $N - 2m$:

$$\mathcal{H} = \bigotimes_{j=1}^{N} \mathbb{C}^2 = \bigoplus_{m=0}^{N} \mathcal{H}(m). \tag{2.3}$$

It is easy to see that

$$\dim \mathcal{H}(m) = \binom{N}{m} = \frac{N!}{m!(N-m)!}.$$

In particular, $\mathcal{H}(0)$ and $\mathcal{H}(N)$ are one-dimensional spaces spanned by the state in which all spins look up (respectively, down). The decomposition (2.3) is very important for what follows. Let us note that the operators $\sigma_z^{(j)}$ map each space $\mathcal{H}(m)$ to itself while the operators $\sigma_\pm^{(j)}$ act as follows: $\sigma_\pm^{(j)} : \mathcal{H}(m) \to \mathcal{H}(m \mp 1)$.

2.1.2 Isotropic Heisenberg model (XXX spin chain)

The main problem is diagonalization of the following operator in $\mathrm{End}(\mathcal{H})$:

$$H(J_x, J_y, J_z) = \frac{1}{2} \sum_{k=1}^{N} (J_x \sigma_x^{(k)} \sigma_x^{(k+1)} + J_y \sigma_y^{(k)} \sigma_y^{(k+1)} + J_z \sigma_z^{(k)} \sigma_z^{(k+1)}), \quad \vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}, \tag{2.4}$$

which can be understood as a $2^N \times 2^N$ matrix of special form. Of physical interest are its eigenvalues and eigenvectors in the limit $N \to \infty$ (the thermodynamic limit).

This operator serves as the Hamiltonian of a quantum-mechanical model of identical atoms with magnetic degrees of freedom (spins). Each spin is assigned to its own atom (site of the chain). The identification $\vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}$ means imposing periodic boundary conditions (a closed chain). The interaction is between the nearest neighbors only. The strength of the interaction is characterized by the constants J_x, J_y, J_z. This model (in the particular case $J_x = J_y = J_z$) was suggested by one of the founders of quantum mechanics W.Heisenberg. The eigenvalues of the Hamiltonian $H(J_x, J_y, J_z)$ (the spectrum) are possible values of energy of the system of spins.

The case when only one of the three constants is nonzero (for example, $J_x = J_y = 0$, $J_z \neq 0$) is straightforwardly reduced to the one-dimensional Ising model which can be easily solved by elementary methods. The case when two of the three constants are nonzero is equivalent to the two-dimensional Ising model, the exact solution of which is already nontrivial.

The following terminology is adopted in the theory of integrable systems. If the model is completely anisotropic (i.e. $J_x \neq J_y \neq J_z \neq J_x$), it is called the XYZ spin chain (magnet), if $J_x = J_y \neq J_z$ it is called the XXZ spin chain, and, finally, if the interaction is the same for all the three directions, i.e., $J_x = J_y = J_z$, it is called the XXX spin chain.

From a purely algebraic point of view, the constants J_x, J_y, J_z can take any values including complex ones. However, an important physical requirement is that the operator $H(J_x, J_y, J_z)$ must be Hermitean. Then the constants J_x, J_y, J_z should be real. Their signs are again not important for algebraic methods but physical properties of the model may essentially depend on their signs. The following problem shows that some different choices of the signs lead to unitary equivalent operators. This allows one to restrict the consideration by the cases when the constants are either all negative (ferromagnet) or all positive (antiferromagnet).

Problem. Let the number of sites N be even. Show that

$$H(J_x, J_y, -J_z) = -\mathcal{U}_z H(J_x, J_y, J_z)(\mathcal{U}_z)^{-1},$$

where $\mathcal{U}_z = \sigma_z^{(2)}\sigma_z^{(4)}\sigma_z^{(6)} \ldots \sigma_z^{(N)}$. Prove also similar identities for the x- and y-directions.

Problem. Find spectrum of the Hamiltonian of the XYZ spin chain for $N = 2$:

$$H = J_x\sigma_x^{(1)}\sigma_x^{(2)} + J_y\sigma_y^{(1)}\sigma_y^{(2)} + J_z\sigma_z^{(1)}\sigma_z^{(2)}.$$

Separately consider the case of the XXX spin chain ($J_x = J_y = J_z = J$).

Below in this section we consider the XXX spin chain in detail. The method to find the eigenvectors and the spectrum of the Hamiltonian was suggested by Bethe in 1931 (ansatz Bethe). The role of this method in the theory of quantum integrable systems goes far beyond the particular model.

We take the Hamiltonian of the XXX spin chain in the form

$$\boxed{H^{\text{xxx}} = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_x^{(k)}\sigma_x^{(k+1)} + \sigma_y^{(k)}\sigma_y^{(k+1)} + \sigma_z^{(k)}\sigma_z^{(k+1)} - \sigma_0^{(k)}\sigma_0^{(k+1)}\right).}$$

Comaring to (2.4), we add the last term proportional to the identity operator. This simply shifts the spectrum by a constant. The common factor is taken to be $-\frac{1}{2}$ (the sign corresponds to the ferromagnetic case) but if necessary the common factor J can be easily restored. This operator can be represented in different equivalent forms:

$$\begin{aligned} H^{\text{xxx}} &= -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_x^{(k)}\sigma_x^{(k+1)}+\sigma_y^{(k)}\sigma_y^{(k+1)}+\sigma_z^{(k)}\sigma_z^{(k+1)}-1\right) \\ &= -\frac{1}{2}\sum_{k=1}^{N}\left(2\sigma_+^{(k)}\sigma_-^{(k+1)}+2\sigma_-^{(k)}\sigma_+^{(k+1)}+\sigma_z^{(k)}\sigma_z^{(k+1)}-1\right) \qquad (2.5) \\ &= -\sum_{k=1}^{N}\mathsf{P}_{k,k+1}\,+\,N. \end{aligned}$$

In the last line $\mathsf{P}_{k,k+1}$ is the permutation operator of kth and $(k+1)$th tensor multipliers in the tensor product, and $\mathsf{P}_{N,N+1}\equiv\mathsf{P}_{N,1}$.

Problem. Prove that all eigenvalues of the operator H^{xxx} are non-negative.

2.1.3 Construction of eigenvectors in the XXX spin chain

Two eigenvectors of H^{xxx} are obvious. From the representation of the Hamiltonian in the form $H^{\text{xxx}}=-\sum_{k=1}^{N}\mathsf{P}_{k,k+1}\,+\,N$ (the last line in (2.5)) it immediately follows that the vectors $|\Omega\rangle := |++++\,\dots\,+\rangle\,,\quad \left|\bar{\Omega}\right\rangle := |----\,\dots\,-\rangle$ are its eigenvectors with the eigenvalue 0:

$$H^{\text{xxx}}\,|\Omega\rangle = H^{\text{xxx}}\left|\bar{\Omega}\right\rangle = 0.$$

The further procedure implies that we choose one of them, say, $|\Omega\rangle$ (all spins up), and construct the other eigenvectors by applying some operators to it. Physicists call the vector $|\Omega\rangle$ *the vacuum* (sometimes bare or false), and the construction of other eigenstates is interpreted as creation of some "excitations" or "quasiparticles" in the vacuum.

An important remark which allows one to simplify the construction of other eigenvectors is that the Hamiltonian of the XXX spin chain commutes with the z-projection of the total spin:

$$[H^{\text{xxx}},\,S_z]=0 \qquad (2.6)$$

(this is also the case for the XXZ-model, but not for XYZ). Again, this can be most easily seen from the representation of the Hamiltonian as sum of permutations. Moreover, in the XXX spin chain the three directions are on equal footing, therefore,

$$[H^{\text{xxx}},\,\vec{S}]=0 \qquad (2.7)$$

(this is already not the case for XXZ). Consequences of this global $SU(2)$-invariance will be discussed later. At the moment we need only the invariance with respect to the Cartan subalgebra given by (2.6). It implies that the operators H^{xxx} and S_z have common eigenvectors. Therefore, the eigenvectors of H^{xxx} have a definite z-projection of spin, i.e., one can find them separately in each sector $\mathcal{H}(m)$ of the decomposition (2.3). We already saw this on the simplest example: $|\Omega\rangle\in\mathcal{H}(0)$.

We have mentioned above that the spectrum of H^{xxx} is non-negative. From this and from the result of the problem below we conclude that the ground state has $E=0$ and N-fold degenerate.

Problem. Show that $H^{\rm xxx}\,|\Omega_m\rangle = 0$, $S_z\,|\Omega_m\rangle = (N-2m)\,|\Omega_m\rangle$, where $|\Omega_m\rangle = S_-^m\,|\Omega\rangle$. How the vectors $|\Omega_m\rangle$ look like?

Let us remark that besides the projections of spin there is yet another operator commuting with the Hamiltonian: the operator e^{iP} of shift by one site of the chain. It acts to the basis vectors as follows:

$$e^{iP}\,|\epsilon_1\rangle_1\,|\epsilon_2\rangle_2\,|\epsilon_3\rangle_3\,|\epsilon_4\rangle_4\;\ldots\;|\epsilon_{n-1}\rangle_{N-1}\,|\epsilon_n\rangle_N = |\epsilon_2\rangle_1\,|\epsilon_3\rangle_2\,|\epsilon_4\rangle_3\,|\epsilon_5\rangle_4\;\ldots\;|\epsilon_N\rangle_{N-1}\,|\epsilon_1\rangle_N\,,$$

where $\epsilon_i = \pm$. It commutes with the operators $\vec{\sigma}^{(j)}$ according to the rule $e^{iP}\,\vec{\sigma}^{(j+1)} = \vec{\sigma}^{(j)}e^{iP}$. Clearly, it commutes with the operators of total spin. Physicists call an eigenvalue of the operator $-i\log e^{iP} = P$ quasimomentum. In what follows we call it simply momentum.

Our task is thus to find common eigenstates of the operators $H^{\rm xxx}$, e^{iP} and S_z. In other words, we will find stationary states of the spin system with definite values of the momentum and z-projection of the total spin.

A single inverted spin. A more complicated (although also simple) case is the N-dimensional space $\mathcal{H}(1)\subset\mathcal{H}$ (one inverted spin) which should contain N eigenvectors. The basis vectors of $\mathcal{H}(1)$ are obtained by action of the operators $\sigma_-^{(j)}$ to the vacuum vector $|\Omega\rangle$. We will find eigenvectors of $H^{\rm xxx}$ in the general form

$$|\Psi^{(1)}\rangle = \sum_{k=1}^{N} a(k)\,\sigma_-^{(k)}\,|\Omega\rangle\,,$$

where $a(k)$ are as yet unknown coefficients satisfying the periodicity condition $a(k+N) = a(k)$. Physicists call $a(k)$ one-particle wave function (in the coordinate representation). Let us introduce the temporary notation $\sigma_-^{(j)}\,|\Omega\rangle = |j\rangle$. It is easy to see that the operator $\mathcal{P}\equiv\sum_k \mathsf{P}_{k,k+1}$ sends the vector $|j\rangle$ to

$$(N-2)\,|j\rangle + |j+1\rangle + |j-1\rangle = N\,|j\rangle + |j+1\rangle + |j-1\rangle - 2\,|j\rangle\,,$$

so that the eigenvalue equation $H^{\rm xxx}V = EV$ with the eigenvalue E is equivalent to the following linear difference equation for the coefficients $a(k)$:

$$-a(k+1) - a(k-1) + 2a(k) = Ea(k). \tag{2.8}$$

The solution can be found in the form $a(k) = e^{ipk}$ with $0\le p<2\pi$, then

$$|\Psi^{(1)}\rangle = |\Psi^{(1)}(p)\rangle = \sum_{k=1}^{N} e^{ipk}\,\sigma_-^{(k)}\,|\Omega\rangle\,,\qquad E = 2(1-\cos p) = 4\sin^2(p/2).$$

However, we should take into account the periodicity condition $a(k+N) = a(k)$. It implies that the parameter p (the momentum) can take only a finite number of values

$$p = p_\ell = \frac{2\pi\ell}{N},\quad \ell = 0,1,2,\ldots,N-1.$$

We have found N eigenvectors of the form

$$|\Psi^{(1)}\rangle = |\Psi_\ell^{(1)}\rangle = \sum_{j=1}^{N} e^{2\pi i\ell j/N}\sigma_-^{(j)}\,|\Omega\rangle\,,\quad \ell = 0,1,2,\ldots,N-1$$

with eigenvalues $E_\ell = 2\Big(1-\cos\frac{2\pi\ell}{N}\Big)$. Note that $E_\ell > 0$ at $\ell \neq 0$ (the state with $\ell = 0$ is $S_-|\Omega\rangle$). These energy levels are degenerate because all states of the form $S_-^m|\Psi_\ell^{(1)}\rangle$ with $1 < m \le N-1$ are eigenstates with the same energy. Obviously, $S_z|\Psi_\ell^{(1)}\rangle = (N-2)|\Psi_\ell^{(1)}\rangle$. It is also easy to check that these states are also eigenstates for the shift operator: $e^{iP}|\Psi^{(1)}(p)\rangle = e^{ip}|\Psi^{(1)}(p)\rangle$ or

$$e^{iP}|\Psi_\ell^{(1)}\rangle = e^{2\pi i\ell/N}|\Psi_\ell^{(1)}\rangle.$$

In the limit $N\to\infty$ possible values of p densely occupy the segment from 0 to 2π. In the solid state physics such states are called magnons. At small p the energy of a magnon as a function of its momentum has the same dependence as for usual free massive non-relativistic particles: $E(p)\approx p^2$. If we recall the constant J, we would obtain $E(p)\approx Jp^2$, so $(2J)^{-1}$ plays the role of mass of these particles.

Problem. Prove that $S_+|\Psi_\ell^{(1)}\rangle = 0$ at $\ell\neq 0$.

Problem. a) Find the norm of the constructed vectors with respect to the scalar product in $\mathcal{H}$ introduced above; b) Prove that $\Big\langle \Psi_l^{(1)}\Big|\, \Psi_{l'}^{(1)}\Big\rangle = 0$ at $l \neq l'$.

Two inverted spins. In the subspace $\mathcal{H}(2)\subset\mathcal{H}$ (two inverted spins) there are $N(N-1)/2$ eigenvectors. We can find them in the form

$$\Big|\Psi^{(2)}\Big\rangle = \sum_{1\le k_1<k_2\le N} a(k_1,k_2)\,\sigma_-^{(k_1)}\sigma_-^{(k_2)}\,|\Omega\rangle\,,$$

where $a(k_1,k_2)$ are as yet unknown coefficients satisfying the periodicity conditions to be taken into account later. Physicists call $a(k_1,k_2)$ two-particle wave function (in the coordinate representation). Below we will write simply $|\Psi\rangle$ instead of $\Big|\Psi^{(2)}\Big\rangle$.

The eigenvalue equation for the operator $H^{\rm xxx} = N - \mathcal{P}$, where $\mathcal{P}\equiv\sum_{k=1}^{N}\mathsf{P}_{k,k+1}$ has the form

$$\mathcal{P}\,|\Psi\rangle = (N-E)\,|\Psi\rangle\,.$$

We should obtain from it relations for the coefficients $a(k_1,k_2)$, similarly to (2.8). For this purpose, multiply both sides from the left by the covector $\langle\Omega|\,\sigma_+^{(n_2)}\sigma_+^{(n_1)}$,

$$\langle\Omega|\,\sigma_+^{(n_2)}\sigma_+^{(n_1)}\mathcal{P}\,|\Psi\rangle = (N-E)\,\langle\Omega|\,\sigma_+^{(n_2)}\sigma_+^{(n_1)}\,|\Psi\rangle\,,$$

and move the permutation operators to the left end, where they will disappear after acting to the left vacuum because the vacuum is invariant under all permutations (all spins look up). The commutation rule is

$$\sigma_+^{(n)}\mathsf{P}_{k,k+1} = \mathsf{P}_{k,k+1}\Big[\sigma_+^{(n)} + \delta_{kn}(\sigma_+^{(n+1)}-\sigma_+^{(n)}) + \delta_{k+1,n}(\sigma_+^{(n-1)}-\sigma_+^{(n)})\Big].$$

Applying it two times, we obtain:

$$\begin{aligned}
&\langle\Omega|\, \sigma_+^{(n_2)}\sigma_+^{(n_1)}\mathsf{P}_{k,k+1} \\
= \;& \langle\Omega|\, \sigma_+^{(n_2)}\sigma_+^{(n_1)} \\
&+ \delta_{kn_1}\, \langle\Omega|\, \sigma_+^{(n_2)}(\sigma_+^{(n_1+1)} - \sigma_+^{(n_1)}) + \delta_{k+1,n_1}\, \langle\Omega|\, \sigma_+^{(n_2)}(\sigma_+^{(n_1-1)} - \sigma_+^{(n_1)}) \\
&+ \delta_{kn_2}\, \langle\Omega|\, (\sigma_+^{(n_2+1)} - \sigma_+^{(n_2)})\sigma_+^{(n_1)} + \delta_{k+1,n_2}\, \langle\Omega|\, (\sigma_+^{(n_2-1)} - \sigma_+^{(n_2)})\sigma_+^{(n_1)} \\
&+ \delta_{kn_1}\delta_{k+1,n_2}\, \langle\Omega|\, (\sigma_+^{(n_2-1)} - \sigma_+^{(n_2)})(\sigma_+^{(n_1+1)} - \sigma_+^{(n_1)}) \\
&+ \delta_{kn_2}\delta_{k+1,n_1}\, \langle\Omega|\, (\sigma_+^{(n_2+1)} - \sigma_+^{(n_2)})(\sigma_+^{(n_1-1)} - \sigma_+^{(n_1)}).
\end{aligned}$$

Next, take sum over k and use the identity $a(n_1, n_2) = \langle\Omega|\, \sigma_+^{(n_2)}\sigma_+^{(n_1)}\, |\Psi\rangle$ (at $n_1 < n_2$). In the case $n_1 < n_2 - 1$ (but $(n_1, n_2) \neq (1, N)$) the last two lines do not work and we get the equation

$$a(n_1{+}1, n_2){+}a(n_1{-}1, n_2){+}a(n_1, n_2{+}1){+}a(n_1, n_2{-}1){-}4a(n_1, n_2) = -Ea(n_1, n_2). \tag{2.9}$$

If $n_1 = n_2 - 1 = n$, the penultimate line comes into play and we get:

$$a(n, n+2) + a(n-1, n+1) - 2a(n, n+1) = -Ea(n, n+1). \tag{2.10}$$

Finally, if $n_1 = 1$, $n_2 = N$, the last line contributes:

$$a(1, N-1) + a(2, N) - 2a(1, N) = -Ea(1, N) \tag{2.11}$$

(recall that $\delta_{N+1,n} = \delta_{1,n}$ due to pariodic boundary conditions).

Let us forget for a while about equations (2.10) and (2.11) which work only for nearest neighbors and try to find a general solution of equation (2.9), extending it to all possible values of n_1, n_2 without any periodicity conditions. Obviously, $a(n_1, n_2) = e^{ip_1n_1+ip_2n_2}$ is one of possible solutions of (2.9) at all n_1, n_2 and for this solution the energy is $E = 2(1-\cos p_1)+2(1-\cos p_2)$ and the momentum is $P = p_1+p_2$. What is the general solution with given energy and momentum? One can consider a linear combination of solutions of the form $e^{\pm ip_1n_1\pm ip_2n_2}$. All of them have the same energy but their superposition is in general not suitable for us because they are not eigenfunctions of the shift operator. However, we can take

$$a(n_1, n_2) = Ae^{ip_1n_1+ip_2n_2} + Be^{ip_2n_1+ip_1n_2} \tag{2.12}$$

with arbitrary A, B; we have still $E = 2(1 - \cos p_1) + 2(1 - \cos p_2)$ and $P = p_1 + p_2$, i.e.,

$$a(n_1 + 1, n_2 + 1) = e^{i(p_1+p_2)}a(n_1, n_2). \tag{2.13}$$

But this solution is not all right: for arbitrary A, B it does not satisfy equation (2.10). Let us see (following Bethe) whether it is possible to choose A and B in such a way that this equation would be also satisfied. Subtract (2.10) from (2.9) at $n_1 = n_2 - 1 = n < N$ in order to exclude E. In this way we get an additional condition

$$a(n, n) + a(n + 1, n + 1) = 2a(n, n + 1), \quad 1 \leq n < N, \tag{2.14}$$

which should be satisfied by the wave function (2.12). Substituting (2.12) into (2.14), we find that (2.12) is a solution if the coefficients A, B are connected by the formula

$$\frac{A}{B} = -\frac{1 + e^{i(p_1+p_2)} - 2e^{ip_1}}{1 + e^{i(p_1+p_2)} - 2e^{ip_2}} := e^{i\theta(p_1,p_2)}.$$

For brevity we will often write $\theta_{12} = \theta(p_1, p_2) = -\theta_{21}$. So, the final answer is

$$\boxed{a(n_1, n_2) = e^{i(p_1n_1+p_2n_2+\frac{\theta_{12}}{2})} + e^{i(p_2n_1+p_1n_2+\frac{\theta_{21}}{2})} \quad (\text{at } n_1 < n_2).} \tag{2.15}$$

Exercise. Check that $2\mathrm{ctg}\,\frac{\theta_{12}}{2} = \mathrm{ctg}\,\frac{p_1}{2} - \mathrm{ctg}\,\frac{p_2}{2}$.

For the infinite chain the parameters p_1, p_2 are arbitrary. But our chain is a ring (N-periodic). There is an obvious periodicity condition

$$a(n_1 + N, n_2 + N) = a(n_1, n_2), \tag{2.16}$$

which simply means that the system can be rotated as a whole 360°, and it will go into itself. This condition implies quantization of the total momentum: $e^{i(p_1+p_2)N} = 1$, i.e., $p_1 + p_2 = \frac{2\pi l}{N}$, $l = 0, 1, \ldots N - 1$.

There is also a more subtle periodicity condition. Let us explain it on the example of $a(1, 2)$. How it is connected with $a(1, N)$? On the ring they should be connected because both cases correspond to states with two neighboring inverted spins. They differ only by a shift by one site. Recalling (2.13), we can therefore write $a(1, 2) = e^{i(p_1+p_2)}a(1, N)$. But according to the same equation (2.13), $e^{i(p_1+p_2)}a(1, N) = a(2, N + 1)$, so that we should require $a(1, 2) = a(2, N + 1)$.

Consider now the amplitudes $a(1, n)$ at $n > 1$. How $a(1, n)$ and $a(1, N - n + 2)$are connected? Both amplitudes correspond to the states in which the inverted spins are separated by $n-1$ edges of the lattice. They differ by a shift by $n-1$ sites and, therefore, $a(1, n) = e^{i(n-1)(p_1+p_2)}a(1, N - n + 2)$. But $e^{i(n-1)(p_1+p_2)}a(1, N - n + 2) = a(n, N + 1)$, and we should require $a(1, n) = a(n, N + 1)$. At last, shifting this condition as a whole along the chain by k steps, we get $a(k + 1, n + k) = a(n + k, N + k + 1)$, which can be written in the form

$$\boxed{a(n_1, n_2) = a(n_2, n_1 + N) \qquad (\text{at } 1 \le n_1 < n_2 \le N).} \tag{2.17}$$

This condition is the periodicity condition under the shift of *one* of the variables (n_1) by N at a fixed value of n_2. Note that the condition $a(n_1 + N, n_2) = a(n_1, n_2)$ *is wrong* because our wave function is defined only at $n_1 < n_2$, and the shift $n_1 \to n_1 + N$ changes the relative position of the arguments.

Another way to understand this periodicity condition is to extend the wave function to the region $n_1 > n_2$ by imposing a natural symmetry requirement under the permutation of variables: $a(n_1, n_2) = a(n_2, n_1)$. It is easy to see that the extended wave function is given by the formula

$$a^{\mathrm{symm}}(n_1, n_2) = e^{i(p_1n_1+p_2n_2+\frac{1}{2}\,\mathrm{sign}(n_2-n_1)\theta_{12})} + e^{i(p_2n_1+p_1n_2+\frac{1}{2}\,\mathrm{sign}(n_2-n_1)\theta_{21})}. \tag{2.18}$$

This wave function satisfies the periodicity condition in the usual form $a^{\text{symm}}(n_1 + N, n_2) = a^{\text{symm}}(n_1, n_2)$. It is equivalent to (2.17).

The periodicity condition implies the restrictions for possible values of p_1, p_2:

$$\begin{cases} e^{ip_1 N} = e^{i\theta(p_1,p_2)} \\ e^{ip_2 N} = e^{-i\theta(p_1,p_2)}. \end{cases} \tag{2.19}$$

This is the simplest example of the system of Bethe equations. Physicists interpret them in the following way. The first magnon, moving around the circle, acquires a phase which should be a multiple of 2π. On the other hand, this phase is a sum of the phase due to the free motion $p_1 N$ and the scattering phase on the second magnon which is equal to $\theta(p_2, p_1)$. The Bethe equations just state that the sum is a multiple of 2π.

In fact the Bethe equations (2.19) are obtained automatically if one takes into account the condition (2.11) (which was not of any use so far). For this we subtract (2.11) from (2.9) at $n_1 = 1$, $n_2 = N$ and obtain the equation

$$a(0, N) + a(1, N + 1) = 2a(1, N).$$

Substituting (2.12), we get (taking into account that $e^{i(p_1+p_2)N} = 1$ and $A/B = e^{i\theta_{12}}$) the same Bethe equatins for p_1, p_2.

Problem. Find solutions of the Bethe equations such that $p_1 = p_2$, and also find the corresponding eigenvectors.

The result of this problem suggests that not all solutions of the Bethe equations correspond to nontrivial eigenstates.

It is convenient to rewrite the Bethe equations in a different parametrization, in which they become algebraic. For example, one can put $e^{ip_1} = z_1$, $e^{ip_2} = z_2$, and then

$$\begin{cases} z_1^N = -\dfrac{1 + z_1 z_2 - 2z_1}{1 + z_1 z_2 - 2z_2} \\[2ex] z_2^N = -\dfrac{1 + z_1 z_2 - 2z_2}{1 + z_1 z_2 - 2z_1}. \end{cases}$$

But in what follows another parametrization will be much more convenient. Instead of p, let us introduce the parameter λ as follows:

$$e^{ip} = \frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} \qquad \text{or} \qquad \lambda = \frac{1}{2} \operatorname{ctg} \frac{p}{2}\,. \tag{2.20}$$

This parameter is sometimes called rapidity. For a reason which will be clear later it is also called *spectral parameter*. Introduce the useful functions

$$\begin{aligned} p(\lambda) &:= -i \log \frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} = -2 \operatorname{arctg}(2\lambda) + \pi \pmod{2\pi}, \\ \varepsilon(\lambda) &:= 2(1 - \cos p(\lambda)) = \frac{1}{\lambda^2 + \frac{1}{4}}, \end{aligned} \tag{2.21}$$

which have the meaning of momentum and energy of a free magnon with spectral parameter λ. A direct calculation shows that

$$e^{i\theta(p_1,p_2)} = \frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}, \tag{2.22}$$

i.e., the scattering phase shift of two magnons depends only on the difference of their rapidities, and this is the main advantage of the λ-parametrization. The system of Bethe equations is written in the form

$$\begin{cases} \left(\dfrac{\lambda_1-\frac{i}{2}}{\lambda_1+\frac{i}{2}}\right)^N = \dfrac{\lambda_1-\lambda_2-i}{\lambda_1-\lambda_2+i} \\ \\ \left(\dfrac{\lambda_2-\frac{i}{2}}{\lambda_2+\frac{i}{2}}\right)^N = \dfrac{\lambda_2-\lambda_1-i}{\lambda_2-\lambda_1+i}. \end{cases} \tag{2.23}$$

Let us see how the solutions look like as $N\to\infty$. We have:

$$\begin{cases} \dfrac{\lambda_1-\frac{i}{2}}{\lambda_1+\frac{i}{2}} = \omega_1 e^{\frac{i}{N}\theta_{12}} \\ \\ \dfrac{\lambda_2-\frac{i}{2}}{\lambda_2+\frac{i}{2}} = \omega_2 e^{-\frac{i}{N}\theta_{12}}, \end{cases}$$

where $\omega_{1,2}$ are two arbitrary roots of 1 of Nth degree. In the limit $N\to\infty$ their arguments independently run in the segment $[0,2\pi)$, and exponential functions in the right hand sides tend to 1, so the variables separate and the equations are solved in a trivial way, with real λ_1,λ_2. Physically the corresponding eigenstates are interpreted as scattering states of two magnons.

However, equations (2.23) have also complex solutions. Set $\lambda_1 = u_1+iv_1$, $\lambda_2 = u_2+iv_2$. Taking module of the first equation, we get:

$$\left(\frac{u_1^2+(v_1-\frac{1}{2})^2}{u_1^2+(v_1+\frac{1}{2})^2}\right)^N = \frac{(u_1-u_2)^2+(v_1-v_2-1)^2}{(u_1-u_2)^2+(v_1-v_2+1)^2}.$$

Assume that $v_1>0$, then the left hand side is exponentially small as $N\to\infty$. So, with exponential precision we have $u_1=u_2$, $v_1-v_2=1$. Taking module of the second equation, we see that $v_2<0$. Multiplying the both equations, we get:

$$\left(\frac{u_1+i(v_1-\frac{1}{2})}{u_1+i(v_1+\frac{1}{2})}\cdot\frac{u_2+i(v_2-\frac{1}{2})}{u_2+i(v_2+\frac{1}{2})}\right)^N = 1.$$

Plugging here $u_1=u_2$, $v_1-v_2=1$, we arrive at the relation

$$\left(\frac{u_1+i(v_1-\frac{3}{2})}{u_1+i(v_1+\frac{1}{2})}\right)^N = 1,$$

hence $v_1=\frac{1}{2}$. So, with the exponential precision as $N\to\infty$ we obtain the family of solutions

$$\lambda_1 = u+\frac{i}{2}, \qquad \lambda_2 = u-\frac{i}{2}. \tag{2.24}$$

Such a solution is called *a string*. In our case it is a string of length 2. The real parameter u is arbitrary. The total momentum of the state and the energy are expressed as

$$\begin{aligned} p(\lambda_1,\lambda_2) &= p(\lambda_1)+p(\lambda_2) = p(u/2) = -i\log\frac{u+i}{u-i}\,, \\ E(\lambda_1,\lambda_2) &= \varepsilon(\lambda_1)+\varepsilon(\lambda_2) = 1-\cos p(\lambda_1,\lambda_2) = \frac{2}{u^2+1}. \end{aligned} \tag{2.25}$$

Exercise. Derive these formulas.

Problem. Prove that $E(u+\frac{i}{2}, u-\frac{i}{2})$ is always less than the energy of two magnons with momenta p_1 and p_2 such that $p_1+p_2 = p(u+\frac{i}{2}, u-\frac{i}{2}) = p(u/2)$. Therefore, such state can be interpreted as a bound state of two magnons. Describe how its wave function looks like.

To summarize, we constructed eigenstates of the operator $H^{\rm xxx}$ of the form $|\Psi(\lambda_1,\lambda_2)\rangle$, where λ_1, λ_2 is a solution of the Bethe equations (2.23). Their momentum and energy are given by the formulas

$$\begin{aligned} p(\lambda_1,\lambda_2) &= p(\lambda_1)+p(\lambda_2), \\ E(\lambda_1,\lambda_2) &= \varepsilon(\lambda_1)+\varepsilon(\lambda_2). \end{aligned}$$

These states are degenerate because all states of the form $S_-^m|\Psi(\lambda_1,\lambda_2)\rangle$ with $1<m\le N-2$ have the same energy. Obviously, $S_z|\Psi(\lambda_1,\lambda_2)\rangle = (N-4)|\Psi(\lambda_1,\lambda_2)\rangle$.

Problem. Prove that $S_+|\Psi(\lambda_1,\lambda_2)\rangle = 0$ (λ_1,λ_2 satisfy the Bethe equations).

More than two inverted spins: general case. Actually, it is not surprising that we managed to solve the problem in the sector with two inverted spins. It is truly remarkable that the same method allows one to go much further and to solve the problem in the sector with arbitrary number m of inverted spins.

We will find eigenvectors of the operator $H^{\rm xxx} = N - \mathcal{P}$ in the subspace $\mathcal{H}(m)$ in the form

$$|\Psi\rangle = \sum_{1\le k_1<\ldots<k_m\le N} a(k_1,\ldots,k_m)\,\sigma_-^{(k_1)}\ldots\,\sigma_-^{(k_m)}\,|\Omega\rangle\,.$$

The eigenvalue equation is the same as before: $\mathcal{P}\,|\Psi\rangle = (N-E)\,|\Psi\rangle$ (recall that $\mathcal{P} = \sum_k \mathsf{P}_{k,k+1}$). Similarly to the case of two inverted spins, multiply the both sides from the left by the covector $\langle\Omega|\,\sigma_+^{(n_1)}\ldots\sigma_+^{(n_m)}$ (at $n_1<n_2<\ldots<n_m$),

$$\langle\Omega|\,\sigma_+^{(n_1)}\ldots\sigma_+^{(n_m)}\mathcal{P}\,|\Psi\rangle = (N-E)\,\langle\Omega|\,\sigma_+^{(n_1)}\ldots\sigma_+^{(n_m)}\,|\Psi\rangle \tag{2.26}$$

and move all parmutation operators to the left end. For brevity, we write the commutation rule in the form

$$\sigma_+^{(n)}\mathsf{P}_{k,k+1} = \mathsf{P}_{k,k+1}(\sigma_+^{(n)}+\beta_k^{(n)}),$$

where

$$\beta_k^{(n)} \equiv \delta_{kn}(\sigma_+^{(n+1)}-\sigma_+^{(n)}) + \delta_{k+1,n}(\sigma_+^{(n-1)}-\sigma_+^{(n)}).$$

The right hand side of (2.26) is $(N-E)a(n_1,\ldots,n_m)$. The left hand side, after moving the permutation operators to the left, is of the form

$$\sum_k \langle\Omega|\,(\sigma_+^{(n_1)}+\beta_k^{(n_1)})(\sigma_+^{(n_2)}+\beta_k^{(n_2)})\ldots(\sigma_+^{(n_m)}+\beta_k^{(n_m)})\,|\Psi\rangle\,.$$

Not very instructive but let us see what this gives after opening the brackets and taking the sum over k (from 1 to N). First, there is the term not containing the operators $\beta_k^{(n)}$; it does not depend on k and gives

$$N \langle \Omega | \, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_m)} \, | \Psi \rangle \, .$$

Second, there are m terms in which the operators β enter once. They are unified in the expression

$$\sum_{\alpha=1}^{m} \langle \Omega | \, \sigma_+^{(n_1)} \ldots \not{\sigma}_+^{(n_\alpha)} \ldots \sigma_+^{(n_m)} \Big(\sum_k \beta_k^{(n_\alpha)} \Big) \, | \Psi \rangle \, .$$

Here the crossed operator means that it is absent at this place. It is easy to see that

$$\sum_k \beta_k^{(n)} = \sigma_+^{(n+1)} + \sigma_+^{(n-1)} - 2\sigma_+^{(n)},$$

, so these terms have the expected form already familiar from the previous cases. Consider now the terms in which there are two operators β:

$$\sum_{\substack{\alpha,\alpha'=1 \\ \alpha<\alpha'}}^{m} \langle \Omega | \, \sigma_+^{(n_1)} \ldots \not{\sigma}_+^{(n_\alpha)} \ldots \not{\sigma}_+^{(n_{\alpha'})} \ldots \sigma_+^{(n_m)} \Big(\sum_k \beta_k^{(n_\alpha)} \beta_k^{(n_{\alpha'})} \Big) \, | \Psi \rangle \, .$$

Problem. Prove that

$$\sum_k \beta_k^{(n_\alpha)} \beta_k^{(n_{\alpha'})} = 0 \quad \text{at } |\alpha' - \alpha| \geq 2 \ (\text{mod } m),$$

$$\sum_k \beta_k^{(n_\alpha)} \beta_k^{(n_{\alpha+1})} = 2 \, \delta_{n_\alpha+1, n_{\alpha+1}} \sigma_+^{(n_\alpha)} \sigma_+^{(n_\alpha+1)} .$$

It follows from this that the bilinear terms in β yield

$$2 \sum_{\alpha=1}^{m} \delta_{n_\alpha+1, n_{\alpha+1}} \, \langle \Omega | \, \sigma_+^{(n_1)} \ldots \, \sigma_+^{(n_m)} \, | \Psi \rangle \, .$$

(Hereafter the index α is understood modulo m.) Finally, let us look on the higher terms in β. The key fact is that

$$\sum_k \beta_k^{(n_{\alpha_1})} \beta_k^{(n_{\alpha_2})} \ldots \beta_k^{(n_{\alpha_r})} = 0 \quad \text{at } r \geq 3 \text{ for all } n_{\alpha_i} .$$

The proof is not difficult; we suggest to prove this identity as a problem. So, all the terms containing more than two operators β in the left hand side of (2.26) vanish! That is why the Bethe's method works for $m > 2$.

Collecting everything together, we have:

$$\langle \Omega | \, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_m)} (\mathcal{P} - N)$$

$$= \sum_{\alpha=1}^{m} \langle \Omega | \, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_{\alpha-1})} \sigma_+^{(n_\alpha+1)} \sigma_+^{(n_{\alpha+1})} \ldots \sigma_+^{(n_m)} + \sum_{\alpha=1}^{m} \langle \Omega | \, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_{\alpha-1})} \sigma_+^{(n_\alpha-1)} \sigma_+^{(n_{\alpha+1})} \ldots \sigma_+^{(n_m)}$$

$$+\,2\sum_{\alpha=1}^{m}(\delta_{n_\alpha+1,n_{\alpha+1}}-1)\,\langle\Omega|\,\sigma_+^{(n_1)}\ldots\sigma_+^{(n_m)}. \qquad (2.27)$$

In the second sum, shift the summation index $\alpha\to\alpha+1$. After taking the scalar product with the vector $|\Psi\rangle$ we get the following equation for $a(n_1,\ldots,n_m)$:

$$\begin{aligned}
&\sum_{\alpha=1}^{m}(1-\delta_{n_\alpha+1,n_{\alpha+1}})\,a(n_1,\ldots,n_{\alpha-1},n_\alpha{+}1,n_{\alpha+1},\ldots,n_m)\\
+\;&\sum_{\alpha=1}^{m}(1-\delta_{n_\alpha,n_{\alpha+1}-1})\,a(n_1,\ldots,n_{\alpha-1},n_\alpha,n_{\alpha+1}{-}1,\ldots,n_m)\\
-\;&2\sum_{\alpha=1}^{m}(1-\delta_{n_\alpha+1,n_{\alpha+1}})\,a(n_1,\ldots,n_m)\;=\;-E\,a(n_1,\ldots,n_m).
\end{aligned}$$

The δ-symbols in the first two sums appear for the following reason. In the first sum the term with $n_\alpha+1=n_{\alpha+1}$ would correspond to the vector $\langle\Omega|\,\sigma_+^{(n_1)}\ldots\sigma_+^{(n_{\alpha}-1)}$, in which there are two operators σ_+, hence this vector vanishes and its contribution should be excluded. Using the shift operators $e^{\pm\partial_{n_\alpha}}$, we can write the equation in somewhat more compact form:

$$\begin{aligned}
\sum_{\alpha=1}^{m}\Big(2-e^{\partial_{n_\alpha}}-e^{-\partial_{n_\alpha}}\Big)\,a(n_1,\ldots,n_m)+\sum_{\alpha=1}^{m}\delta_{n_\alpha+1,n_{\alpha+1}}\Big(2-e^{\partial_{n_\alpha}}-e^{-\partial_{n_\alpha}}\Big)\,a(n_1,\ldots,n_m)\\
=\;E\,a(n_1,\ldots,n_m).
\end{aligned}\qquad (2.28)$$

As in the case $m=2$, we will find solutions among the functions $a(n_1,\ldots,n_m)$, which satisfy the equation

$$\sum_{\alpha=1}^{m}\Big(2-e^{\partial_{n_\alpha}}-e^{-\partial_{n_\alpha}}\Big)\,a(n_1,\ldots,n_m)=\;E\,a(n_1,\ldots,n_m) \qquad (2.29)$$

for all n_i. The remaining terms in the left hand side will vanish if we impose the additional conditions

$$a(n_1,\ldots,n_\alpha,n_\alpha,\ldots,n_m)+a(n_1,\ldots,n_\alpha{+}1,n_\alpha{+}1,\ldots,n_m)=2a(n_1,\ldots,n_\alpha,n_\alpha{+}1,\ldots,n_m). \qquad (2.30)$$

The general solution of the first equation, which is an eigenstate for the shift operator, is

$$a(n_1,\ldots,n_m)=\sum_{\sigma\in S_m}A_\sigma\exp\Big(i\sum_{j=1}^{m}p_{\sigma(j)}n_j\Big).$$

The sum is taken over all $m!$ permutations of the indices $\{1,2,\ldots,m\}$. The parameters p_α and the coefficients A_σ are arbitrary. The momentum and the energy are given by the formulas

$$P=\sum_{\alpha=1}^{m}p_\alpha\,,\qquad E=2\sum_{\alpha=1}^{m}(1-\cos p_\alpha).$$

Similarly to the case $m=2$, the conditions (2.30) impose certain relations for the coefficients A_σ which can be uniquely solved up to a common multiplier. The detailed analysis is an instructive exercise (do it at least for $m=3$).

The result has the form (at $1 \le n_1 < n_2 < \ldots < n_N \le N$)

$$\boxed{a(n_1, n_2, \ldots, n_m) = \sum_{\sigma \in S_m} \exp\Big(i \sum_{j=1}^{m} p_{\sigma(j)} n_j + \frac{i}{2} \sum_{j<k} \theta_{\sigma(j)\sigma(k)}\Big),} \tag{2.31}$$

where the phases $\theta_{jk} \equiv \theta(p_j, p_k)$ are defined by the same formula

$$e^{i\theta(p_j,p_k)} = -\frac{1 + e^{i(p_j+p_k)} - 2e^{ip_j}}{1 + e^{i(p_j+p_k)} - 2e^{ip_k}}$$

as before. The expression (2.31) is called *the Bethe wave function*. Similarly to (2.18), one can extend the wave function (2.31) to other sectors of the configuration space symmetrically:

$$a^{\mathrm{symm}}(n_1, n_2, \ldots, n_m) = \sum_{\sigma \in S_m} \exp\Big(i \sum_{j=1}^{m} p_{\sigma(j)} n_j + \frac{i}{2} \sum_{j<k} \mathrm{sign}\,(n_k - n_j) \theta_{\sigma(j)\sigma(k)}\Big). \tag{2.32}$$

The periodicity condition is now as follows:

$$\boxed{a(n_1, n_2, \ldots, n_m) = a(n_2, n_3, \ldots, n_1 + N),} \tag{2.33}$$

which leads to the system of Bethe equations

$$e^{ip_j N} = \prod_{k \ne j} e^{i\theta(p_j,p_k)}, \quad j = 1, \ldots, m. \tag{2.34}$$

In the λ-parametrization these equations read

$$\boxed{\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right)^N = \prod_{k \ne j} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}, \quad j = 1, \ldots, m.} \tag{2.35}$$

Problem. Find spectrum of the Hamiltonian of the XXX spin chain on 3 sites with periodic boundary conditions:

$$H^{\mathrm{xxx}} = -\frac{1}{2} \sum_{k=1}^{3} \left(\vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - \mathsf{I}\right), \qquad \vec{\sigma}^{(4)} \equiv \vec{\sigma}^{(1)}.$$

Determine also the degeneracy of the levels.

Problem (difficult). Prove that all eigenstates $\left|\Psi^{(m)}\right\rangle$ corresponding to solutions of the Bethe equations satisfy $S_+ \left|\Psi^{(m)}\right\rangle = 0$.

Solutions of the system (2.35) as $N \to \infty$ are analyzed similarly to the case of two magnons. Real solutions correspond to the states of m independent magnons. Complex solutions form “strings”. Their lengths can be $2M + 1 \le m$, where $M \in \frac{1}{2}\mathbb{Z}_{\ge 0}$. The numbers λ_k for a string of length $2M + 1$ are

$$\lambda_k = u + ik, \quad k = -M, -M + 1, \ldots, M - 1, M,$$

where u is an arbitrary real number. For example, string of length 2 is $\{u+\frac{i}{2}, u-\frac{i}{2}\}$, string of length 3 is $\{u+i, u, u-i\}$ and so on. Strings of length $2M+1>1$ are interpreted as bound states of $2M+1$ magnons. Magnons themselves are formally strings of length 1 (in this case $M=0$).

Problem. Find possible form of string solutions of the Bethe equations in the limit $N\to\infty$.

So, in the eigenstate of general form the Bethe numbers λ_j form strings of different lengths. Let ν_M be the number of strings of length $2M+1$ and $\lambda_{j,M}$ ($j=1,\ldots,\nu_M$) be real parts of the parameters λ forming the string indexed by j. Let Q be the total number of strings (including strings of length 1). We have:

$$Q=\sum_{M\geq 0}\nu_M\,,\qquad m=\sum_{M\geq 0}(2M+1)\nu_M.$$

The set of integer numbers $\{m, Q, \{\nu_M\}\}$ connected by these relations and Q real numbers $\lambda_{j,M}$ characterize the state. We call such set of parameters *configuration*. The energy and momentum of the state corresponding to a given configuration is a sum of Q terms each of them is energy and momentum of a separate string (this is true with the exponential precision as $N\to\infty$).

Problem. Show that energy and momentum of the string of length $2M+1$ with $\lambda_k=u+ik$ ($k=-M,-M+1,\ldots,M-1,M$) with exponential precision as $N\to\infty$ are given by the formulas

$$\begin{aligned}
E&=\frac{1}{2M+1}\,\varepsilon\Big(\frac{u}{2M+1}\Big)=\frac{2M+1}{u^2+(M+\frac{1}{2})^2}=\frac{2}{2M+1}\,(1-\cos P),\\
P&=p\Big(\frac{u}{2M+1}\Big)=-2\mathrm{arctg}\,\frac{u}{M+\frac{1}{2}}+\pi\quad(\mathrm{mod}\,2\pi).
\end{aligned}\tag{2.36}$$

At a fixed m and $N\to\infty$ the real parts of the parameters $\lambda_{j,M}$ are arbitrary. However, if m tends to ∞ together with N in such a way that m/N is fixed, then the parameters $\lambda_{j,M}$ of a given configuration satisfy a system of equations which is obtained from the original Bethe equations as follows. For a given string of length $2M+1$ multiply the Bethe equations corresponding to the parameters λ_j which form this string. In the right hand side, multiply the factors $\frac{\lambda_j-\lambda_k-i}{\lambda_j-\lambda_k+i}$ over k according to the decomposition of λ to the strings in a given configuration. Let us introduce the notation

$$V_0(\lambda)=\frac{\lambda-i}{\lambda+i}.$$

Problem. Prove the identities

$$\prod_{l=-M}^{M}V_0(2\lambda+2il)=V_0\Big(\frac{2\lambda}{2M+1}\Big),$$

$$\prod_{l=-M}^{M}V_0(\lambda+il)=V_0\Big(\frac{\lambda}{M}\Big)V_0\Big(\frac{\lambda}{M{+}1}\Big)\equiv V_M(\lambda),$$

$$\prod_{l_1=-M_1}^{M_1}\prod_{l_2=-M_2}^{M_2}V_0(\lambda+i(l_1+l_2))=\prod_{L=|M_1-M_2|}^{M_1+M_2}V_L(\lambda)\equiv V_{M_1,M_2}(\lambda).$$

Problem. Prove that the system of equations for real parts of the parameters $\lambda_{j,M}$ has the form

$$V_0^N\Big(\frac{\lambda_{j,M_1}}{M_1+\frac{1}{2}}\Big) = \prod_{M_2} \prod_{\substack{k=1 \\ (k,M_2)\neq(j,M_1)}}^{\nu_{M_2}} V_{M_1,M_2}(\lambda_{j,M_1}-\lambda_{k,M_2}). \tag{2.37}$$

So, as $N\to\infty$ the eigenstates of the Hamiltonian of the XXX spin chain are states of independent magnons and their bound states (strings of length greater than 1).

2.1.4 Ground state of antiferromagnetic chain

Consider now the Hamiltonian of the XXX spin chain with the opposite sign:

$$H^{\mathrm{xxx,AF}} = \frac{1}{2}\sum_{k=1}^{N}\Big(\sigma_x^{(k)}\sigma_x^{(k+1)} + \sigma_y^{(k)}\sigma_y^{(k+1)} + \sigma_z^{(k)}\sigma_z^{(k+1)} - \sigma_0^{(k)}\sigma_0^{(k+1)}\Big).$$

Accordingly, one should change sign in all expressions for energies of m-magnon states. The vacuum $|\Omega\rangle$ now has the *maximal* energy (equal to 0) and all states with some number of magnons with nonzero momenta will have negative energies. Physicists say that in this case $|\Omega\rangle$ is a false vacuum while the true (physical) vacuum, i.e. the state with the minimal energy is obtained by filling of the false vacuum by enough number of magnons. This is similar to the Dirac sea.

In this section we assume that N is even. Then the ground state belongs to the sector with zero total projection of spin, i.e. it has $m = N/2$ inverted spins. We will show that the Bethe equations allow one to obtain an exhaustive information about this state as $N\to\infty$. A complete description of excited states over the physical vacuum is also possible (it is given in [6]), but we will restrict ourselves by the analysis of the ground state. One can show that in the ground state all λ_j are real, i.e. there are no strings of lengths greater than 1.

So, we consider the system of equations (2.35) and pass to logarithms in it:

$$N\log\frac{\lambda_j-\frac{i}{2}}{\lambda_j+\frac{i}{2}} = \sum_{k=1,\neq j}^{N/2}\log\frac{\lambda_j-\lambda_k-i}{\lambda_j-\lambda_k+i} + 2\pi i q_j, \quad j=1,\ldots,\frac{N}{2},$$

or

$$Np(\lambda_j) = \sum_{k=1,\neq j}^{N/2} p\Big(\frac{\lambda_j-\lambda_k}{2}\Big) - 2\pi q_j,$$

where q_j are some integer numbers. Recalling that $2\lambda = -\mathrm{tg}\,\frac{p-\pi}{2}$ (see (2.20)), we can write these equations in the form

$$\mathrm{arctg}(2\lambda_j) = \frac{\pi}{N}Q_j + \frac{1}{N}\sum_{k=1}^{N/2}\mathrm{arctg}(\lambda_j-\lambda_k), \quad j=1,\ldots,\frac{N}{2},$$

where integer (at odd $N/2$) or half-integer (at even $N/2$) numbers Q_j are connected with q_j as $Q_j = \frac{1}{4}(3N+2)-q_j$. A detailed analysis shows that in the ground state the numbers

Q_j monotonically increase with j in the interval $\left[-\frac{N}{4}+\frac{1}{2}, \frac{N}{4}-\frac{1}{2}\right]$, i.e., $Q_{j+1}-Q_j=1$. In the limit $N\to\infty$ we can substitute

$$\frac{Q_j}{N}\to x, \quad \lambda_j\to\lambda(x), \quad -\frac{1}{4}\le x\le\frac{1}{4},$$

where $\lambda(x)$ is a monotonic continuous function and $\lambda(\pm\frac{1}{4})=\pm\infty$. In the limit, we obtain the integral equation

$$\operatorname{arctg} 2\lambda(x)=\pi x+\int_{-1/4}^{1/4}\operatorname{arctg}\left(\lambda(x)-\lambda(x')\right)dx' \tag{2.38}$$

for it. However, it is more convenient to work with the function

$$\rho(\lambda)=\Big(\frac{d\lambda(x)}{dx}\Big)^{-1}\Big|_{x=x(\lambda)} \tag{2.39}$$

(here $x(\lambda)$ is the function inverse to $\lambda(x)$), which has the meaning of normalized density of the numbers λ_j in the interval $[\lambda,\, \lambda+d\lambda]$. This can be easily understood by writing

$$N\rho(\lambda)d\lambda=N(x(\lambda+d\lambda)-x(\lambda))=N\frac{dx}{d\lambda}d\lambda=Ndx.$$

By the definition of $x(\lambda)$, the right hand side is the amount of the numbers Q_j in the interval $[x, x+dx]$. Therefore, the left hand side gives the amount of the numbers λ_j in the interval $[\lambda,\, \lambda+d\lambda]$. It is clear that the function $\rho(\lambda)$ satisfies the normalization condition

$$\int_{-\infty}^{+\infty}\rho(\lambda)\,d\lambda=\frac{1}{2}\,. \tag{2.40}$$

The density function allows one to replace sums by integrals in the limit $N\to\infty$ according to the rule

$$\sum_j f(\lambda_j)=N\int_{-\infty}^{\infty}f(\lambda)\rho(\lambda)d\lambda.$$

Differentiating (2.38) with respect to x, we get the integral equation for $\rho(\lambda)$:

$$\pi\rho(\lambda)+\int_{-\infty}^{+\infty}\frac{\rho(\mu)\,d\mu}{(\lambda-\mu)^2+1}=\frac{2}{4\lambda^2+1}. \tag{2.41}$$

It can be solved by the Fourier transformation. Integrate both parts with the function $e^{i\lambda\xi}$ and take into account that

$$\int_{-\infty}^{\infty}\frac{e^{i\lambda\xi}d\lambda}{\lambda^2+1}=\pi e^{-|\xi|}.$$

Then we obtain an algebraic equation for the Fourier image $\hat\rho(\xi)=\int_{-\infty}^{\infty}e^{i\lambda\xi}\rho(\lambda)d\lambda$. Its solution has the form $\hat\rho(\xi)=\dfrac{1}{2\cosh(\xi/2)}$, hence

$$\rho(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda\xi}\,d\xi}{2\cosh(\xi/2)}=\frac{1}{2\cosh(\pi\lambda)}. \tag{2.42}$$

The energy and momentum of the ground state are found as

$$E_0 = -\sum_{k=1}^{N/2} \varepsilon(\lambda_k) = -N \int_{-\infty}^{\infty} \varepsilon(\lambda)\rho(\lambda)d\lambda,$$

$$P_0 = \sum_{k=1}^{N/2} p(\lambda_k) = N \int_{-\infty}^{\infty} p(\lambda)\rho(\lambda)d\lambda.$$

Exercise. Calculate these integrals and show that

$$E_0 = -2N\log 2, \qquad P_0 = \pi N/2 \ (\mathrm{mod}\, 2\pi).$$

2.1.5 Anisotropic (XXZ) spin chain

The Hamiltonian of the XXZ spin chain has the form

$$\boxed{H^{\mathrm{xxz}} = -\frac{1}{2}\sum_{k=1}^{N} \Big(\sigma_x^{(k)}\sigma_x^{(k+1)} + \sigma_y^{(k)}\sigma_y^{(k+1)} + \Delta(\sigma_z^{(k)}\sigma_z^{(k+1)} - 1)\Big),} \tag{2.43}$$

where Δ is called the anisotropy parameter. The diagonalization of this operator is only a little bit more difficult than in the isotropic case. Indeed, writing

$$H^{\mathrm{xxz}} = H^{\mathrm{xxx}} - \frac{1}{2}(\Delta - 1)\Big(\sum_{k=1}^{N} \sigma_z^{(k)}\sigma_z^{(k+1)} - N\mathsf{I}\Big),$$

we see that the additional part acts to the basis vectors diagonally, and so its effect is easy to take into account.

Introduce the temporary notation $\Big| k_1, k_2, \ldots, k_m \Big\rangle \equiv \sigma_-^{(k_1)}\sigma_-^{(k_2)} \ldots \sigma_-^{(k_m)} |\Omega\rangle$ (as usual, $1 \le k_1 \le \ldots \le k_m \le N$). From the obvious equality

$$\sigma_z^{(k)} \Big| k_1, k_2, \ldots, k_m \Big\rangle = \Big(1 - 2\sum_{\alpha=1}^{m} \delta_{k,k_\alpha}\Big) \Big| k_1, k_2, \ldots, k_m \Big\rangle$$

it easily follows that

$$\Big(\sum_{k=1}^{N} \sigma_z^{(k)}\sigma_z^{(k+1)} - N\Big) \Big| k_1, k_2, \ldots, k_m \Big\rangle = 4\sum_{\alpha=1}^{m} \Big(\delta_{k_\alpha+1,k_{\alpha+1}} - 1\Big) \Big| k_1, k_2, \ldots, k_m \Big\rangle .$$

This additional contribution changes the expression for the covector

$$\langle\Omega|\, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_m)} H^{\mathrm{xxz}} = \langle\Omega|\, \sigma_+^{(n_1)} \ldots \sigma_+^{(n_m)} \Big[N\mathsf{I} - \mathcal{P} - \frac{1}{2}(\Delta - 1)\Big(\sum_{k=1}^{N} \sigma_z^{(k)}\sigma_z^{(k+1)} - N\mathsf{I}\Big)\Big]$$

similar to (2.27) and the third sum in the right hand side will acquire the coefficient 2Δ instead of 2. Therefore, the equation (2.29) and the conditions (2.30) acquire the form

$$\sum_{\alpha=1}^{m} \Big(2\Delta - e^{\partial_{n_\alpha}} - e^{-\partial_{n_\alpha}}\Big)\, a(n_1, \ldots, n_m) = \ E\, a(n_1, \ldots, n_m) \tag{2.44}$$

$$
\begin{aligned}
a(n_1,\ldots,n_\alpha,n_\alpha,\ldots,n_m) &+ a(n_1,\ldots,n_\alpha{+}1,n_\alpha{+}1,\ldots,n_m) \\
&= 2\Delta a(n_1,\ldots,n_\alpha,n_\alpha{+}1,\ldots,n_m).
\end{aligned} \tag{2.45}
$$

The general solution of the first equation which is simultaneously an eigenvector for the shift operator is the same as in the isotropic case:

$$
a(n_1,\ldots,n_m) = \sum_{\sigma\in S_m} A_\sigma \exp\Bigl(i\sum_{j=1}^m p_{\sigma(j)}n_j\Bigr).
$$

The total momentum is again given by the sum of the p_α's but the expression for energy should be modified:

$$
P = \sum_{\alpha=1}^m p_\alpha\,, \qquad E = 2\sum_{\alpha=1}^m(\Delta - \cos p_\alpha).
$$

The conditions (2.45) imply similar relations for the coefficients A_σ, and they can be uniquely resolved up to a common multiplier.

The Bethe wave function (at $1 \le n_1 < n_2 < \ldots < n_N \le N$) has the same general form (2.31) as before:

$$
a(n_1,n_2,\ldots,n_m) = \sum_{\sigma\in S_m} \exp\Bigl(i\sum_{j=1}^m p_{\sigma(j)}n_j + \frac{i}{2}\sum_{j<k}\theta_{\sigma(j)\sigma(k)}\Bigr), \tag{2.46}
$$

but the phases $\theta_{jk} \equiv \theta(p_j,p_k)$ are now defined by the formula

$$
e^{i\theta(p_j,p_k)} = -\frac{1 + e^{i(p_j+p_k)} - 2\Delta e^{ip_j}}{1 + e^{i(p_j+p_k)} - 2\Delta e^{ip_k}}.
$$

The periodicity condition (2.33) leads to the same Bethe equations for p_i in the form (2.34).

There exist also analogs of the λ-parametrization. They are different in the cases $\Delta > 1$ and $0 < \Delta < 1$. Assume first that $\Delta > 1$. The parameter λ is introduced by the formula

$$
e^{ip} = \frac{\sin\eta(\lambda + \frac{i}{2})}{\sin\eta(\lambda - \frac{i}{2})} \qquad \text{or} \qquad \mathrm{ctg}\,\frac{p}{2} = \coth(\eta/2)\,\mathrm{tg}\,(\eta\lambda), \tag{2.47}
$$

where η is connected with Δ by the relation

$$
\Delta = \cosh\eta. \tag{2.48}
$$

For the phase shift we have

$$
e^{i\theta_{12}} = \frac{\sin\eta(\lambda_1 - \lambda_2 + i)}{\sin\eta(\lambda_1 - \lambda_2 - i)}. \tag{2.49}
$$

The functions $p(\lambda)$ and $\varepsilon(\lambda)$ are (cf. (2.21)):

$$
\begin{aligned}
p(\lambda) &= -2\,\mathrm{arctg}\left(\frac{\mathrm{tg}\,(\eta\lambda)}{\tanh(\eta/2)}\right) + \pi, \\
\varepsilon(\lambda) &= \frac{2\,\sinh^2\eta}{\cosh\eta - \cos(2\eta\lambda)}.
\end{aligned} \tag{2.50}
$$

They coincide with (2.21) in the limit $\eta \to 0$. The Bethe equations for the XXZ spin chain in the λ-parametrization are

$$\boxed{\left(\frac{\sin\eta(\lambda_j - \frac{i}{2})}{\sin\eta(\lambda_j + \frac{i}{2})}\right)^N = \prod_{k\neq j} \frac{\sin\eta(\lambda_j - \lambda_k - i)}{\sin\eta(\lambda_j - \lambda_k + i)}, \quad j = 1, \ldots, m.} \tag{2.51}$$

The formulas for the case $|\Delta| < 1$ are obtained from the ones written above by the substitution $\eta = i\gamma$, then $\Delta = \cos\gamma$.

Problem. Find string solutions of Bethe equations for the XXZ spin chain at $m = 2$ in the limit $N \to \infty$ and express energy and momentum of the corresponding states through the real parts of λ_j's.

2.2 Bethe ansatz for one-dimensional Bose gas with point-like interaction

The aim of this section is to show how the coordinate Bethe ansatz works for continuous models.

Consider N quantum particles on the line which interact only when any two of them are in one and the same point. Such "point-like interaction" is mathematically described by the δ-function potential. The Hamiltonian in the coordinate representation is

$$\hat{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1\leq j<k\leq N} \delta(x_j - x_k). \tag{2.52}$$

We have put $\hbar = 1$, mass of particle $= \frac{1}{2}$. At $c = 0$ we have a system of free (non-interacting) particles. At $c > 0$ the interaction is repulsive (the particles do not like to be at the same point), at $c < 0$ it is attractive. Below we consider the repulsive case $c > 0$.

According to the quantum mechanical rules, the momentum operator of this system of particles is

$$\hat{P}_N = -i\sum_{j=1}^{N} \frac{\partial}{\partial x_j}. \tag{2.53}$$

It commutes with the Hamiltonian because the interaction depends only on the differences of the coordinates (the total momentum is conserved). The stationary Schrodinger equation for the wave function Ψ is

$$\hat{H}_N \Psi(x_1, \ldots, x_N) = E\Psi(x_1, \ldots, x_N). \tag{2.54}$$

We are going to find common eigenfunctions of the Hamiltonian and the total momentum and the corresponding eigenvalues (the energy spectrum).

In the secondary quantization formalism, the model is described by the Hamiltonian

$$\hat{H} = \int \Big(\partial_x \psi^\dagger(x) \partial_x \psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\Big), \tag{2.55}$$

where $\psi^\dagger$, ψ are creation and annihilation operators of the Bose particles. They satisfy the commutation relations

$$[\psi(x,t), \psi^\dagger(y,t)] = \delta(x-y), \quad [\psi(x,t), \psi(y,t)] = [\psi^\dagger(x,t), \psi^\dagger(y,t)] = 0.$$

The equation of motion reads

$$i\partial_t\psi = -\partial_x^2\psi + 2c\psi^\dagger\psi\psi. \tag{2.56}$$

It is called the quantum nonlinear Schrodinger equation. That is why the model of particles is often called the quantum nonlinear Schrodinger equation. The Fock vacuum is defined by the equation $\psi(x)\,|0\rangle = 0$. In the N-particle sector the eigenstates of the Hamiltonian are constructed as

$$|\Psi\rangle = \int dx_1 \dots dx_N \Psi(x_1, \dots, x_N)\psi^\dagger(x_1)\dots\psi^\dagger(x_N)\,|0\rangle\,,$$

where $\Psi(x_1, \dots, x_N)$ is the wave function satisfying equation (2.54).

2.2.1 The Bethe wave function

In this section we work in the sector with a fixed number of particles. Let us start with the trivial case of one particle. We have:

$$\Psi(x_1) = e^{ip_1x_1}\,, \qquad E = p_1^2.$$

If we impose the periodic boundary conditions $\Psi(x_1 + L) = \Psi(x_1)$, then the momentum p_1 is quantized: $p_1 = 2\pi\ell/L$, $\ell \in \mathbb{Z}_{\geq 0}$. For one particle the interaction is absent.

For two particles the problem is more interesting:

$$-(\partial_{x_1}^2 + \partial_{x_2}^2)\Psi(x_1, x_2) + 2c\delta(x_1 - x_2)\Psi(x_1, x_2) = E\Psi(x_1, x_2).$$

At $x_1 = x_2$ there is a singularity. The wave function is continuous but its derivative has a jump. At $x_1 \neq x_2$ there is no interaction and the particles behave as free ones, i.e., for example at $x_1 < x_2$ we can find the wave function in the form

$$\Psi(x_1, x_2) = A_{12}e^{i(p_1x_1+p_2x_2)} + A_{21}e^{i(p_1x_2+p_2x_1)},$$

so that $-(\partial_{x_1}^2 + \partial_{x_2}^2)\Psi = E\Psi$, $E = p_1^2 + p_2^2$.

The delta-function potential is equivalent to certain boundary condition at $x_1 = x_2 - 0$. Simple arguments show that this condition has the form

$$(\partial_{x_2} - \partial_{x_1} - c)\,\Psi\Big|_{x_1=x_2-0} = 0, \tag{2.57}$$

hence

$$\frac{A_{21}}{A_{12}} = \frac{p_1 - p_2 - ic}{p_1 - p_2 + ic}. \tag{2.58}$$

In order to derive this condition, we pass to the coordinates $x = x_2 - x_1$, $X = \frac{1}{2}(x_2 + x_1)$, then $\partial_{x_2} - \partial_{x_1} = 2\partial_x$, $\partial_{x_1}^2 + \partial_{x_2}^2 = 2\partial_x^2 + \frac{1}{2}\partial_X^2$, and the Schrodinger equation is written in the form

$$-2\partial_x^2\Psi + 2c\delta(x)\Psi = (E + \tfrac{1}{2}\,\partial_X^2)\Psi.$$

The terms in the right hand side remain finite at $x = 0$. The left hand side also should be finite, so the singularity coming from the δ-function should cancel by the discontinuity of the derivative of Ψ at $x = 0$. In other words, integrate both sides over small interval $[0, \varepsilon]$ and tend $\varepsilon \to 0$:

$$\int_0^\varepsilon (-2\partial_x^2 \Psi + 2c\delta(x)\Psi)dx = 0.$$

In the right hand side we write 0 because in the limit $\varepsilon \to 0$ the integral in the right hand side vanish. We obtain, therefore,

$$-2\partial_x \Psi\Big|_0^\varepsilon + 2c\int_0^\varepsilon \delta(x)\Psi dx = 0.$$

How to understand the integral of the δ-function over the segment such that its left end is the support of the δ-function (the point $x = 0$)? In most presentation of the Bethe method in the literature, the integration here goes over the segment $|x| \le \varepsilon$. This, on one hand, removes the problem, but, on the other, requires extension of the wave function to the domain $x_1 > x_2$ in which it originally was not defined. Certainly, the symmetric extension is implied but we believe that it is important that there exists an argument which allows one to remain in the sector $x_1 \le x_2$. If so, the integral one of the limits of which is the support of the δ-function, should be understood as *half* of the integral over the segment containing the support. (Only half of the δ-function will work). Similarly, $\partial_x \Psi(0)$ should be understood as 0 by symmetry. Therefore, our condition tells us that

$$2 \lim_{x \to +0} \partial_x \Psi(x) = c\Psi(0),$$

which is the same as (2.57).

A more formal way to derive this condition is to find the solution for all x_1, x_2 but to impose the symmetry $\Psi(x_1, x_2) = \Psi(x_2, x_1)$. Namely, let us find the solution in the form

$$\Psi(x_1, x_2) = f(x_1, x_2)\theta(x_2 - x_1) + f(x_2, x_1)\theta(x_1 - x_2),$$

where $\theta(x)$ is the step function ($\theta(x) = 1$ at $x > 0$ and $\theta(x) = 0$ at $x < 0$), and

$$f(x_1, x_2) = A_{12}e^{i(p_1x_1+p_2x_2)} + A_{21}e^{i(p_1x_2+p_2x_1)}$$

is the function which we denoted as Ψ before. Note that if we add the condition $\theta(0) = \frac{1}{2}$ to the definition of the step function, then the function Ψ will be continuous at $x_1 = x_2$. Since $\partial_x\theta(x) = \delta(x)$, a direct calculation yields

$$(\partial_{x_1}^2 + \partial_{x_2}^2)\Psi(x_1, x_2) = \theta(x_2 - x_1)(\partial_{x_1}^2 + \partial_{x_2}^2)f(x_1, x_2) + \theta(x_1 - x_2)(\partial_{x_1}^2 + \partial_{x_2}^2)f(x_2, x_1)$$
$$- 2\delta(x_1 - x_2)(\partial_{x_1} - \partial_{x_2})\,[f(x_1, x_2) - f(x_2, x_1)] - 2\delta'(x_1 - x_2)\,[f(x_1, x_2) - f(x_2, x_1)].$$

Note that the term with δ' do not cancel. It should be understood in the sense of distributions (generalized functions) as

$$\delta'(x_1 - x_2)\varphi(x_1, x_2) = -\frac{1}{2}\,\delta(x_1 - x_2)(\partial_{x_1} - \partial_{x_2})\varphi(x_1, x_2),$$

which is valid for any antisymmetric function φ because integrals of both sides with any smooth test function are equal. Taking this into account, we get:

$$(\partial_{x_1}^2 + \partial_{x_2}^2)\Psi(x_1, x_2) = \theta(x_2 - x_1)(\partial_{x_1}^2 + \partial_{x_2}^2)f(x_1, x_2) + \theta(x_1 - x_2)(\partial_{x_1}^2 + \partial_{x_2}^2)f(x_2, x_1)$$

$$-\,\delta(x_1-x_2)(\partial_{x_1}-\partial_{x_2})\,[f(x_1,x_2)-f(x_2,x_1)].$$

Since

$$(\partial_{x_1}-\partial_{x_2})\,[f(x_1,x_2)-f(x_2,x_1)] = i(A_{12}-A_{21})(p_1-p_2)(e^{i(p_1x_1+p_2x_2)}+e^{i(p_2x_1+p_1x_2)}),$$

and one can put $x_2 = x_1$ in the terms with the δ-function, we have finally:

$$\hat{H}_N\Psi = (p_1^2+p_2^2)\Psi + 2\delta(x_1-x_2)\Big(c(A_{12}+A_{21}) + i(A_{12}-A_{21})(p_1-p_2)\Big)e^{i(p_1+p_2)x_1}.$$

The condition of vanishing of the last term gives the same relation (2.58).

In the case of N particles the wave function in the sector $0 \le x_1 < x_2 < \ldots < x_N \le L$ has the form

$$\Psi = \sum_{\sigma\in S_N} A_\sigma \exp\Big(i\sum_{j=1}^{N} p_{\sigma(j)}x_j\Big) \tag{2.59}$$

with boundary conditions

$$\Big(\partial_{x_{j+1}}-\partial_{x_j}-c\Big)\Psi\Big|_{x_j=x_{j+1}-0} = 0, \qquad j=1,2,\ldots,N-1.$$

These conditions are obtained precisely in the same way as for two particles.

Theorem. The exact eigenfunction of the N-particle Hamiltonian (2.52) in the sector $0 \le x_1 \le x_2 \le \ldots \le x_N \le L$ has the form

$$\Psi = C\prod_{m<n}\Big(i\partial_{x_n}-i\partial_{x_m}-ic\Big)\det_{1\le j,k\le N}\Big(e^{ip_jx_k}\Big). \tag{2.60}$$

For the proof one should check that it is of the form (2.59) and the boundary conditions are satisfied. The former is evident from the expansion of the determinant,

$$\Psi = C\sum_{\sigma\in S_N}(-1)^{[\sigma]}\prod_{m<n}(p_{\sigma(m)}-p_{\sigma(n)}-ic)\,\exp\Big(i\sum_{j=1}^{N}p_{\sigma(j)}x_j\Big), \tag{2.61}$$

and the latter is more convenient to check in the form (2.60) not expanding the determinant. Verify, for example, the condition $(\partial_{x_2}-\partial_{x_1}-c)\Psi\Big|_{x_1=x_2-0} = 0$. For this we note that $\Psi = -i(\partial_{x_2}-\partial_{x_1}+c)\tilde{\Psi}$, where

$$\tilde{\Psi} = C\prod_{j=3}^{N}\Big(i\partial_{x_1}-i\partial_{x_j}-ic\Big)\Big(i\partial_{x_2}-i\partial_{x_j}-ic\Big)\prod_{3\le m<n\le N}\Big(i\partial_{x_m}-i\partial_{x_n}-ic\Big)\det_{1\le j,k\le N}\Big(e^{ip_jx_k}\Big).$$

The function $\tilde{\Psi}$ is obviously symmetric with respect to the permutation $x_1 \leftrightarrow x_2$. Rewriting the left hand side of our condition $(\partial_{x_2}-\partial_{x_1}-c)\Psi\Big|_{x_1=x_2-0} = 0$ in the form $-i\left((\partial_{x_2}-\partial_{x_1})^2-c^2\right)\tilde{\Psi}$, we see that it is antisymmetric with respect to the permutation $x_1 \leftrightarrow x_2$, and, therefore, it is equal to 0.

Corollary. The exact symmetrized eigenfunction of the N-particle Hamiltonian (2.52) has the form

$$\Psi_N^{\mathrm{symm}} = C_N\sum_{\sigma\in S_N}(-1)^{[\sigma]}\prod_{m<n}\Big(p_{\sigma(n)}-p_{\sigma(m)}+ic\,\mathrm{sign}\,(x_m-x_n)\Big)\,\exp\Big(i\sum_{j=1}^{N}p_{\sigma(j)}x_j\Big). \tag{2.62}$$

In order to indicate the dependence on coordinates and momenta, we will write $\Psi_N^{\rm symm} = \Psi_N^{\rm symm}(\{x_i\}|\{p_i\})$.

The normalization constant C_N can be defined from the normalization condition

$$\int_{\mathbb{R}^N} dx_1 \dots dx_N \Psi_N^{\rm symm}(\{x_i\}|\{p_i\}) \Psi_N^{\rm symm}(\{x_i\}|\{p'_i\}) = (2\pi)^N \prod_{j=1}^N \delta(p_i - p'_i). \tag{2.63}$$

The condition of completeness is also valid:

$$\int_{\mathbb{R}^N} dp_1 \dots dp_N \Psi_N^{\rm symm}(\{x_i\}|\{p_i\}) \Psi_N^{\rm symm}(\{x'_i\}|\{p_i\}) = (2\pi)^N \prod_{j=1}^N \delta(x_i - x'_i). \tag{2.64}$$

Problem. Show that the normalization constant C_N is given by

$$C_N = \left\{ N! \prod_{j<k} \left[(p_j - p_k)^2 + c^2 \right] \right\}^{-1/2} \tag{2.65}$$

and check the normalization condition and the condition of completeness.

The limiting case $c \to \infty$ corresponds to the system of impenetrable Bose particles. In this case the wave function in the sector $x_1 < \dots < x_N$ coincides with the wave function of N free fermions $\det_{jk}(e^{ip_j x_k})$, and $\Psi^{\rm symm} \propto \det_{jk}(e^{ip_j x_k}) \prod_{j<k} \operatorname{sign}(x_j - x_k)$.

So, the exact eigenfunctions of the N-particle Hamiltonian (2.52) (Bethe wave functions) have the form (2.61), (2.60), or, in the symmetrized version (2.62). They are parametrized by N numbers $p_1, \dots, p_N$. The momentum and energy for them are

$$P = \sum_{j=1}^N p_j \,, \qquad E = \sum_{j=1}^N p_j^2. \tag{2.66}$$

Note that from (2.60) itis clear that the wave function vanishes if there is a pair of indices $j \neq k$ such that $p_j = p_k$ (the Pauli principle for one-dimensional bosons).

2.2.2 Bethe equations

Imposing periodic boundary conditions on a segment of length L leads to constraints for possible values of the parameters p_j. Arguments similar to the ones used for the derivation of Bethe equations in the spin chain lead to the condition

$$\boxed{\begin{array}{c} \Psi(x_1, x_2, \dots, x_N) = \Psi(x_2, x_3, \dots x_N, x_1 + L) \\ \\ \text{in the sector} \quad x_1 < x_2 < \dots < x_N. \end{array}} \tag{2.67}$$

It is equivalent to the system of Bethe equations for the p_j's:

$$e^{ip_j L} = \prod_{k \neq j} \frac{p_j - p_k + ic}{p_j - p_k - ic} \,, \qquad j = 1, \dots, N. \tag{2.68}$$

One can show that contrary to the case of the spin chains, all their solutions for $c > 0$ are real.

Problem. For the system of three identical Bose particles with the Hamiltonian

$$\hat{H}_3 = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le 3} \delta(x_j - x_k)$$

a) construct common eigenstates of the Hamiltonian and the momentum operator $\hat{P} = -i \sum_{j=1}^{3} \frac{\partial}{\partial x_j}$,

b) impose periodic boundary conditions on the segment $[0, L]$ and obtain the Bethe equations.

Problem. Find eigenstates and the energy spectrum for two Bose particles with the Hamiltonian

$$\hat{H}_2 = -\sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2} + 2c\delta(x_j - x_k)$$

on the segment $[0, L]$ with impenetrable ends (this means that the wave function vanishes if at least one of the particles is at the ends of the segment).

Note that after the substitution $p_j = \lambda_j$ the right hand sides of the Bethe equations for the XXX spin chain and the Bose gas coincide. In fact this is a particular case of a general rule that possible form of the right hand sides of Bethe equations is determined by general laws of integrability (which are still behind the scene) and allows one to classify quantum integrable models while the left hand sides may contain arbitrary functions of λ which determine concrete models within a given class. So, we see that the XXX spin chain and the Bose gas belong to the same class while the XXZ spin chain to another one. An analogy with representation theory is appropriate: the right hand sides of Bethe equations are similar to structure constants of an algebra and left hand sides to a particular representation.

Having this in mind, we will write the Bethe equations for the Bose gas in terms of λ:

$$\boxed{e^{i\lambda_j L} = \prod_{k \neq j} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}, \qquad j = 1, \ldots, N.} \tag{2.69}$$

In the Bose gas model the parameter λ is connected with momentum in the most simple way.

For the analysis of these equations it is useful to take logarithm:

$$L\lambda_j + \sum_{k \neq j} \tilde{\Phi}(\lambda_j - \lambda_k) = 2\pi \tilde{n}_j. \tag{2.70}$$

Here $\tilde{\Phi}(\lambda) = i \log \frac{\lambda + ic}{\lambda - ic}$, and $\tilde{n}_j$ are some integer numbers. Instead of $\tilde{\Phi}(\lambda)$ it is convenient to introduce the odd function

$$\Phi(\lambda) = \tilde{\Phi}(\lambda) + \pi = 2\mathrm{arctg}\big(\lambda/c\big) \tag{2.71}$$

and redefine the numbers $\tilde{n}_j$ as follows: $n_j = \tilde{n}_j + \frac{1}{2}(N-1)$ (at even N they are half-integer). Our equations will have the form

$$\boxed{L\lambda_j + \sum_k \Phi(\lambda_j - \lambda_k) = 2\pi n_j} \tag{2.72}$$

(note that one can extend the sum to all values of k because $\Phi(0) = 0$). the representation in the form (2.72) is better because one can prove (see below) that solutions of this system such that $\lambda_j \neq \lambda_k$ at $j \neq k$ exist and are uniquely defined by sets of integer or half-integer distinct numbers n_j.

Lemma. If $n_j > n_k$, then $\lambda_j > \lambda_k$; if $n_j = n_k$, then $\lambda_j = \lambda_k$.

For the proof consider the difference of two equations of the system:

$$L(\lambda_j - \lambda_k) + 2\sum_{l=1}^{N}\Big(\operatorname{arctg}\left((\lambda_j - \lambda_l)/c\right) - \operatorname{arctg}\left((\lambda_k - \lambda_l)/c\right)\Big) = 2\pi(n_j - n_k).$$

Since arctg is a monotonically increasing function, the left hand side is positive if and only if $\lambda_j > \lambda_k$ and is zero if and only if $\lambda_j = \lambda_k$. Therefore, at $n_j > n_k$ one should have $\lambda_j > \lambda_k$, and at $n_j = n_k$ $\lambda_j = \lambda_k$.

Lemma. The energy $E = \sum \lambda_j^2$ in the model with N particles is minimal in the state with the following set of n_j:

$$n_j = -\frac{N+1}{2} + j\,, \qquad j = 1, 2, \ldots, N. \tag{2.73}$$

In other words, in the ground state the numbers n_j fill the interval from $-\frac{1}{2}(N-1)$ to $+\frac{1}{2}(N-1)$ without holes. This is more or less clear from the symmetry argument but one can find a rigorous proof in the literature.

Problem. Prove that in the state which is characterized by the numbers n_j the total momentum is equal to $P = \dfrac{2\pi}{L}\sum_j n_j$.

2.2.3 The Yang function

Let us move all terms in the Bethe equations (2.72) to the left hand side and denote

$$\mathcal{B}_j(\lambda_1, \ldots, \lambda_N) \equiv L\lambda_j - 2\pi n_j + \sum_k \Phi(\lambda_j - \lambda_k),$$

then the Bethe equations state that $\mathcal{B}_j = 0$.

Problem. Show that

$$\frac{\partial \mathcal{B}_j}{\partial \lambda_l} = \frac{\partial \mathcal{B}_l}{\partial \lambda_j}. \tag{2.74}$$

This implies that there exists a function $\mathcal{Y}(\lambda_1, \ldots, \lambda_N)$ such that $\mathcal{B}_j = \partial\mathcal{Y}/\partial\lambda_j$. It is called the Yang function and plays an important role. The Bethe equations are conditions that the Yang function has an extremum: $\partial\mathcal{Y}/\partial\lambda_j = 0$. For the Bose gas model

$$\mathcal{Y} = \frac{L}{2}\sum_{j=1}^{N}\lambda_j^2 - 2\pi\sum_{j=1}^{N} n_j\lambda_j + \frac{1}{2}\sum_{j,k=1}^{N}\Phi_1(\lambda_j - \lambda_k), \tag{2.75}$$

where

$$\Phi_1(\lambda) = \int_0^\lambda \Phi(\mu)d\mu = 2\lambda \operatorname{arctg}(\lambda/c) - c\log\Big(1 + \frac{\lambda^2}{c^2}\Big).$$

In fact the possibility to represent the Bethe equations as a variational principle is a general fact and exists also for other models. The Bose gas is distinguished by the fact that in this case the Yang function has especially good properties which allow one to prove rigorously the following important statement.

Theorem. Solutions of the system (2.72) exist and are uniquely defined by sets of integer or half-integer numbers n_j.

As was already said, the Bethe equations (2.72) are obtained as conditions for an extremum of the Yang function. The statement of the theorem follows from the fact that the Yang function is convex; thus the extremum is a minimum and it is unique. To prove that the Yang function is convex, it is enough to show that the matrix $\mathcal{Y}_{jk} = \partial^2 \mathcal{Y}/\partial\lambda_j\partial\lambda_k$ (Hessian) is positively defined, i.e., all its eigenvalues are positive. This matrix has the form

$$\mathcal{Y}_{jk} = \delta_{jk}\Big(L + \sum_{l=1}^{N} K(\lambda_j - \lambda_l)\Big) - K(\lambda_j - \lambda_k),$$

where

$$K(\lambda - \mu) = \Phi'(\lambda - \mu) = \frac{2c}{(\lambda - \mu)^2 + c^2}\,. \tag{2.76}$$

For any real vector v_j we have

$$\sum_{jk} \mathcal{Y}_{jk} v_j v_k = L \sum_j v_j^2 + \frac{1}{2} \sum_{k \neq j} K(\lambda_j - \lambda_k)(v_j - v_k)^2 > 0$$

(here it is important that $c > 0$). This means that the matrix $\mathcal{Y}_{jk}$ is positively defined for all λ_j.

For XXX and XXZ spin chain the Yang function is in general not convex and rigorous proofs of similar statements are problematic.

The matrix $\mathcal{Y}_{jk}$ is also important for the following reason. The squared norm of the Bethe state of the Bose gas in a finite volume is expressed through its determinant. Namely, for the function Ψ_N^{symm} (2.62) with the constant C_N as in (2.65) and momenta $p_j = \lambda_j$ satisfying the Bethe equations it holds

$$\int_0^L \ldots \int_0^L |\Psi_N^{\mathrm{symm}}|^2 \, dx_1 \ldots dx_N = \det_{1 \leq j,k \leq N} \mathcal{Y}_{jk}. \tag{2.77}$$

We stress that this formula is valid *only* for the states in which the parameters λ_j satisfy the Bethe equations. A direct verification of this formula is non-trivial already in the case $N = 2$.

Problem. Find the explicit form of the Yang function for the XXX spin chain.

2.2.4 Solution of Bethe equations in the thermodynamic limit

The thermodynamic limit is the limit $N \to \infty$, $L \to \infty$ such that $\rho_0 = N/L$ (the mean density of the particles) remains finite.

The ground state. Let us begin with the ground state (the vacuum state). According to the above lemma, the numbers n_j should be as in (2.73). The total momentum is zero. The numbers n_j and λ_j are placed symmetrically with respect to 0 forming an analog of the Dirac sea. In the limit the numbers λ_j densely fill an interval $(-\Lambda, \Lambda)$. Their density is given by the formula

$$\rho(\lambda_j) = \lim_{N,L\to\infty} \frac{1}{L(\lambda_{j+1} - \lambda_j)}. \tag{2.78}$$

This function allows one to substitute sums by integrals according to the rule

$$\sum_j f(\lambda_j) = L \int_{-\Lambda}^{\Lambda} f(\lambda)\rho(\lambda)d\lambda.$$

take to neighboring equations (2.72) (for $j+1$ and j) and subtract them:

$$L(\lambda_{j+1} - \lambda_j) + \sum_k [\Phi(\lambda_{j+1} - \lambda_k) - \Phi(\lambda_j - \lambda_k)] = 2\pi.$$

The difference $\lambda_{j+1} - \lambda_j$ is small and we can expand the expression in the Taylor series up to the first term. After that, divide both sides by $L(\lambda_{j+1} - \lambda_j)$:

$$1 + \frac{1}{L}\sum_k \Phi'(\lambda_j - \lambda_k) = \frac{2\pi}{L(\lambda_{j+1} - \lambda_j)}. \tag{2.79}$$

Substituting the sum by the integral, recalling the definition of density (2.78) and the function $K(\lambda - \mu)$ (2.76), we obtain the following integral equation for the density function:

$$\boxed{2\pi\rho(\lambda) - \int_{-\Lambda}^{\Lambda} K(\lambda - \mu)\rho(\mu)d\mu = 1.} \tag{2.80}$$

A more formal way to derive this equation is to define the density function by the formula $\rho(\lambda) = \frac{1}{L}\sum_{j=1}^{N} \delta(\lambda - \lambda_j)$ and to represent (2.72) as an integral equation

$$\lambda + \int_{-\Lambda}^{\Lambda} \Phi(\lambda - \mu)\rho(\lambda)d\lambda = \frac{2\pi n(\lambda)}{L}$$

from the very beginning. Here $n(\lambda) = \sum_j \theta(\lambda - \lambda_j)$ is a non-decreasing step function which takes into account how many momenta are places from the left of the point λ. Taking the λ-derivative, we arrive at (2.80). The singular function $\rho(\lambda)$ becomes continuous in the limit.

Contrary to the XXX spin chain, the solution to the integral equation can not be expressed through known functions (because of the fact that the limits of integration are finite) but can be effectively found numerically. One can also prove some rigorous statements about the solution. We will not discuss this point here.

The quantity Λ (the Fermi momentum) is determined from the normalization condition

$$\rho_0 = \frac{N}{L} = \int_{-\Lambda}^{\Lambda} \rho(\lambda)d\lambda. \tag{2.81}$$

The ground state energy is given by the formula

$$E^{(0)} = L \int_{-\Lambda}^{\Lambda} \lambda^2 \rho(\lambda)d\lambda. \tag{2.82}$$

Elementary excitations. Let us show, on the simplest example, how the Bethe ansatz technique allows one to construct excited states (eigenstates of the Hamiltonian with energy $E > E^{(0)}$). Physically the most interesting are low-lying excitations, i.e. such that $E - E^{(0)}$ remains finite as $L \to \infty$. They are interpreted as physically observable "dressed" particles and their scattering states. Original particles, in terms of which the Hamiltonian was written, are called "bare" particles.

Let us consider the states with a fixed number of particles. The simplest excitations correspond to the choice of the numbers n_j as follows:

$$\{n_j\} = \Big\{ -\tfrac{N-1}{2}, -\tfrac{N-3}{2}, \dots, \tfrac{N-1}{2} - m^h - 1, \tfrac{N-1}{2} - m^h + 1, \dots, \tfrac{N-1}{2}, \tfrac{N-1}{2} + m^p \Big\}.$$

Here m^h and m^p are positive integer numbers. This can be thought of as creation of a "particle" at m^p and a "hole" at m^h (the latter means that the number $\frac{N-1}{2} - m^h$ is absent in the sequence). Assume that $m^h < \frac{N-1}{2}$. Accordingly, one can introduce the momentum of the added particle λ^p as the Bethe root corresponding to the added munber $\frac{N-1}{2} + m^p$ and the momentum of the hole λ^h as the Bethe root corresponding to the number $\frac{N-1}{2} - m^h$ in the vacuum solution.

It is important to note that the total momentum of the excited state is not equal to $\lambda^p - \lambda^h$ because in the new solution of Bethe equations the values of all other parameters λ_j are slightly shifted comparing to their vacuum values. But since there are many of them (of order $O(N)$), the total contribution of such shifts can be $O(1)$, i.e., of the same order as $\lambda^p - \lambda^h$. Physicists say that $\lambda^p - \lambda^h$ is a "bare" momentum while the observed ("dressed") momentum is due to interaction. Note that at $c = \infty$ they coincide.

In order to find the response of the "Dirac sea" to creation of the pair particle-hole, we subtract the Bethe equations for the excited and vacuum states and expand in small $\delta\lambda_j = \tilde{\lambda}_j - \lambda_j = O(1/L)$. Here λ_j is jth Bethe root for the ground state and $\tilde{\lambda}_j$ is jth Bethe root for the excited state. We obtain, for $1 \le j \le N - m^h - 1$:

$$(\tilde{\lambda}_j - \lambda_j)L + \sum_k \Big[\Phi(\tilde{\lambda}_j - \tilde{\lambda}_k) - \Phi(\lambda_j - \lambda_k) \Big] = \Phi(\lambda_j - \lambda^h) - \Phi(\tilde{\lambda}_j - \lambda^p).$$

The terms with integer numbers n_j cancel because they are the same for all such j. The terms corresponding to the particle and the hole are moved to the right hand side. After expanding in small $\delta\lambda_j = \tilde{\lambda}_j - \lambda_j$, these equations become (in the leading order):

$$\delta\lambda_j L + \delta\lambda_j \sum_k \Phi'(\lambda_j - \lambda_k) - \sum_k \Phi'(\lambda_j - \lambda_k)\delta\lambda_k = \Phi(\lambda_j - \lambda^h) - \Phi(\lambda_j - \lambda^p),$$

where in the right hand side we changed $\tilde{\lambda}_j$ to λ_j because the difference appears only in the next order. The first two terms in the left hand side can be transformed using (2.79):

$$2\pi \frac{\tilde{\lambda}_j - \lambda_j}{\lambda_{j+1} - \lambda_j} - \frac{1}{L} \sum_k K(\lambda_j - \lambda_k) \frac{\tilde{\lambda}_k - \lambda_k}{\lambda_{k+1} - \lambda_k} L(\lambda_{k+1} - \lambda_k) = \Phi(\lambda_j - \lambda^h) - \Phi(\lambda_j - \lambda^p).$$

Let us introduce the *shift function*

$$F(\lambda_j | \lambda^h, \lambda^p) = \lim_{N,L \to \infty} \frac{\tilde{\lambda}_j - \lambda_j}{\lambda_{j+1} - \lambda_j} = \lim_{N,L \to \infty} \Big(L \delta\lambda_j \rho(\lambda_j) \Big) \tag{2.83}$$

and write the result as the integral equation

$$\boxed{2\pi F(\lambda|\lambda^h,\lambda^p) - \int_{-\Lambda}^{\Lambda} K(\lambda-\mu)F(\mu|\lambda^h,\lambda^p)d\mu = \Phi(\lambda-\lambda^h) - \Phi(\lambda-\lambda^p).} \tag{2.84}$$

Here originally $\lambda^h < \Lambda$, $\lambda^p > \Lambda$ but the solution can be analytically continued to the whole real axis. From the right hand side it follows that

$$F(\lambda|\lambda^h,\lambda^p) = f(\lambda|\lambda^p) - f(\lambda|\lambda^h),$$

where $f(\lambda|\mu)$ satisfies the integral equation

$$2\pi f(\lambda|\mu) - \int_{-\Lambda}^{\Lambda} K(\lambda-\nu)f(\nu|\mu)d\nu = -\Phi(\lambda-\mu). \tag{2.85}$$

From the fact that $\Phi(\lambda)$ is an odd function, it easily follows that

$$f(-\lambda|-\mu) = -f(\lambda|\mu). \tag{2.86}$$

The function $f(\lambda|\mu)$ will be also called the shift function. Its physical meaning will be clarified below.

The excited state energy is

$$\begin{aligned} E - E^{(0)} &= (\lambda^p)^2 - (\lambda^h)^2 + \sum_j(\tilde{\lambda}_j^2 - \lambda_j^2) = (\lambda^p)^2 - (\lambda^h)^2 + 2\sum_j \lambda_j\delta\lambda_j \\ &= (\lambda^p)^2 - (\lambda^h)^2 + 2\int_{-\Lambda}^{\Lambda} \lambda F(\lambda|\lambda^h,\lambda^p)d\lambda = \varepsilon(\lambda^p) - \varepsilon(\lambda^h), \end{aligned} \tag{2.87}$$

where

$$\varepsilon(\lambda) = \lambda^2 - h + 2\int_{-\Lambda}^{\Lambda} \mu f(\mu|\lambda)d\mu, \tag{2.88}$$

and the constant h is choisen from the condition that $\varepsilon(\pm\Lambda) = 0$. Note that $\varepsilon(\lambda)$ is an even function because of (2.86). The momentum is

$$P = \lambda^p - \lambda^h + \sum_j(\tilde{\lambda}_j - \lambda_j) = \lambda^p - \lambda^h + \int_{-\Lambda}^{\Lambda} F(\lambda|\lambda^h,\lambda^p)d\lambda. \tag{2.89}$$

Multiplying both sides of equation (2.84) by $\rho(\lambda)$ and integrating over λ, we get

$$\int_{-\Lambda}^{\Lambda} F(\lambda|\lambda^h,\lambda^p)d\lambda = \int_{-\Lambda}^{\Lambda}\Big(\Phi(\lambda-\lambda^h) - \Phi(\lambda-\lambda^p)\Big)\rho(\lambda)d\lambda,$$

hence we obtain another formula for the excited state momentum:

$$P = \lambda^p - \lambda^h + \int_{-\Lambda}^{\Lambda}\Big(\Phi(\lambda-\lambda^h) - \Phi(\lambda-\lambda^p)\Big)\rho(\lambda)d\lambda. \tag{2.90}$$

The integral term is the “dressed” part.

Consider now the case of several particles and holes. So far we have assumed that the numbers of particles and holes are the same. This restriction can be removed if one does not consider the number N of bare particles as fixed. Suppose there are m particles and

n holes with the parameters λ_a^p, λ_b^h. The shift function is introduced in a way similar to (2.83):

$$F(\lambda_j|\{\lambda_b^h\},\{\lambda_a^p\}) = \lim_{N,L\to\infty} \frac{\lambda'_j - \lambda_j}{\lambda_{j+1} - \lambda_j}, \tag{2.91}$$

where now λ'_j are Bethe roots in the presense of m particles and n holes. The integral equation for this shift function can be derived in a similar way:

$$2\pi F(\lambda|\{\lambda_a^h\},\{\lambda_a^p\}) - \int_{-\Lambda}^{\Lambda} K(\lambda-\mu)F(\mu|\{\lambda_a^h\},\{\lambda_a^p\})d\mu = \sum_{b=1}^{n} \Phi(\lambda-\lambda_b^h) - \sum_{a=1}^{m} \Phi(\lambda-\lambda_a^p). \tag{2.92}$$

The solution is

$$F(\lambda|\{\lambda_b^h\},\{\lambda_a^p\}) = \sum_{a=1}^{m} f(\lambda|\lambda_a^p) - \sum_{b=1}^{n} f(\lambda|\lambda_b^h). \tag{2.93}$$

The velocity of sound. Consider the one-particle excitation in more detail. The excitations constructed above correspond to massless particles because they have the linear dispersion law $E = v_s P$ as $P \to 0$, where v_s has the meaning of velocity of sound. In order to see this and to obtain a formula for the velocity of sound, some calculations are necessary. According to the definition,

$$v_s = \left.\frac{\partial E}{\partial P}\right|_{P=0} = \left.\frac{\varepsilon'(\lambda)}{P'(\lambda)}\right|_{\lambda=\Lambda}, \tag{2.94}$$

where P is regarded as a function of $\lambda = \lambda^p$ (see (2.90)). From (2.90) we have:

$$P'(\Lambda) = 1 + \int_{-\Lambda}^{\Lambda} K(\Lambda-\lambda)\rho(\lambda)d\lambda = 2\pi\rho(\Lambda). \tag{2.95}$$

Differentiating equation (2.80) and then integrating by parts, we find:

$$2\pi\rho'(\lambda) = \int_{-\Lambda}^{\Lambda} K(\lambda-\mu)\rho'(\mu)d\mu - \rho(\Lambda)\Big(K(\lambda-\Lambda) - K(\lambda+\Lambda)\Big) \tag{2.96}$$

(here we take into account that $\rho(\lambda)$ is an even function). Differentiating equation (2.85) with respect to μ, we obtain:

$$K(\lambda-\mu) = 2\pi\dot{f}(\lambda|\mu) - \int_{-\Lambda}^{\Lambda} K(\lambda-\nu)\dot{f}(\nu|\mu)d\nu,$$

where

$$\dot{f}(\lambda|\mu) = \frac{\partial f(\lambda|\mu)}{\partial\mu}.$$

Substitute this into (2.96) instead of $K(\lambda \pm \Lambda)$. In this way we get the homogeneous integral equation

$$2\pi g(\lambda) = \int_{-\Lambda}^{\Lambda} K(\lambda-\mu)g(\mu)d\mu$$

for the function $g(\lambda) = \rho'(\lambda) + \rho(\Lambda)(\dot{f}(\lambda|\Lambda) - \dot{f}(\lambda|-\Lambda))$, which has the solution $g(\lambda) = 0$, i.e.,

$$\rho'(\lambda) = -\rho(\Lambda)\Big(\dot{f}(\lambda|\Lambda) - \dot{f}(\lambda|-\Lambda)\Big). \tag{2.97}$$

Take the integral in the normalization condition (2.81) by parts,

$$\rho_0 = 2\Lambda\rho(\Lambda) - \int_{-\Lambda}^{\Lambda} \lambda\rho'(\lambda)d\lambda,$$

and substitute here $\rho'(\lambda)$ from (2.97). We obtain:

$$\rho_0 = 2\Lambda\rho(\Lambda) + \rho(\Lambda)\int_{-\Lambda}^{\Lambda} \Big(\lambda\dot{f}(\lambda|\Lambda) - \lambda\dot{f}(\lambda|-\Lambda)\Big)d\lambda$$

$$= \rho(\Lambda)\left(2\Lambda + 2\int_{-\Lambda}^{\Lambda} \lambda\dot{f}(\lambda|\Lambda)d\lambda\right)$$

(the second equality follows from $\dot{f}(\lambda|-\Lambda) = \dot{f}(-\lambda|\Lambda)$ by virtue of (2.86)). Comparing this with (2.88), we conclude that $\rho_0 = \rho(\Lambda)\varepsilon'(\Lambda)$, and so for the velocity of sound we find the formula

$$v_s = \frac{\rho_0}{2\pi\rho^2(\Lambda)} \tag{2.98}$$

(see (2.94), (2.95)).

Dressing equation for the function $\varepsilon(\lambda)$. The function $\varepsilon(\lambda)$ (2.88) satisfies a linear integral equation. Let us derive it. First derive the equation for the derivative $\varepsilon'(\lambda)$. We have, differentiating the definition (2.88):

$$\varepsilon'(\lambda) = 2\lambda + 2\int_{-\Lambda}^{\Lambda} \mu\dot{f}(\mu|\lambda)d\mu, \tag{2.99}$$

where the function $\dot{f}$ satisfies the equation

$$\dot{f}(\lambda|\mu) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} K(\lambda-\nu)\dot{f}(\nu|\mu)d\nu = \frac{1}{2\pi}K(\lambda-\mu)$$

(see (2.85)). The solution can be written in the form

$$\dot{f}(\lambda|\mu) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} R(\lambda,\nu)K(\nu-\mu)d\nu, \tag{2.100}$$

where $R(\lambda,\nu)$ is the kernel of the integral operator inverse to $I - \frac{1}{2\pi}\hat{K}$ with the kernel $\delta(\lambda-\nu) - \frac{1}{2\pi}K(\lambda-\nu)$. Since the kernel $K(\lambda-\mu)$ is symmmetric, the kernel $R(\lambda,\mu)$ is also symmetric. By definition,

$$\int_{-\Lambda}^{\Lambda} R(\lambda,\nu)\left(\delta(\nu-\mu) - \frac{1}{2\pi}K(\nu-\mu)\right)d\nu = \delta(\lambda-\mu),$$

hence, comparing this with (2.100), we find:

$$\dot{f}(\lambda|\mu) = R(\lambda,\mu) - \delta(\lambda-\mu).$$

Now let us plug this into (2.99):

$$\varepsilon'(\lambda) = 2\int_{-\Lambda}^{\Lambda} R(\lambda,\mu)\mu d\mu,$$

which is equivalent to the integral equation

$$\varepsilon'(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} K(\lambda-\mu)\varepsilon'(\mu)d\mu = 2\lambda,$$

or, integrating by parts,

$$\varepsilon'(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} K'(\lambda-\mu)\varepsilon(\mu)d\mu = 2\lambda \tag{2.101}$$

(recall that $\varepsilon(\pm\Lambda) = 0$). From comparison of (2.85) at $\mu \to \pm\infty$ and (2.80) it follows that $f(\lambda|\pm\infty) = \pm\pi\rho(\lambda)$, and because $\rho(\lambda)$ is an even function, from (2.88) it follows that

$$\varepsilon(\lambda) = \lambda^2 - h + o(1) \quad \text{as } \lambda \to \infty.$$

Therefore, integrating (2.101), we get the integral equation for $\varepsilon(\lambda)$:

$$\boxed{\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} K(\lambda-\mu)\varepsilon(\mu)d\mu = \lambda^2 - h.} \tag{2.102}$$

Sometimes this equation is called the *dressing equation* because the energy of a "bare" particle λ^2 is "dressed" to the energy of a physical ("dressed") particle by means of the integral operator with the kernel $K(\lambda-\mu)$. Similarly, the integral equation (2.80) with the same kernel dresses the density $\frac{1}{2\pi}$ of non-interacting particles to the density $\rho(\lambda)$.

Scattering of physical particles. The physical particles do not have internal degrees of freedom and their interaction is reduced to a phase shift. Using simple arguments, one can find this phase shift and to give a physical meaning to the shift function $f(\lambda|\mu)$.

Let us calculate the phase shift $\Delta(\lambda_2, \lambda_1)$ for scattering of two particles with parameters $\lambda_1^p = \lambda_1, \lambda_2^p = \lambda_2$. It is equal to the difference of the phase φ_{21} which the second particle acquires when moves from 0 to L in the presence of the first one and the phase φ_2 which the second particle would acquire without the first one: $\Delta(\lambda_2, \lambda_1) = \varphi_{21} - \varphi_2$. We have:

$$\varphi_2 = L\lambda_2 + \sum_k \Phi(\lambda_2 - \tilde{\lambda}_k),$$

$$\varphi_{21} = L\lambda_2 + \sum_k \Phi(\lambda_2 - \lambda'_k) + \Phi(\lambda_2 - \lambda_1),$$

where $\tilde{\lambda}_k$ are Bethe roots with the presence of only the second particle while λ'_k are Bethe roots in the presence of both particles. Then

$$\Delta(\lambda_2, \lambda_1) = \Phi(\lambda_2 - \lambda_1) + \sum_k \Big(\Phi(\lambda_2 - \lambda'_k) - \Phi(\lambda_2 - \tilde{\lambda}_k)\Big)$$

$$= \Phi(\lambda_2 - \lambda_1) - \sum_k K(\lambda_2 - \lambda_1)\Big(F(\lambda_k|\{\lambda_1, \lambda_2\}) - F(\lambda_k|\lambda_2)\Big)(\lambda_{k+1} - \lambda_k).$$

Substituting the sum by the integral and using (2.93), we obtain:

$$\Delta(\lambda_2, \lambda_1) = \Phi(\lambda_2 - \lambda_1) - \int_{-\Lambda}^{\Lambda} K(\lambda_2 - \mu) f(\mu|\lambda_1) d\mu. \tag{2.103}$$

Comparing this with (2.85), we find:

$$\Delta(\lambda_2, \lambda_1) = -2\pi f(\lambda_2|\lambda_1), \tag{2.104}$$

which gives a physical meaning to the function $f(\lambda|\mu)$ and justifies its name "shift function". Note that the equation (2.85) can be interpreted as dressing equation for the phase shift which dresses the bare phase shift $\Phi(\lambda_2 - \lambda_1)$ to the phase shift $\Delta(\lambda_2 - \lambda_1)$ of physical particles.

2.2.5 Thermodynamics of the model at finite temperature

Particles and holes. The bijection between states and sets of distinct integer or half-integer numbers $\{n_j\}_{j=1}^N$ allows for a relatively simple analysis of thermodynamics of the model at finite temperature. For simplicity we assume that n_j are integer numbers.

Let a set of real numbers $\{\lambda_j\}_{j=1}^N$ be a solution of Bethe equations. Consider the function

$$y(\lambda) = \lambda + \frac{1}{L}\sum_{k=1}^{N} \Phi(\lambda - \lambda_k). \tag{2.105}$$

It monotonically increases with λ and $y(\pm\infty) = \pm\infty$. Let $\{\bar{n}_j\}$ be the set of integer numbers complimentary to $\{n_j\}_{j=1}^N$: $\{\bar{n}_j\} = \mathbb{Z} \setminus \{n_j\}_{j=1}^N$. The following terminology is convenient.

- Those $\lambda_j \in \mathbb{R}$ for which $y(\lambda_j) = \dfrac{2\pi n_j}{L}$ are called *particles*;

- Those $\lambda_k \in \mathbb{R}$ for which $y(\lambda_k) = \dfrac{2\pi \bar{n}_k}{L}$ are called *holes*.

Solutions to the equation $y(\lambda) \in 2\pi\,\mathbb{Z}/L$ (for given $\{n_j\}_{j=1}^N$) are sometimes called vacancies. Obviously, a vacancy is either a particle or a hole. Particles are sometimes called occupied vacancies and holes are called free vacancies.

The density of particles ρ is defined by equation (2.78) as before. A similar density function $\bar{\rho}$ can be introduced for holes. At zero temperature, there were no holes in the interval from $-\Lambda$ to Λ or the number of holes was finite. That is why the function $\bar{\rho}$ is of no use at zero temperature. At finite temperature both functions are nontrivial on the whole real axis. So, we have:

- Density of particles: $L\rho(\lambda)d\lambda$ is the number of particles on the segment $[\lambda, \lambda + d\lambda]$;

- Density of holes: $L\bar{\rho}(\lambda)d\lambda$ is the number of holes on the segment $[\lambda, \lambda + d\lambda]$.

The density of vacancies is then $\rho(\lambda) + \bar{\rho}(\lambda)$. From the definitions it immediately follows that $y'(\lambda) = 2\pi(\rho(\lambda) + \bar{\rho}(\lambda))$. Differentiating (2.105), we get the integral equation which connects densities of particles and holes:

$$\boxed{2\pi(\rho(\lambda) + \bar{\rho}(\lambda)) - \int K(\lambda - \mu)\rho(\mu)\, d\mu = 1.} \tag{2.106}$$

Hereafter, the integration is from $-\infty$ to ∞. Recall that $K(\lambda) = \dfrac{2c}{\lambda^2 + c^2}$. The meaning of this equation is similar to Bethe equations in the form (2.72): given the density of holes $\bar\rho$ (this is analogous to the choice of a certain sequence of the numbers n_j in (2.72)), the density of particles ρ can be found from the integral equation.

Macroscopic description: entropy. The method of statistical thermodynamics consists of not tracking individual states (by means of the numbers n_j) but of passing to description in terms of densities of particles and holes. To a given macroscopic state (a fixed function ρ) there correspond many different microscopic states (sets of the numbers n_j). Indeed, there are

$$\delta\mathcal{N}(\lambda) = \frac{\big[L(\rho(\lambda) + \bar\rho(\lambda)d\lambda]!}{\big[L\rho(\lambda)d\lambda\big]!\,\big[L\bar\rho(\lambda)d\lambda\big]!}$$

possibilities to place $L\rho(\lambda)d\lambda$ particles in $L(\rho(\lambda)+\bar\rho(\lambda)d\lambda$ vacancies, with fixed macroscopic density functions. In the thermodynamic limit it is a large number. The Stirling formula $\log n! = n\log n - n + \ldots$ yields the contribution to the entropy $\delta S_N(\lambda) = \log \delta\mathcal{N}(\lambda)$:

$$\delta S_N(\lambda) = \Big[(\rho(\lambda) + \bar\rho(\lambda))\log(\rho(\lambda) + \bar\rho(\lambda)) - \rho(\lambda)\log\rho(\lambda) - \bar\rho(\lambda)\log\bar\rho(\lambda)\Big]L d\lambda.$$

The total entropy is calculated as

$$S_N = \int \delta S_N d\lambda = L\int \Big[(\rho + \bar\rho)\log(\rho + \bar\rho) - \rho\log\rho - \bar\rho\log\bar\rho\Big]d\lambda. \qquad (2.107)$$

Integral equation. In the thermodynamic limit the total energy

$$E_N = L\int \lambda^2\rho(\lambda)\,d\lambda \qquad (2.108)$$

depends only on the macroscopic density ρ. This allows one to pass in the partition function from summation over n_j to a "functional integration" over $\rho(\lambda)$, taking into account the entropy:

$$Z_N = \sum_{n_1<n_2<\ldots<n_N} e^{-\beta E_N(\{n_j\})} = \int [D\rho]\, e^{S_N - \beta E_N}.$$

Here $\beta = 1/T$ is the inverse temperature. As usual in statistical thermodynamics, when $N\to\infty$ the main contribution comes from the states for which $e^{S_N-\beta E_N}$ has a maximum. In other words, we should find extremum off the functional

$$S_N - \beta E_N = -L\int d\lambda\,\Big[\beta\lambda^2\rho - (\rho + \bar\rho)\log(\rho + \bar\rho) + \rho\log\rho + \bar\rho\log\bar\rho\Big]$$

of density ρ under the condition that the main density is kept constant:

$$\int \rho(\lambda)\,d\lambda = N/L = \rho_0. \qquad (2.109)$$

Introducing the Lagrange multiplier βh and varying over ρ with the condition (2.106), we have:

$$\int d\lambda\,\Big[\beta(\lambda^2 - h)\delta\rho + \delta\rho\log\rho + \delta\bar\rho\log\bar\rho - (\delta\rho+\delta\bar\rho)\log(\rho+\bar\rho)\Big] = 0$$

for all $\delta\rho$. After simple transformations we obtain the condition that the variation is zero:

$$\beta(\lambda^2 - h) + \log\frac{\rho(\lambda)}{\bar{\rho}(\lambda)} - \frac{1}{2\pi}\int K(\lambda-\mu)\log\Big(1 + \frac{\rho(\mu)}{\bar{\rho}(\mu)}\Big)\,d\mu = 0. \tag{2.110}$$

Let us introduce the notation

$$\frac{\bar{\rho}(\lambda)}{\rho(\lambda)} := e^{\beta\varepsilon(\lambda)}, \tag{2.111}$$

then the above condition takes the form

$$\boxed{\varepsilon(\lambda) = \lambda^2 - h - \frac{1}{2\pi\beta}\int K(\lambda-\mu)\log\Big(1 + e^{-\beta\varepsilon(\mu)}\Big)\,d\mu.} \tag{2.112}$$

It is a nonlineaar integral equation for the function $\varepsilon(\lambda)$, in which h is a parameter. The solution scheme of the original problem is as follows: substitute $\varepsilon(\lambda)$ determined by this integral equation (parametrically depending on h) into the relation (2.106), which becomes a Fredholm integral equation for ρ:

$$2\pi\Big(1 + e^{\beta\varepsilon(\lambda)}\Big)\,\rho(\lambda) = 1 + \int K(\lambda-\mu)\rho(\mu)\,d\mu. \tag{2.113}$$

The solution ρ should be substituted into the normalization condition (2.109), which gives ρ_0 from a given h. The functions ρ and ε are enough (in principle) to find all thermodynamic quantities of interest.

In the limit $T \to 0$ ($\beta \to \infty$) the nonlinear integral equation (2.112) becomes linear (2.102), and the function $\varepsilon(\lambda)$ introduced in this section becomes the function $\varepsilon(\lambda)$ which enters equation (2.102) (the energy of elementary excitation).

Free energy and chemical potential. As an example, let us show how to derive expressions for free energy and pressure. The entropy can be found from (2.107) with the substituted definition of the function ε:

$$S_N = L\int(\rho+\bar{\rho})\log\Big(1 + e^{-\beta\varepsilon}\Big)d\lambda + L\beta\int\varepsilon\rho\,d\lambda. \tag{2.114}$$

Next, multiply both sides of (2.112) by $\rho(\lambda)$ and integrate over λ. Using (2.113), we represent the result in the form

$$L\int(\rho+\bar{\rho})\log\Big(1 + e^{-\beta\varepsilon}\Big)\,d\lambda = L\beta\int(\lambda^2-\varepsilon)\rho\,d\lambda - \beta Nh + \frac{L}{2\pi}\int\log\Big(1 + e^{-\beta\varepsilon}\Big)\,d\lambda, \tag{2.115}$$

where the left hand side just coincides with the first term in the right hand side of (2.114). Hence the formula for entropy (2.114) can be simplified:

$$S_N = L\beta\int\lambda^2\rho\,d\lambda - \beta Nh + \frac{L}{2\pi}\int\log\Big(1 + e^{-\beta\varepsilon}\Big)\,d\lambda. \tag{2.116}$$

For the free energy $F_N = E_N - TS_N$ (recall that $\beta = 1/T$) we then get the expression

$$\boxed{F_N = Nh - \frac{LT}{2\pi}\int\log\Big(1 + e^{-\varepsilon/T}\Big)\,d\lambda.} \tag{2.117}$$

The pressure is found by the formula

$$\mathcal{P} = -\Big(\frac{\partial F_N}{\partial L}\Big)_T = -N\frac{\partial h}{\partial L} - \frac{L}{2\pi}\int \frac{d\lambda}{1+e^{\varepsilon/T}}\frac{\partial \varepsilon}{\partial h}\frac{\partial h}{\partial L} + \frac{T}{2\pi}\int \log\left(1+e^{-\varepsilon/T}\right)d\lambda.$$

The partial derivative $\partial\varepsilon/\partial h$ can be found by differentiating the integral equation (2.112) with respect to the parameter h. We obtain:

$$-\frac{\partial \varepsilon(\lambda)}{\partial h} = 1 - \frac{1}{2\pi}\int K(\lambda-\mu)\,\frac{\partial\varepsilon(\mu)/\partial h}{1+e^{\varepsilon(\mu)/T}}\,d\mu.$$

Comparing with (2.113), we see that the quantity $-\frac{1}{2\pi}\frac{\partial\varepsilon/\partial h}{1+e^{\varepsilon/T}}$ as a function of λ satisfies the same integral equation (2.113) as $\rho(\lambda)$. From the uniqueness of solution (which can be proved separately) we conclude that

$$\frac{\partial \varepsilon}{\partial h} = -2\pi\rho\left(1+e^{\varepsilon/T}\right).$$

Plugging this into the right hand side of the formula for pressure and using the normalization condition, we see that the first two terms in the right hand side cancel each other. Therefore, we obtain the following remarkable result:

$$\boxed{\mathcal{P} = \frac{T}{2\pi}\int \log\left(1+e^{-\varepsilon(\lambda)/T}\right)d\lambda.} \tag{2.118}$$

Comparing with (2.117), we observe the known thermodynamic formula

$$F_N = -L\mathcal{P} + Nh$$

from which it is clear that h has the meaning of chemical potential of the system. In fact one could start from a given h rather than ρ_0, then the solution scheme is a little bit simpler.

What is the meaning of the function $\varepsilon(\lambda)$? It turns out that $\varepsilon(\lambda)$ is the energy of excitation over the state of thermodynamic equilibrium. Intuitively, this can be understood from the fact that the ratio of the number of occupied vacancies (i.e. particles) to the total number of vacancies in a small interval $\delta\lambda$ is equal to

$$\frac{\rho(\lambda)}{\rho(\lambda)+\bar\rho(\lambda)} = \frac{1}{1+e^{\beta\varepsilon(\lambda)}}$$

which coincides with the Fermi-Dirac distribution. Note also that the obtained formulas for the pressure and free energy look like written for a gas of non-interacting fictitious "Fermi-particles" with energies $\varepsilon(\lambda)$. In particular, at $c=\infty$ these fictitious particles can be identified with real ones in terms of which the original Hamiltonian is written.

At this point we finish the description of the thermodynamics of the model. A deeper analysis is out of the scope of these lectures.

3 Vertex models of statistical mechanics on two-dimensional lattice

This section serves as an intermediate step from studying particular models to analysis of general algebraic structures of quantum integrability. Here we discuss vertex models on square lattice. These are models of absolutely different nature (and from a different branch of physics) comparing to the models considered in the previous section. However, their exact solution is also possible using the Bethe method. At the same time vertex models are closely connected with spin chains and allow one to construct a family of commuting integrals of motion for the latter, i.e., operators that commute with each other and the Hamiltonian. In the context of vertex models, the main general notions and objects of the quantum inverse scattering method (such as transfer matrix, R-matrix and other) become most natural.

3.1 General vertex model on square lattice

Consider the square lattice of size $N \times M$ rolled into a torus, i.e. such that the end rows and columns are identified. To each edge an arrow is assigned. On horizontal edges, the arrows look either to the left or to the right, and on each vertical edge the arrows look either up or down. Since each vertex is surrounded by 4 edges, we have $2^4 = 16$ possible configurations of arrows around vertex (16 types of vertices). Let us assign to each type of vertex indexed by $j = 1, \dots, 16$ a number ε_j (energy of the local configuration). The total energy of the lattice is then a sum of local energies over all vertices:

$$E = \sum_{j=1}^{16} N_j \varepsilon_j ,$$

where N_j is the number of vertices of the type j in a given configuration. The quantities $w_j = e^{-\varepsilon_j/T}$ (local Boltzmann weights) are often more convenient. We assume that they are the same for all vertices. The partition function is

$$Z = \sum e^{-E/T} = \sum \prod_j w_j^{N_j} ,$$

where the summation is over all configurations of arrow on the lattice and E is the total energy of the configuration. Usually the free energy per one vertex in the thermodynamic limit is of the main interest:

$$f = -T \lim_{M,N\to\infty} \frac{\log Z}{MN}$$

For the clear reason, such model is called 16-vertex model. For general values of the Boltzmann weights it does not have an exact solution.

Note that Boltzmann weights of some local configurations can be equal to zero (their energy is then $+\infty$). This means that such local configurations at the vertex are forbidden. In such a way one introduces 8-vertex and 6-vertex models which already admit an exact solution (see below).

Instead of arrows one may use spin variables $\sigma = \pm 1$ which live on *edges* of the lattice: if an arrow looks to the right or up, then $\sigma = +1$, if it looks to the left or down, then

$\sigma = -1$. To each local configuration there correspond 4 spin variables $\alpha, \alpha', \beta, \beta'$ taking values ± 1. The variables α, α' live on vertical edges while β, β' on horizontal ones (Fig. 1). Let us denote the local Boltzmann weight of such configuration as

$$R_{\alpha}^{\alpha'}(\beta, \beta') \qquad \text{or} \qquad R_{\alpha\beta}^{\alpha'\beta'}.$$

Consider some horizontal row of the lattice and adjacent vertical edges at the top and bottom of it. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ be spin variables on the vertical edges at the bottom row and $\{\alpha_1', \alpha_2', \ldots, \alpha_N'\}$ at the top row (Fig. 2). At the moment, let them be fixed and let us find the partition function of such horizontal "slice" of the lattice which we denote as

$$T_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\alpha_1', \alpha_2', \ldots, \alpha_N'} \quad \text{or, for brevity} \quad T_{\{\alpha\}}^{\{\alpha'\}}.$$

To calculate it, one should multiply the local Boltzmann weights and take the sum over all states on the horizontal edges:

$$T_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\alpha_1', \alpha_2', \ldots, \alpha_N'} = \sum_{\beta_1, \ldots, \beta_N} R_{\alpha_1}^{\alpha_1'}(\beta_1, \beta_2) R_{\alpha_2}^{\alpha_2'}(\beta_2, \beta_3) \ldots R_{\alpha_N}^{\alpha_N'}(\beta_N, \beta_1). \tag{3.1}$$

It is useful to regard the quantity $T_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\alpha_1', \alpha_2', \ldots, \alpha_N'}$ as matrix element of an operator T which acts in the space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes N}$, in the basis $|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle$:

$$\mathsf{T}\,|\alpha_1 \alpha_2, \ldots, \alpha_N\rangle \;=\; T_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\alpha_1', \alpha_2', \ldots, \alpha_N'}\,|\alpha_1' \alpha_2', \ldots, \alpha_N'\rangle$$

(hereafter, we assume summation over repeated indices). Then

$$Z = \mathrm{tr}_{\mathcal{H}}\, \mathsf{T}^M,$$

where the trace is taken in the space $\mathcal{H}$. So, in order to find the partition function it is enough to find eigenvalues of the matrix T. To determine the free energy per site in the limit $N, M \to \infty$ it is enough to know the largest eigenvalue as $N \to \infty$. Because of its importance, the matrix T has a special name. It is called the *transfer matrix* because it describes transition from one horizontal row to the next one. Diagonalization of the transfer matrix is the first main problem in the theory of vertex models. The second main problem is finding correlation functions but it is much more difficult.

Let us discuss the structure of the transfer matrix in more detail. The set of quantities $R_{\alpha}^{\alpha'}(\beta, \beta')$ can be regarded as a 2×2 matrix $R_{\alpha}^{\alpha'}$ with respect to the indices β, β' whose matrix elements are in their turn 2×2 matrices (with indices α, α'):

$$(R_{\alpha}^{\alpha'})_{\beta\beta'} = R_{\alpha}^{\alpha'}(\beta, \beta').$$

In other words, one can regard $R_{\alpha}^{\alpha'}(\beta, \beta')$ as a block matrix. Then the right hand side of equation (3.1) is nothing else than the matrix product in the horizontal (common for all matrices) space $\mathbb{C}^2$ (it is called the *auxiliary space*) with subsequent taking trace in it:

$$T_{\{\alpha\}}^{\{\alpha'\}} = T_{\alpha_1, \alpha_2, \ldots, \alpha_N}^{\alpha_1', \alpha_2', \ldots, \alpha_N'} = \mathrm{tr}_{\mathbb{C}^2}\Big(R_{\alpha_1}^{\alpha_1'} R_{\alpha_2}^{\alpha_2'} \ldots R_{\alpha_N}^{\alpha_N'}\Big).$$

So, the elementary building block is the set of Boltzmann weights $R_{\alpha}^{\alpha'}(\beta, \beta') = R_{\alpha\beta}^{\alpha'\beta'}$. They can be unified in a 4×4 matrix which can be regarded as the matrix of a linear operator in the tensor product of two two-dimensional spaces. This linear operator acts as

$$\mathsf{R}: \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2.$$

In the basis $|\beta\rangle \otimes |\alpha\rangle$ it acts in the following way:

$$\mathsf{R}: |\beta\rangle \otimes |\alpha\rangle \mapsto R^{\alpha\beta}_{\alpha'\beta'} |\beta'\rangle \otimes |\alpha'\rangle .$$

The matrix R in the basis $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$, $|-\rangle \otimes |-\rangle$ is written as

$$\mathsf{R} = \begin{pmatrix} R^{++}_{++} & R^{-+}_{++} & R^{+-}_{++} & R^{--}_{++} \\ R^{++}_{-+} & R^{-+}_{-+} & R^{+-}_{-+} & R^{--}_{-+} \\ R^{++}_{+-} & R^{-+}_{+-} & R^{+-}_{+-} & R^{--}_{+-} \\ R^{++}_{--} & R^{-+}_{--} & R^{+-}_{--} & R^{--}_{--} \end{pmatrix}.$$

Note that although physically the Boltzmann weights should be real non-negative numbers, from the algebraic point of view it is convenient to regard them as arbitrary complex variables.

Finally, let us explain how to write down the transfer matrix without indices. Given the tensor product $V_1 \otimes \ldots \otimes V_N$ of identical spaces $V_i \cong \mathbb{C}^2$, let R_{ij} be the operator acting on the product $V_i \otimes V_j$ as R and as the identity operator on the other tensor factors. Then

$$\mathsf{T} = \mathrm{tr}_{V_0}\Big(\mathsf{R}_{01}\mathsf{R}_{02} \ldots \mathsf{R}_{0N}\Big).$$

The operator under the trace $\mathcal{T} = \mathsf{R}_{01}\mathsf{R}_{02} \ldots \mathsf{R}_{0N}$ has a special name, too. For historical reasons (following an analogy with the inverse scattering method) it is called the quantum monodromy matrix. It is naturally represented as a 2×2 matrix in the auxiliary space whose elements are operators in the space $\mathcal{H}$:

$$\mathcal{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathsf{T} = A + D.$$

The operators A, B, C, D have the meaning of transfer matrices for the chain with open ends (with fixed spins at the ends).

3.2 The 6-vertex model

Let us consider only local configurations at a vertex such that the number of incoming arrows is the same as the number of outgoing ones, and declare the other configurations forbidden (put their Boltzmann weights equal to zero). There are 6 such configurations (Fig. 3). They come in pairs which correspond to inverting all the arrows. In the symmetric 6-vertex model, the Boltzmann weights are the same for the two configurations in each pair. So, there are three independent parameters in the model, and the number of essential parameters is two because the dependence on the common factor is trivial.

3.2.1 The matrix of Boltzmann weights of the symmetric 6-vertex model

The matrix of local Boltzmann weights of the symmetric 6-vertex model has the form

$$\mathsf{R} = \begin{pmatrix} R_+^+(+,+) & 0 & 0 & 0 \\ 0 & R_-^-(+,+) & R_-^+(+,-) & 0 \\ 0 & R_+^-(-,+) & R_+^+(-,-) & 0 \\ 0 & 0 & 0 & R_-^-(-,-) \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

It is called the R-matrix. Sometimes other ways to write it down are more convenient:

$$\begin{aligned} \mathsf{R} &= \begin{pmatrix} \frac{a+b}{2} + \frac{a-b}{2}\sigma_z & c\sigma_- \\ c\sigma_+ & \frac{a+b}{2} - \frac{a-b}{2}\sigma_z \end{pmatrix} \\ &= \frac{a+b}{2}\sigma_0 \otimes \sigma_0 + \frac{a-b}{2}\sigma_z \otimes \sigma_z + \frac{c}{2}\sigma_y \otimes \sigma_y + \frac{c}{2}\sigma_x \otimes \sigma_x. \end{aligned}$$

The matrix of Boltzmann weights at jth site of the lattice can be written as

$$\begin{aligned} \mathsf{R}_{0j} &= \begin{pmatrix} \frac{a+b}{2} + \frac{a-b}{2}\sigma_z^{(j)} & c\sigma_-^{(j)} \\ c\sigma_+^{(j)} & \frac{a+b}{2} - \frac{a-b}{2}\sigma_z^{(j)} \end{pmatrix} \\ &= \frac{a+b}{2}\sigma_0^{(0)} \otimes \sigma_0^{(j)} + \frac{a-b}{2}\sigma_z^{(0)} \otimes \sigma_z^{(j)} + \frac{c}{2}\sigma_y^{(0)} \otimes \sigma_y^{(j)} + \frac{c}{2}\sigma_x^{(0)} \otimes \sigma_x^{(j)}. \end{aligned}$$

Problem. Show that the vector $|\Omega\rangle = |+ + + \ldots +\rangle$ is an eigenvector for the transfer matrix $\mathsf{T} = \mathrm{tr}_{V_0}\Big(\mathsf{R}_{01}\mathsf{R}_{02}\ldots\mathsf{R}_{0N}\Big)$ and find the eigenvalue. (It can be shown that at $a > b+c$ this is the maximal eigenvalue.)

Problem. Show that the transfer matrix of the 6-vertex model commutes with the operator of the cyclic shift e^{iP} and the operator $S_z = \sum_{j=1}^{N}\sigma_z^{(j)}$: $[\mathsf{T}, e^{iP}] = [\mathsf{T}, S_z] = 0$.

The commutation relation $[\mathsf{T}, S_z] = 0$ means that the number of arrow looking down (an analog of inverted spins) is conserved under the action of the operator T, i.e., it is the same in all slices. Therefore, one can find eigenvectors in sectors with fixed number of inverted arrows. For example, the eigenvectors in the N-dimensional subspace with one inverted arrow can be found in the form

$$\sum_{n=1}^{N} z^n \sigma_-^{(n)} |\Omega\rangle\,,$$

where z is a complex number such that $z^N = 1$ (due to the periodic boundary condition). The explicit construction of eigenvectors can be performed in the sector with arbitrary number of inverted arrows, and it turns out to be identical with the solution of the XXZ spin chain by coordinate Bethe ansatz! Of course there is a deep hidden reason for this and this fact suggests that the transfer matrix of the 6-vertex model commutes with the Hamiltonian of the XXZ spin chain, and the Bethe method gives their common eigenvectors.

In the general 16-vertex model the vector $|\Omega\rangle$ is not an eigenvector and the number of inverted arrows is not conserved. The same is true for the 8-vertex model, in which only configurations with even number of incoming arrows are allowed. It turns out, however, that the 8-vertex model is exactly solvable, as well as the 6-vertex one, but the coordinate Bethe ansatz is not applicable to it. The solution of the 8-vertex model was obtained by other methods, which are more algebraic (developed in the works of Baxter and the former Leningrad school). We will first look at these methods in a simpler example of the 6-vertex model.

3.2.2 Commutig transfer matrices and the Yang-Baxter equation

A key to the algebraic solution of the 6-vertex model is finding a commutative family of transfer matrices. Namely, we will show that the transfer matrices of the models for which

$$\Delta = \frac{a^2+b^2-c^2}{2ab}$$

has the same value commute.

So, we ask when the transfer matrices

$$\mathsf{T} = \mathrm{tr}_{V_0}\mathcal{T} = \mathrm{tr}_{V_0}\Big(\mathsf{R}_{01}\mathsf{R}_{02}\ldots\mathsf{R}_{0N}\Big), \quad \mathsf{T}' = \mathrm{tr}_{V_0}\mathcal{T}' = \mathrm{tr}_{V_0}\Big(\mathsf{R}'_{01}\mathsf{R}'_{02}\ldots\mathsf{R}'_{0N}\Big)$$

commute with each other. Here R' is the R-matrix with the parameters (a', b', c'). The products $\mathsf{T}\mathsf{T}'$ and $\mathsf{T}'\mathsf{T}$ can be written as

$$\mathsf{T}\mathsf{T}' = \mathrm{tr}_{V_0\otimes V_0}\Big(\mathcal{T}\otimes\mathcal{T}'\Big), \qquad \mathsf{T}'\mathsf{T} = \mathrm{tr}_{V_0\otimes V_0}\Big(\mathcal{T}'\otimes\mathcal{T}\Big).$$

In the right hand sides there are tensor products of $\mathcal{T}$-matrices while their elements are multiplied as operators $\mathcal{H}$, taking into account the order. For example,

$$\mathcal{T}\otimes\mathcal{T}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' & BA' & BB' \\ AC' & AD' & BC' & BD' \\ CA' & CB' & DA' & DB' \\ CC' & CD' & DC' & DD' \end{pmatrix}.$$

For commutativity of the transfer matrices it is enough that there is a non-degenerate number-valued 4×4 matrix M such that

$$\mathcal{T}'\otimes\mathcal{T} = \mathsf{M}(\mathcal{T}\otimes\mathcal{T}')\mathsf{M}^{-1} \quad \text{or} \quad \mathsf{M}(\mathcal{T}\otimes\mathcal{T}') = (\mathcal{T}'\otimes\mathcal{T})\mathsf{M}.$$

Then the traces of $\mathcal{T}\otimes\mathcal{T}'$ and $\mathcal{T}'\otimes\mathcal{T}$ will be the same due to cyclicity of the trace.

Let P be the permutation operator in the space $\mathbb{C}^2\otimes\mathbb{C}^2$: $\mathsf{P}\,u\otimes v = v\otimes u$. (In the previous section this operator was denoted as P_{12}.) Assuming that the matrix elements of $\mathcal{T}$ commute with matrix elements of $\mathcal{T}'$, we would have $\mathsf{P}(\mathcal{T}\otimes\mathcal{T}') = (\mathcal{T}'\otimes\mathcal{T})\,\mathsf{P}$, i.e., in this simplest case, when the traces trivially commute, we would have $\mathsf{M} = \mathsf{P}$. In the non-commuting case we will find the matrix M in the form $\mathsf{M} = \mathsf{P}\mathsf{R}''$, where R'' is some number-valued matrix (the meaning of such notation will be clear below).

The "intertwining" relation

$$\mathsf{P}\mathsf{R}''(\mathcal{T}\otimes\mathcal{T}') = (\mathcal{T}'\otimes\mathcal{T})\mathsf{P}\mathsf{R}'' \tag{3.2}$$

will be of the main importance for us. It is useful to rewrite it in a little bit different form. Denote $\mathcal{T}_1 = \mathcal{T} \otimes 1$, $\mathcal{T}_2 = 1 \otimes \mathcal{T}$, then $\mathcal{T} \otimes \mathcal{T}' = \mathcal{T}_1\mathcal{T}_2'$, and $\mathcal{T}' \otimes \mathcal{T} = \mathsf{P}\,\mathcal{T}_2'\mathcal{T}_1\,\mathsf{P}$. Multiplying both sides of our relation by P from the left, we write it in the form

$$\mathsf{R}''_{12}\mathcal{T}_1\mathcal{T}_2' = \mathcal{T}_2'\mathcal{T}_1\mathsf{R}''_{12}.$$

The indices remind in which spaces the operators act. The matrix R''_{12} acts in the tensor product of the first and the second spaces.

The matrix R'' can be found by imposing a stronger sufficient condition that such a matrix exists for each R-matrix multiplier of the $\mathcal{T}$-matrix, namely,

$$\mathsf{P}\mathsf{R}''(\mathsf{R} \otimes \mathsf{R}') = (\mathsf{R}' \otimes \mathsf{R})\mathsf{P}\mathsf{R}'' \qquad \text{or} \qquad \mathsf{R}''_{12}\mathsf{R}_{13}\mathsf{R}'_{23} = \mathsf{R}'_{23}\mathsf{R}_{13}\mathsf{R}''_{12}.$$

Both sides of this equation are 8×8 number-valued matrices and one may hope that it can be solved. In the notation with indices we have

$$\sum_{\mu\nu\lambda} R''^{\,\nu\mu}_{\,\beta\gamma} R'^{\,\lambda\beta'}_{\,\alpha\mu} R^{\alpha'\gamma'}_{\lambda\nu} = \sum_{\mu\nu\lambda} R^{\lambda\mu}_{\alpha\beta} R'^{\,\alpha'\nu}_{\,\lambda\gamma} R''^{\,\gamma'\beta'}_{\,\mu\nu} \tag{3.3}$$

(recall that $R^{\alpha'}_{\alpha}(\beta, \beta') = R^{\alpha'\beta'}_{\alpha\beta}$). Besides, we suppose that the matrices R, R', R'' have the same structure and differ only by values of the parameters: (a, b, c) for R, (a', b', c') for R' and (a'', b'', c'') for R''.

The condition (3.3) is called the Yang-Baxter equation. One can represent it graphically (Fig. 1). It is a system of 64 equations with 3 unknown variables (non-zero elements of the matrix R''). Our task is to find whether one can choose R, R' in such a way that the system be solvable.

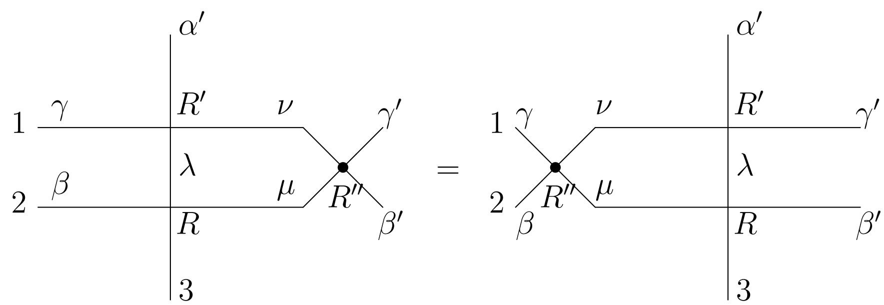

Figure 1: The Yang-Baxter equation

First of all, notice that because of the property $R^{\alpha'\beta'}_{\alpha\beta} = 0$ at $\alpha + \beta \neq \alpha' + \beta'$ many equations become trivial identities $0 = 0$. Something non-zero in the both sides appear only if $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$. As a result, only 20 nontrivial equations are left which are reduced to 10 if one takes into account the symmetry $|+\rangle \leftrightarrow |-\rangle$. Four of these 10 equations are satisfied identically while the other ones form three pairs of equivalent equations. So, there are only 3 nontrivial equations which have the form

$$\begin{cases} bc'a'' = cb'c'' + ac'b'' \\ ca'a'' = cb'b'' + ac'c'' \\ ba'c'' = cc'b'' + ab'c''. \end{cases} \tag{3.4}$$

Consider them as a system of linear homogeneous equations with unknowns a'', b'', c''. A non-zero solution exists if the determinant of the system vanishes. A direct calculation shows that this condition is equivalent to

$$\frac{a^2+b^2-c^2}{2ab} = \frac{a'^2+b'^2-c'^2}{2a'b'}.$$

Similarly, regarding (3.4) as a system of homogeneous equations with unknowns a', b', c', we get

$$\frac{a^2+b^2-c^2}{2ab} = \frac{a''^2+b''^2-c''^2}{2a''b''}.$$

Therefore, we have shown that if the quantity

$$\Delta = \frac{a^2+b^2-c^2}{2ab}$$

is the same for all R-matrices R, R', R'', then the corresponding transfer matrices commute.

For what follows it is extremely convenient to parametrize the Boltzmann weights a, b, c in the following way:

$$\begin{cases} a = \rho \sinh(u+\eta) \\ b = \rho \sinh u \\ c = \rho \sinh \eta. \end{cases} \tag{3.5}$$

Then $\Delta = \cosh \eta$, and the transfer matrices commute for different values of u, ρ (and the same η). The commutativity at different ρ is trivial because it is a common multiplier. We will put $\rho = 1$. The variable u is called *spectral parameter*, and the R-matrix as well as all other objects are usually regarded as functions of u: $\mathsf{R} = \mathsf{R}(u)$, $\mathsf{T} = \mathsf{T}(u)$ and so on. (It is implied that u can vary while η is fixed, then $[\mathsf{T}(u), \mathsf{T}(u')] = 0$).

The spectral parameters of the R-matrices entering the yang-Baxter equation turn out to be connected. Substituting the parametrization (3.5) for each R-matrix (with u, u', u'' and the same η) into the conditions (3.4), we get

$$u = u' + u''.$$

Then the Yang-Baxter equation acquires the symmetric form

$$\mathsf{R}_{12}(u_1-u_2)\mathsf{R}_{13}(u_1-u_3)\mathsf{R}_{23}(u_2-u_3) = \mathsf{R}_{23}(u_2-u_3)\mathsf{R}_{13}(u_1-u_3)\mathsf{R}_{12}(u_1-u_2) \tag{3.6}$$

(for the given parametrization it is an identity). The quantum monodromy matrix is constructed as

$$\mathcal{T}(u) = \mathsf{R}_{01}(u)\mathsf{R}_{02}(u)\ldots\mathsf{R}_{0N}(u). \tag{3.7}$$

It satisfies the intertwining relation (3.2)

$$\boxed{\check{\mathsf{R}}(u-u')(\mathcal{T}(u)\otimes\mathcal{T}(u')) = (\mathcal{T}(u')\otimes\mathcal{T}(u))\check{\mathsf{R}}(u-u')} \tag{3.8}$$

with the R-matrix

$$\check{\mathsf{R}}(u) = \mathsf{P}\mathsf{R}(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}.$$

It guarantees that the transfer matrices $\mathsf{T}(u)$ (traces of the quantum monodromy matrices in the auxiliary space) commute at different values of the spectral parameter.

For convenience we give here the explicit form of the R-matrix in the parametrization (3.5) with $\rho = 1$:

$$\begin{aligned} \mathsf{R}(u) = \mathsf{R}(u,\eta) &= \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix} \\ &= \begin{pmatrix} \sinh\left(u+\frac{\eta}{2}+\frac{\eta}{2}\,\sigma_z\right) & \sinh\eta\,\sigma_- \\ \\ \sinh\eta\,\sigma_+ & \sinh\left(u+\frac{\eta}{2}-\frac{\eta}{2}\,\sigma_z\right) \end{pmatrix}. \end{aligned} \tag{3.9}$$

Note that $\check{\mathsf{R}}(u,\eta) = \mathsf{R}(\eta,u)$ and $\mathsf{R}(0) = \sinh\eta\;\mathsf{P}$ or, writing with indices,

$$R_{\alpha\beta}^{\alpha'\beta'}(0) = R_{\alpha}^{\alpha'}(0|\beta,\beta') = \sinh\eta\,\delta_{\alpha\beta'}\delta_{\alpha'\beta}. \tag{3.10}$$

The R-matrix (3.9) obeys the following important property: if $\mathbf{g} = \mathrm{diag}\,(g_1, g_2)$ is a diagonal matrix, then

$$\mathsf{R}_{12}(u)\mathbf{g}_1\mathbf{g}_2 = \mathbf{g}_1\mathbf{g}_2\mathsf{R}_{12}(u). \tag{3.11}$$

This property allows one to generalize the definition of the transfer matrix to *twisted boundary conditions*:

$$\mathsf{T}(u) = \mathrm{tr}_0\Big(\mathbf{g}_0\mathsf{R}_{01}(u)\ldots\mathsf{R}_{0N}(u)\Big). \tag{3.12}$$

The Yang-Baxter equation and property (3.11) guarantee that $\mathsf{T}(u)$ form a commutative family of transfer matrices.

Problem. Prove property (3.11).

3.2.3 Connection of the 6-vertex model with the XXZ spin chain

Using (3.10), we see that the operator $\mathsf{T}(0)$ is proportional to the cyclic shift of the lattice by one site:

$$\boxed{\mathsf{T}(0) = (\sinh\eta)^N e^{-iP}.} \tag{3.13}$$

Indeed,

$$\begin{aligned} T_{\{\alpha\}}^{\{\alpha'\}}(0) &= (\sinh\eta)^N \sum_{\{\beta\}} \delta_{\alpha_1\beta_2}\delta_{\alpha'_1\beta_1}\,\delta_{\alpha_2\beta_3}\delta_{\alpha'_2\beta_2}\,\ldots\,\delta_{\alpha_N\beta_1}\delta_{\alpha'_N\beta_N} \\ &= (\sinh\eta)^N\delta_{\alpha'_1\alpha_N}\delta_{\alpha'_2\alpha_1}\ldots\delta_{\alpha'_N\alpha_{N-1}} \\ &= (\sinh\eta)^N\,\left(e^{-iP}\right)_{\{\alpha\}}^{\{\alpha'\}}. \end{aligned}$$

The connection with the XXZ spin chain is based on the remarkable fact that the Hamiltonian of the latter is contained in the family of the transfer matrices $\mathsf{T}(u)$ of the

6-vertex model; namely $H^{\mathrm{xxz}} \propto \mathsf{T}^{-1}(0)\partial_u \mathsf{T}(u)\Big|_{u=0}$ + const. Here is the exact formula:

$$\boxed{H^{\mathrm{xxz}} = -\sinh\eta \,\frac{d}{du}\log \mathsf{T}(u)\Big|_{u=0} \;+\; N\cosh\eta.} \tag{3.14}$$

The proof is a direct calculation. Below we give some details. By definition, we have:

$$\frac{d}{du}T^{\{\alpha'\}}_{\{\alpha\}}(u)\Big|_{u=0}$$

$$= \sum_{j=1}^{N}\sum_{\{\beta\}} R^{\alpha_1'\beta_2}_{\alpha_1\beta_1}(0)\dots R^{\alpha_{j-1}'\beta_j}_{\alpha_{j-1}\beta_{j-1}}(0)\frac{d}{du}R^{\alpha_j'\beta_{j+1}}_{\alpha_j\beta_j}(u)\Big|_{u=0}R^{\alpha_{j+1}'\beta_{j+2}}_{\alpha_{j+1}\beta_{j+1}}(0)\dots R^{\alpha_N'\beta_1}_{\alpha_N\beta_N}(0).$$

Under the sum all factors except the jth one are permutation operators of the type (3.10), and

$$\mathsf{R}'(0) = \frac{d\mathsf{R}(u)}{du}\Big|_{u=0} = \begin{pmatrix} \cosh\eta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cosh\eta \end{pmatrix} = \frac{\cosh\eta+1}{2}\,1\otimes 1 + \frac{\cosh\eta-1}{2}\,\sigma_z\otimes\sigma_z$$

$$= 1\otimes 1 + \frac{\cosh\eta-1}{2}\Big(1\otimes 1 + \sigma_z\otimes\sigma_z\Big).$$

Writing this with indices, we have

$$\frac{d}{du}R^{\alpha'\beta'}_{\alpha\beta}(u)\Big|_{u=0} = \frac{\Delta+1}{2}\,\delta_{\alpha\alpha'}\delta_{\beta\beta'} \;+\; \frac{\Delta-1}{2}\,(\sigma_z)_{\alpha\alpha'}(\sigma_z)_{\beta\beta'}\,, \qquad \Delta = \cosh\eta.$$

Now we are ready to complete the calculation:

$$\sinh\eta\left(\mathsf{T}^{-1}(0)\frac{d}{du}\mathsf{T}(u)\Big|_{u=0}\right)^{\{\alpha'\}}_{\{\alpha\}}$$

$$= \sum_{j=1}^{N}\delta_{\alpha_1\alpha_1'}\dots\delta_{\alpha_{j-1}\alpha_{j-1}'}\cdot\frac{d}{du}R^{\alpha_j'\alpha_{j+1}'}_{\alpha_{j+1}\alpha_j}(u)\Big|_{u=0}\cdot\delta_{\alpha_{j+2}\alpha_{j+2}'}\dots\delta_{\alpha_N\alpha_N'}.$$

It remains to transform

$$\frac{d}{du}R^{\alpha_j'\alpha_{j+1}'}_{\alpha_{j+1}\alpha_j}(u)\Big|_{u=0} = (\mathsf{P}\mathsf{R}'(0))^{\alpha_{j+1}'\alpha_j'}_{\alpha_{j+1}\alpha_j} = \left(\mathsf{P} + \frac{\Delta-1}{2}\,\mathsf{P}(1+\sigma_z^{(j)}\sigma_z^{(j+1)})\right)^{\alpha_{j+1}'\alpha_j'}_{\alpha_{j+1}\alpha_j}$$

and, taking into account that $\mathsf{P}(1+\sigma_z\otimes\sigma_z) = 1+\sigma_z\otimes\sigma_z$, to obtain:

$$\sinh\eta\,\mathsf{T}^{-1}(0)\frac{d}{du}\mathsf{T}(u)\Big|_{u=0} = \sum_{j=1}^{N}\left(\mathsf{P}_{j,j+1} + \frac{\Delta-1}{2}(1+\sigma_z^{(j)}\sigma_z^{(j+1)})\right),$$

which is equivalent to (3.14).

Problem. Find $\partial_u^2 \log \mathsf{T}(u)\Big|_{u=0}$.

3.2.4 Asymmetric 6-vertex model

In the asymmetric 6-vertex model, the Boltzmann weights of the local configurations which differ by inverting all arrows are different. In the natural basis the matrix of Boltzmann weights is

$$\mathsf{R} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c' & b' & 0 \\ 0 & 0 & 0 & a' \end{pmatrix}.$$

The standard argument shows that the partition function of the model with periodic boundary conditions depends only on the product cc', so we can put $c' = c$ from the very beginning without loss of generality. The asymmetric 6-vertex model can be treated as a symmetric one in some vertical and horizontal external fields. It turns out that the asymmetric model with the horizontal external field h ($a/a' = b/b' = e^{2h}$) is equivalent to the symmetric model ($a' = a$, $b' = b$) with twisted boundary conditions (3.12) which preserve integrability. The twist matrix has the form $\mathbf{g} = \mathrm{diag}\,(e^{Nh}, e^{-Nh})$.

The matrix of Boltzmann weights of the asymmetric model with horizontal field h and vertical field v is as follows:

$$\mathsf{R}^{h,v}_{0i}(u) \;=\; e^{\frac{1}{2}\,h\sigma_z^{(0)}} e^{\frac{1}{2}\,v\sigma_z^{(i)}} \mathsf{R}_{0i}(u)\, e^{\frac{1}{2}\,h\sigma_z^{(0)}} e^{\frac{1}{2}\,v\sigma_z^{(i)}}$$

$$= \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix}_0 \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}_i \mathsf{R}_{0i}(u) \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix}_0 \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}_i . \tag{3.15}$$

The explicit form of the "asymmetric" R-matrix in the trigonometric parametrization is

$$\mathsf{R}^{h,v}(u) = \begin{pmatrix} e^{h+v}\sinh(u+\eta) & 0 & 0 & 0 \\ 0 & e^{h-v}\sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & e^{-h+v}\sinh u & 0 \\ 0 & 0 & 0 & e^{-h-v}\sinh(u+\eta) \end{pmatrix}. \tag{3.16}$$

The Yang-Baxter equation together with the invariance with respect to the Cartan subgroup (3.11) implies the following Yang-Baxter equation for the asymmetric R-matrices with the same parameter η:

$$\mathsf{R}^{-v',\,v}_{12}(u-u')\,\mathsf{R}^{h,v}_{13}(u)\,\mathsf{R}^{h,v'}_{23}(u') = \mathsf{R}^{h,v'}_{23}(u')\,\mathsf{R}^{h,v}_{13}(u)\,\mathsf{R}^{-v',\,v}_{12}(u-u'). \tag{3.17}$$

We will consider the asymmetric 6-vertex model with periodic boundary conditions in the horizontal direction. The transfer matrix of the model is defined as usual:

$$\mathsf{T}^{h,v}(u) = \mathrm{tr}_0\Big(\mathsf{R}^{h,v}_{01}(u)\,\mathsf{R}^{h,v}_{02}(u)\,\ldots\,\mathsf{R}^{h,v}_{0N}(u)\Big). \tag{3.18}$$

From the yang-Baxter equation it follows that the transfer matrices with different u and v (but with the same η and h) commute: $[\mathsf{T}^{h,v}(u),\,\mathsf{T}^{h,v'}(u')] = 0$. It is easy to see that the dependence of the transfer matrix on the vertical field v is very simple:

$$\mathsf{T}^{h,v}(u) = e^{v\mathbf{S}_z}\mathsf{T}^{h,0}(u), \tag{3.19}$$

where

$$\mathbf{S}_z = \sum_{i=1}^{N} \sigma_z^{(i)} = \mathbf{M}_1 - \mathbf{M}_2 \tag{3.20}$$

is the operator which counts the (conserved) difference between the total number of arrows looking up ($\mathbf{M}_1 = \frac{1}{2}\sum_{i=1}^{N}(1+\sigma_z^{(i)})$) and down ($\mathbf{M}_2 = \frac{1}{2}\sum_{i=1}^{N}(1-\sigma_z^{(i)})$). Note that $\mathbf{M}_1 + \mathbf{M}_2 = N\mathsf{I}$, where I is the identity operator.

The transfer matrix of the asymmetric model with periodic boundary conditions is connected with the transfer matrix of the symmetric model with *twisted* boundary conditions by a similarity transformation. Set

$$\mathsf{U} = \mathbf{1} \otimes e^{h\sigma_z} \otimes e^{2h\sigma_z} \otimes \ldots \otimes e^{(N-1)h\sigma_z} = \exp\left(\sum_{j=1}^{N}(j-1)h\sigma_z^{(j)}\right).$$

From the invariance with respect to the Cartan subgroup (3.11) it follows that

$$\mathsf{U}\mathsf{T}^{h,v}(u)\mathsf{U}^{-1} = e^{v\mathbf{S}_z}\mathsf{T}^{(h)}(u),$$

where

$$\mathsf{T}^{(h)}(u) = \mathrm{tr}_0\Big(e^{Nh\sigma_z^{(0)}}\mathsf{R}_{01}(u)\,\mathsf{R}_{02}(u)\,\ldots\,\mathsf{R}_{0N}(u)\Big) \tag{3.21}$$

is the transfer matrix of the symmetric model with the boundary conditions twisted by the diagonal group element $\mathbf{g} = e^{Nh\sigma_z}$.

3.3 The 8-vertex model

3.3.1 Elliptic parametrization of the R-matrix

The matrix of Boltzmann weights of the symmetric 8-vertex model is

$$\mathsf{R} = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}.$$

It contains 3 independent parameters (not counting a common multiplier). There exists a parametrization of this R-matrix similar to (3.9) but in elliptic rather than trigonometric functions. A third parameter (the elliptic modular parameter τ) is added to the two existing ones in (3.9) (u and η). Repeating the arguments from section 3.2.2, one can show that the transfer matrices constructed with the help of R-matrices for the 8-vertex model commute if they have the same quantities

$$\Gamma = \frac{cd}{ab}, \qquad \Delta = \frac{a^2+b^2-c^2-d^2}{2ab}. \tag{3.22}$$

For the elliptic parametrization of the R-matrix of the 8-vertex model we use the Jacobi theta-functions

$$\begin{aligned}
\theta_1(u|\tau) &= -i\sum_{k\in\mathbb{Z}}(-1)^k q^{(k+\frac{1}{2})^2} e^{\pi i(2k+1)u},\\
\theta_2(u|\tau) &= \sum_{k\in\mathbb{Z}} q^{(k+\frac{1}{2})^2} e^{\pi i(2k+1)u},\\
\theta_3(u|\tau) &= \sum_{k\in\mathbb{Z}} q^{k^2} e^{2\pi iku},\\
\theta_4(u|\tau) &= \sum_{k\in\mathbb{Z}}(-1)^k q^{k^2} e^{2\pi iku},
\end{aligned} \tag{3.23}$$

where $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, and $q = e^{\pi i\tau}$. Their infinite product representations are also useful:

$$\begin{aligned}
\theta_1(u|\tau) &= 2q^{\frac{1}{4}}\sin\pi u\prod_{n\geq 1}(1-q^{2n})(1-q^{2n}e^{2\pi iu})(1-q^{2n}e^{-2\pi iu}),\\
\theta_2(u|\tau) &= 2q^{\frac{1}{4}}\cos\pi u\prod_{n\geq 1}(1-q^{2n})(1+q^{2n}e^{2\pi iu})(1+q^{2n}e^{-2\pi iu}),\\
\theta_3(u|\tau) &= \prod_{n\geq 1}(1-q^{2n})(1+q^{2n-1}e^{2\pi iu})(1+q^{2n-1}e^{-2\pi iu}),\\
\theta_4(u|\tau) &= \prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1}e^{2\pi iu})(1-q^{2n-1}e^{-2\pi iu}).
\end{aligned} \tag{3.24}$$

It is seen from here that θ_1 and θ_2 become respectively sin and cos as $q \to 0$ while θ_3 and θ_4 become constants (equal to 1). The function θ_1 is odd while the other three are even. Theta-functions satisfy a large number of non-trivial identities which we use below without comments. All these identities with proofs can be found in [15].

It is convenient for us to denote the Pauli matrices σ_x, σ_y, σ_z as σ_1, σ_2, σ_3 ($\sigma_0 = 1$). The matrix of Boltzmann weights of the 8-vertex model in the elliptic parametrization has the form

$$\mathsf{R}(u) = \mathsf{R}(u;\eta,\tau) = \sum_{a=0}^{3} W_a(u)\,\sigma_a\otimes\sigma_a, \tag{3.25}$$

where

$$W_a(u) = W_a(u;\eta,\tau) = \theta_1(\eta|\tau)\,\frac{\theta_{5-a}\Big(u+\frac{\eta}{2}\Big|\tau\Big)}{2\theta_{5-a}\Big(\frac{\eta}{2}\Big|\tau\Big)}, \quad a = 0,\ldots,3, \tag{3.26}$$

and the index of the theta-functions is understood modulo 4 (for example, $\theta_5 = \theta_1$). This R-matrix acts in the tensor product $V_1\otimes V_2$ ($V_i \cong \mathbb{C}^2$) and can be also denoted as $\mathsf{R}_{12}(u) = \sum_{a=0}^{3} W_a(u)\,\sigma_a^{(1)}\sigma_a^{(2)}$. (Pictorially, the first space is associated with the horizontal line and the second one with the vertical line.) In the matrix form we have:

$$\mathsf{R}(u) = \begin{pmatrix} W_0(u)+W_3(u) & 0 & 0 & W_1(u)-W_2(u) \\ 0 & W_0(u)-W_3(u) & W_1(u)+W_2(u) & 0 \\ 0 & W_1(u)+W_2(u) & W_0(u)-W_3(u) & 0 \\ W_1(u)-W_2(u) & 0 & 0 & W_0(u)+W_3(u) \end{pmatrix}$$
$$= \begin{pmatrix} a^{8\mathrm{v}}(u) & 0 & 0 & d^{8\mathrm{v}}(u) \\ 0 & b^{8\mathrm{v}}(u) & c^{8\mathrm{v}}(u) & 0 \\ 0 & c^{8\mathrm{v}}(u) & b^{8\mathrm{v}}(u) & 0 \\ d^{8\mathrm{v}}(u) & 0 & 0 & a^{8\mathrm{v}}(u) \end{pmatrix}, \tag{3.27}$$

where the explicit form of the matrix elements is

$$\begin{aligned} a^{8\mathrm{v}}(u) &= \frac{2\theta_4(\eta|2\tau)\,\theta_1(u+\eta|2\tau)\,\theta_4(u|2\tau)}{\theta_2(0|\tau)\,\theta_4(0|2\tau)}, \\ b^{8\mathrm{v}}(u) &= \frac{2\theta_4(\eta|2\tau)\,\theta_4(u+\eta|2\tau)\,\theta_1(u|2\tau)}{\theta_2(0|\tau)\,\theta_4(0|2\tau)}, \\ c^{8\mathrm{v}}(u) &= \frac{2\theta_1(\eta|2\tau)\,\theta_4(u+\eta|2\tau)\,\theta_4(u|2\tau)}{\theta_2(0|\tau)\,\theta_4(0|2\tau)}, \\ d^{8\mathrm{v}}(u) &= \frac{2\theta_1(\eta|2\tau)\,\theta_1(u+\eta|2\tau)\,\theta_1(u|2\tau)}{\theta_2(0|\tau)\,\theta_4(0|2\tau)}. \end{aligned} \tag{3.28}$$

It is easy to see that when the spectral parameter is shifted by the quasiperiods 1 and τ, the R-matrix transforms as follows:

$$\begin{aligned} \mathsf{R}_{12}(u+1) &= -\sigma_3^{(1)}\mathsf{R}_{12}(u)\sigma_3^{(1)}, \\ \mathsf{R}_{12}(u+\tau) &= -e^{-\pi i(2u+\eta+\tau)}\sigma_1^{(1)}\mathsf{R}_{12}(u)\sigma_1^{(1)}. \end{aligned} \tag{3.29}$$

It can be shown that this R-matrix satisfies the Yang-Baxter equation (3.6) and commutes with $\sigma_a \otimes \sigma_a$:

$$\sigma_a \otimes \sigma_a \mathsf{R}(u) = \mathsf{R}(u)\sigma_a \otimes \sigma_a, \quad a = 1, 2, 3. \tag{3.30}$$

The quantities Γ and Δ for the R-matrix (3.27) are

$$\Gamma = \frac{\theta_1^2(\eta|2\tau)}{\theta_4^2(\eta|2\tau)}, \quad \Delta = \frac{\theta_4^2(0|2\tau)}{\theta_4^2(\eta|2\tau)}\,\frac{\theta_2(\eta|\tau)}{\theta_2(0|\tau)}. \tag{3.31}$$

Below we also need the following properties of the R-matrix (3.27):

$$\begin{aligned} &\mathsf{R}_{12}(-u;-\eta,\tau) = -\mathsf{R}_{12}(u;\eta,\tau), \\ &\mathsf{R}_{12}^{t_1 t_2}(u) = \mathsf{R}_{12}(u), \\ &\mathsf{R}_{12}(u-\eta;\eta,\tau) = e^{\pi i(2u-\eta+\tau)}\mathsf{R}_{12}^{t_1}(u+\tau+1;-\eta,\tau), \end{aligned} \tag{3.32}$$

where t_i means transposition in the i-th space.

Let us mention properties of the R-matrix (3.27) under the modular transformation $\tau \to -1/\tau$. Using the formulas for modular transformations of the theta-functions, we find:

$$\mathsf{R}\Big(\frac{u}{\tau};\frac{\eta}{\tau},-\frac{1}{\tau}\Big) = -i\sqrt{-i\tau}\, e^{\pi i(u^2+u\eta+\eta^2)/\tau}\,\tilde{\mathsf{R}}(u;\eta,\tau), \tag{3.33}$$

where

$$\tilde{\mathsf{R}}(u;\eta,\tau) = \theta_1(\eta|\tau)\sum_{a=0}^{3}\frac{\theta_{a+1}\Big(u+\frac{\eta}{2}|\tau\Big)}{2\theta_{a+1}\Big(\frac{\eta}{2}|\tau\Big)}\,\sigma_a\otimes\sigma_a = \begin{pmatrix} \tilde{a}(u) & 0 & 0 & \tilde{d}(u) \\ 0 & \tilde{b}(u) & \tilde{c}(u) & 0 \\ 0 & \tilde{c}(u) & \tilde{b}(u) & 0 \\ \tilde{d}(u) & 0 & 0 & \tilde{a}(u) \end{pmatrix}.$$

The explicit form of the matrix elements is as follows:

$$\begin{aligned}
\tilde{a}(u) &= \frac{\theta_2\Big(\frac{\eta}{2}\Big|\frac{\tau}{2}\Big)\theta_2\Big(\frac{u}{2}\Big|\frac{\tau}{2}\Big)\theta_1\Big(\frac{u+\eta}{2}\Big|\frac{\tau}{2}\Big)}{\theta_2\Big(0\Big|\frac{\tau}{2}\Big)\,\theta_4(0|\tau)},\\
\tilde{b}(u) &= \frac{\theta_2\Big(\frac{\eta}{2}\Big|\frac{\tau}{2}\Big)\theta_1\Big(\frac{u}{2}\Big|\frac{\tau}{2}\Big)\theta_2\Big(\frac{u+\eta}{2}\Big|\frac{\tau}{2}\Big)}{\theta_2\Big(0\Big|\frac{\tau}{2}\Big)\,\theta_4(0|\tau)},\\
\tilde{c}(u) &= \frac{\theta_1\Big(\frac{\eta}{2}\Big|\frac{\tau}{2}\Big)\theta_2\Big(\frac{u}{2}\Big|\frac{\tau}{2}\Big)\theta_2\Big(\frac{u+\eta}{2}\Big|\frac{\tau}{2}\Big)}{\theta_2\Big(0\Big|\frac{\tau}{2}\Big)\,\theta_4(0|\tau)},\\
\tilde{d}(u) &= -\frac{\theta_1\Big(\frac{\eta}{2}\Big|\frac{\tau}{2}\Big)\theta_1\Big(\frac{u}{2}\Big|\frac{\tau}{2}\Big)\theta_1\Big(\frac{u+\eta}{2}\Big|\frac{\tau}{2}\Big)}{\theta_2\Big(0\Big|\frac{\tau}{2}\Big)\,\theta_4(0|\tau)}.
\end{aligned} \tag{3.34}$$

3.3.2 Connection with the XYZ spin chain

From (3.25), (3.26) it follows that $\mathsf{R}(0) = \theta_1(\eta|\tau)\mathsf{P}$, so $\mathsf{T}(0)$, as in the case of the 6-vertex model, is proportional to the cyclic shift by one site. The logarithmic derivative $\partial_u \log \mathsf{T}(u)\Big|_{u=0}$ contains the Hamiltonian H^{XYZ} of the anisotropic Heisenberg spin chain. To see this, one should perform some calculations similar to those done in section 3.2.3. We have:

$$\theta_1(\eta|\tau)\bigg(\frac{d}{du}\log \mathsf{T}(u)\Big|_{u=0}\bigg)_{\{\alpha\}}^{\{\alpha'\}} = \sum_{j=1}^{N}\delta_{\alpha_1\alpha_1'}\ldots\delta_{\alpha_{j-2}\alpha_{j-2}'}\frac{d}{du}R_{\alpha_j\alpha_{j-1}}^{\alpha_{j-1}'\alpha_j'}(u)\Big|_{u=0}\delta_{\alpha_{j+1}\alpha_{j+1}'}\ldots\delta_{\alpha_N\alpha_N'}.$$

Now note that $\mathsf{PR}(u) = \check{\mathsf{R}}(u) = \sum_{a=0}^{3}\check{W}_a(u)\,\sigma_a\otimes\sigma_a$, where

$$\check{W}_0 = \frac{1}{2}\,(a+c),\quad \check{W}_1 = \frac{1}{2}\,(b+d),\quad \check{W}_2 = \frac{1}{2}\,(b-d),\quad \check{W}_3 = \frac{1}{2}\,(a-c),$$

then

$$R_{\alpha_j\alpha_{j-1}}^{\alpha_{j-1}'\alpha_j'}(u) = \sum_{a=0}^{3}\check{W}_a(u)(\sigma_a)_{\alpha_{j-1}\alpha_{j-1}'}(\sigma_a)_{\alpha_j\alpha_j'}.$$

For the logarithmic derivative we obtain:

$$\partial_u \log \mathsf{T}(u)\Big|_{u=0} = \theta_1^{-1}(\eta|\tau) \sum_{j=1}^{N} \sum_{a=0}^{3} \check{W}_a'(0)\sigma_a^{(j)}\sigma_a^{(j+1)}.$$

A simple calculation using some identities for the Jacobi theta-functions shows that

$$\check{W}_a(u) = \theta_1(u|\tau)\, \frac{\theta_{5-a}\left(\frac{u}{2}+\eta\middle|\tau\right)}{2\theta_{5-a}\left(\frac{u}{2}\middle|\tau\right)}, \tag{3.35}$$

so

$$\check{W}_0'(0) = \frac{1}{2}\,\theta_1'(\eta|\tau), \quad \check{W}_1'(0) = \frac{1}{2}\,\theta_1'(0|\tau)\,\frac{\theta_4(\eta|\tau)}{\theta_4(0|\tau)},$$

$$\check{W}_2'(0) = \frac{1}{2}\,\theta_1'(0|\tau)\,\frac{\theta_3(\eta|\tau)}{\theta_3(0|\tau)}, \quad \check{W}_3'(0) = \frac{1}{2}\,\theta_1'(0|\tau)\,\frac{\theta_2(\eta|\tau)}{\theta_2(0|\tau)}.$$

(Note that $\check{\mathsf{R}}(u;\eta,\tau) = \mathsf{R}(\eta;u,\tau)$, which is obvious from (3.28).) We conclude that

$$\partial_u \log \mathsf{T}(u)\Big|_{u=0} = \frac{\theta_1'(0|\tau)}{2\theta_1(\eta|\tau)}\, H^{\mathrm{XYZ}} + J_0 N \mathsf{I}, \tag{3.36}$$

where $J_0 = \frac{1}{2}\,\theta_1'(\eta|\tau)/\theta_1(\eta|\tau)$, and the Hamiltonian of the XYZ spin chain is

$$H^{\mathrm{XYZ}} = \sum_{j=1}^{N}\Big(J_x\sigma_x^{(j)}\sigma_x^{(j+1)} + J_y\sigma_y^{(j)}\sigma_y^{(j+1)} + J_z\sigma_z^{(j)}\sigma_z^{(j+1)}\Big)$$

with the constants

$$J_x = \frac{\theta_4(\eta|\tau)}{\theta_4(0|\tau)}, \quad J_y = \frac{\theta_3(\eta|\tau)}{\theta_3(0|\tau)}, \quad J_z = \frac{\theta_2(\eta|\tau)}{\theta_2(0|\tau)}.$$

Note that $J_x : J_y : J_z = (1+\Gamma) : (1-\Gamma) : \Delta$ (see (3.31)).

3.3.3 The result of diagonalization of the transfer matrix

The diagonalization of the transfer matrix of the 8-vertex model first done by Baxter is a complicated nontrivial procedure. We will learn it below in section 4.3. Here we will give the final answer. The eigenvalues at even $N = 2n$ (for odd N the exact solution is problematic) are given by the expression

$$T(u) = e^{i\pi\eta\nu}\theta_1^N(u+\eta|\tau)\prod_{k=1}^{n}\frac{\theta_1(u-u_i-\eta|\tau)}{\theta_1(u-u_i|\tau)} + e^{-i\pi\eta\nu}\theta_1^N(u|\tau)\prod_{k=1}^{n}\frac{\theta_1(u-u_i+\eta|\tau)}{\theta_1(u-u_i|\tau)}, \tag{3.37}$$

where n numbers u_i satisfy the system of n Bethe equations

$$e^{2i\pi\eta\nu}\left(\frac{\theta_1(u_j+\eta|\tau)}{\theta_1(u_j|\tau)}\right)^N = \prod_{k=1,\neq j}\frac{\theta_1(u_j-u_k+\eta|\tau)}{\theta_1(u_j-u_k-\eta|\tau)}. \tag{3.38}$$

The parameter ν can take some discrete values, and $\nu = 0$ is among them.

3.3.4 Trigonometric degenerations of the elliptic R-matrix

Let us discuss trigonometric degenerations of the formulas from this section. In the limit $\tau \to +i\infty$ ($q \to 0$) the elliptic R-matrix degenerates into the standard trigonometric R-matrix of the 6-vertex model:

$$\mathsf{R}(u) \to 2q^{\frac{1}{4}} \begin{pmatrix} \sin\pi(u+\eta) & 0 & 0 & 0 \\ 0 & \sin\pi u & \sin\pi\eta & 0 \\ 0 & \sin\pi\eta & \sin\pi u & 0 \\ 0 & 0 & 0 & \sin\pi(u+\eta) \end{pmatrix} + O(q^{\frac{5}{4}}).$$

For the correspondence with formulas from the previous section one should substitute $\eta \to i\eta$ and $u \to iu$.

Another trigonometric degeneration is obtained as $\tau \to 0$ ($q \to 1$). In order to see what happens in this case, it is convenient to pass to the modular-transformed R-matrix (3.33). Using formulas (3.29), we find:

$$\frac{i}{4} \lim_{\tau\to+i\infty} (-i\tau)^{-1/2} q^{-1/4} \mathsf{R}\Big(\frac{u}{\tau}; \frac{\eta}{\tau}, -\frac{1}{\tau}\Big) = \tilde{\mathsf{R}}^{\rm trig}(u;\eta) = \begin{pmatrix} \tilde{a}^{\rm trig} & 0 & 0 & \tilde{d}^{\rm trig} \\ 0 & \tilde{b}^{\rm trig} & \tilde{c}^{\rm trig} & 0 \\ 0 & \tilde{c}^{\rm trig} & \tilde{b}^{\rm trig} & 0 \\ \tilde{d}^{\rm trig} & 0 & 0 & \tilde{a}^{\rm trig} \end{pmatrix}.$$

The matrix elements are

$$\tilde{a}^{\rm trig} = \cos\tfrac{\eta}{2}\,\cos\tfrac{u}{2}\,\sin\tfrac{u+\eta}{2},$$

$$\tilde{b}^{\rm trig} = \cos\tfrac{\eta}{2}\,\sin\tfrac{u}{2}\,\cos\tfrac{u+\eta}{2},$$

$$\tilde{c}^{\rm trig} = \sin\tfrac{\eta}{2}\,\cos\tfrac{u}{2}\,\cos\tfrac{u+\eta}{2},$$

$$\tilde{d}^{\rm trig} = -\sin\tfrac{\eta}{2}\,\sin\tfrac{u}{2}\,\sin\tfrac{u+\eta}{2}.$$

In the limit we get a 8-vertex model in which not all Boltzmann weights are independent (only two parameters are left).

Let us mention that besides these standard trigonometric degenerations there exists a more tricky trigonometric limit of the elliptic R-matrix which gives the so-called 7-vertex model. It can be obtained if before taking the limit one performs the gauge transformation

$$\mathsf{R}(u) \to \mathsf{R}^q(u) = G_1G_2\mathsf{R}(u)(G_1G_2)^{-1}, \quad G_1 = G\otimes 1, \quad G_2 = 1\otimes G$$

with the matrix

$$G = \begin{pmatrix} q^{\frac{1}{4}}\gamma^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{4}}\gamma^{\frac{1}{2}} \end{pmatrix},$$

so that $G_1G_2 = \mathrm{diag}\left(q^{\frac{1}{2}}\gamma^{-1}, 1, 1, q^{-\frac{1}{2}}\gamma\right)$, where γ is an arbitrary parameter.

Problem. Prove that the R-matrix $\mathsf{R}^q(u)$ satisfies the Yang-Baxter equation (3.6).

Now we can define the R-matrix

$$\mathsf{R}^{7\rm v}(u) = \frac{1}{2} \lim_{q\to 0} q^{-\frac{1}{4}}\, \mathsf{R}^q(u), \tag{3.39}$$

which also satisfies the Yang-Baxter equation. Its explicit form is

$$\mathsf{R}^{7\mathrm{v}}(u) = \begin{pmatrix} a^{7\mathrm{v}}(u) & 0 & 0 & 0 \\ 0 & b^{7\mathrm{v}}(u) & c^{7\mathrm{v}}(u) & 0 \\ 0 & c^{7\mathrm{v}}(u) & b^{7\mathrm{v}}(u) & 0 \\ d^{7\mathrm{v}}(u) & 0 & 0 & a^{7\mathrm{v}}(u) \end{pmatrix},$$

where

$$a^{7\mathrm{v}}(u) = \sin\pi(u+\eta), \quad b^{7\mathrm{v}}(u) = \sin\pi u, \quad c^{7\mathrm{v}}(u) = \sin\pi\eta,$$
$$d^{7\mathrm{v}}(u) = 4\gamma^2 \sin\pi\eta\, \sin\pi(u+\eta)\sin\pi u.$$

4 Algebraic Bethe ansatz

4.1 Algebraic Bethe ansatz for the 6-vertex model

Let us consider the 6-vertex model with trigonometric R-matrix. The intertwining relation

$$\check{\mathsf{R}}(u-u')(\mathcal{T}(u)\otimes\mathcal{T}(u')) = (\mathcal{T}(u')\otimes\mathcal{T}(u))\check{\mathsf{R}}(u-u') \tag{4.1}$$

or, which is the same,

$$\mathsf{R}(u-u')\mathcal{T}_1(u)\mathcal{T}_2(u') = \mathcal{T}_2(u')\mathcal{T}_1(u)\mathsf{R}(u-u') \tag{4.2}$$

plays the main role in the theory of integrable models of statistical physics on two-dimensional lattice as well as of integrable models of solid state physics and quantum field theory. From the algebraic point of view, it defines the commutation relations between generators of an infinite-dimensional algebra (quantum affine algebra) generated by coefficients of the matrix elements of $\mathcal{T}(u)$ in the expansion in powers of u. The Yang-Baxter equation is equivalent to associativity of this algebra. The realization of the intertwining relation for the 6-vertex model by matrices of Boltzman weights means a choice of its special finite-dimensional representation. The construction of eigenvectors of the transfer-matrix by means of algebraic properties of the operators is called algebraic Bethe ansatz. Below we will describe the main points of this construction in the case when the R-matrix has the form

$$\check{\mathsf{R}}(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix} \tag{4.3}$$

with $a = \sinh(u+\eta)$, $b = \sinh u$, $c = \sinh\eta$.

The matrix elements of the quantum monodromy matrix

$$\mathcal{T}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

are some operators acting in the space $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes N}$. The relations containing in (4.1) gives commutation rules of these operators. The equality (4.1) in the detailed matrix form looks as follows:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} A(u)A(v) & A(u)B(v) & B(u)A(v) & B(u)B(v) \\ A(u)C(v) & A(u)D(v) & B(u)C(v) & B(u)D(v) \\ C(u)A(v) & C(u)B(v) & D(u)A(v) & D(u)B(v) \\ C(u)C(v) & C(u)D(v) & D(u)C(v) & D(u)D(v) \end{pmatrix}$$

$$= \begin{pmatrix} A(v)A(u) & A(v)B(u) & B(v)A(u) & B(v)B(u) \\ A(v)C(u) & A(v)D(u) & B(v)C(u) & B(v)D(u) \\ C(v)A(u) & C(v)B(u) & D(v)A(u) & D(v)B(u) \\ C(v)C(u) & C(v)D(u) & D(v)C(u) & D(v)D(u) \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where $a = a(u - v)$, $b = b(u - v)$, $c = c(u - v)$. Below we will write explicitly only those relations which will be used for calculations in what follows.

First, we have $[A(u),\, A(v)] = 0$, $[B(u),\, B(v)] = 0$, and so on. Second, the commutation relations

$$a(u-v)B(u)A(v) = c(u-v)B(v)A(u) + b(u-v)A(v)B(u) \tag{4.4}$$

$$a(u-v)B(v)D(u) = c(u-v)B(u)D(v) + b(u-v)D(u)B(v) \tag{4.5}$$

hold.

Exercise. Write down all commutation relations for the operators $A(u)$, $B(u)$, $C(u)$, $D(u)$ containing in (4.1).

Problem. Prove that

$$\det_q \mathcal{T}(u) = A(u+\eta)D(u) - B(u+\eta)C(u) \tag{4.6}$$

is the central element of the algebra generated by the matrix elements of the quantum monodromy matrix, i.e. it commutes with all the generators. This element is called *quantum determinant*.

For the rational degeneration of the R-matrix (4.3) (in the limit $\epsilon \to 0$ after the substitution $u \to \epsilon u, \eta \to \epsilon\eta$), when $a = u + \eta$, $b = u$, $c = \eta$, the commutation relations containing in (4.1) can be compactly written in the form

$$[\mathcal{T}_{ij}(u),\, \mathcal{T}_{kl}(v)] = \frac{\eta}{u-v}\Big(\mathcal{T}_{kj}(v)\mathcal{T}_{il}(u) - \mathcal{T}_{kj}(u)\mathcal{T}_{il}(v)\Big) \tag{4.7}$$

for all $i, j, k, l = 1, 2$.

Let us consider the vector $|\Omega\rangle = |+ + + \ldots +\rangle$. It is easy to see that it is an eigenvector for the operators $A(u)$ and $D(u)$ while $C(u)$ annihilates it:

$$A(u)\,|\Omega\rangle = a^N(u)\,|\Omega\rangle\,, \quad D(u)\,|\Omega\rangle = b^N(u)\,|\Omega\rangle\,, \quad C(u)\,|\Omega\rangle = 0.$$

The vector $|\Omega\rangle$ is called a generating (or vacuum) vector because all other eigenvectors of the transfer matrix will be obtained by applying the operators $B(u)$ to it.

Recall that the transfer matrix is trace in the auxiliary space $V_0 \cong \mathbb{C}^2$ of product of R-matrices:

$$\mathsf{T}(u) = \mathrm{tr}_0\Big(\mathsf{R}_{01}(u)\mathsf{R}_{02}(u)\ldots\mathsf{R}_{0N}(u)\Big).$$

The following basic theorem yields an algebraic constructionof eigenvectors and eigenvalues of the transfer matrix.

Theorem. *The vectors*

$$|\Phi(u_1,\ldots,u_n)\rangle = \prod_{j=1}^{n} B(u_j)\ |\Omega\rangle$$

are eigenvectors of the transfer matrix ($\mathsf{T}(u)\,|\Phi\rangle = T(u)\,|\Phi\rangle$) provided that u_j satisfy the system of Bethe equations

$$\frac{a^N(u_j)}{b^N(u_j)} = (-1)^{n-1} \prod_{k=1,\neq j}^{n} \frac{a(u_j-u_k)}{a(u_k-u_j)} \tag{4.8}$$

or

$$\left(\frac{\sinh(u_j+\eta)}{\sinh u_j}\right)^N = \prod_{k=1,\neq j}^{n} \frac{\sinh(u_j-u_k+\eta)}{\sinh(u_j-u_k-\eta)},$$

and the corresponding eigenvalue $T(u) = T(u;u_1,\ldots,u_n)$ is given by the formula

$$T(u;u_1,\ldots,u_n) = a^N(u) \prod_{j=1}^{n} \frac{a(u_j-u)}{b(u_j-u)} + b^N(u) \prod_{j=1}^{n} \frac{a(u-u_j)}{b(u-u_j)}$$

or

$$T(u;u_1,\ldots,u_n) = \sinh^N(u+\eta) \prod_{j=1}^{n} \frac{\sinh(u-u_j-\eta)}{\sinh(u-u_j)} + \sinh^N u \prod_{j=1}^{n} \frac{\sinh(u-u_j+\eta)}{\sinh(u-u_j)}.$$

Proof. Let us rewrite the commutation relations (4.4), (4.5) in a more convenient form:

$$A(u)B(v) = \frac{a(v-u)}{b(v-u)}\,B(v)A(u) - \frac{c(v-u)}{b(v-u)}\,B(u)A(v),$$

$$D(u)B(v) = \frac{a(u-v)}{b(u-v)}\,B(v)D(u) - \frac{c(u-v)}{b(u-v)}\,B(u)D(v).$$

With the help of these relations, one can transform the expression

$$(A(u)+D(u)) \prod_{j=1}^{n} B(u_j)\ |\Omega\rangle\,,$$

moving $A(u)$ and $D(u)$ through $B(u_j)$ to the right until it meets the vacuum vector $|\Omega\rangle$ which is an eigenvector for them. In doing so, one obtains 2^n terms which can be combined in the expressions of the form

$$T(u) \prod_{k=1}^{n} B(u_k)\ |\Omega\rangle \quad \text{and} \quad \Lambda_j(u)\,B(u) \prod_{k=1,\neq j}^{n} B(u_k)\ |\Omega\rangle \tag{4.9}$$

with some coefficients $T(u)$ and $\Lambda_j(u)$. The first of them is obtained if in the process of commutation one takes into account only first terms in the right hand sides of the commutation relations. Indeed, if one takes the second term at some step, then the operator $B(u)$ appears which can not disappear after that. Therefore, $T(u)$ is given by the expression for the eigenvalue written above. However, we can not yet say that our vector is an eigenvector because there are "bad terms" – vectors of the form $B(u)\prod_{k=1,\neq j}^{n} B(u_k)\,|\Omega\rangle$ with the coefficients $\Lambda_j(u)$. The coefficient $\Lambda_j(u)$ can be found explicitly by the following trick. Taking advantage of the commutativity of the operators B, one can put $B(u_j)$ to the first place and, moving first $A(u)+D(u)$ through $B(u_j)$ use the second terms in the commutation relations and then, moving $A(u)$ or $D(u)$ further to the right, use only the first terms. The result is

$$\Lambda_j(u) = -\frac{c}{b(u_j-u)}\left(a^N(u_j)\prod_{k=1,\neq j}^{n}\frac{a(u_k-u_j)}{b(u_k-u_j)} - b^N(u_j)\prod_{k=1,\neq j}^{n}\frac{a(u_j-u_k)}{b(u_j-u_k)}\right). \tag{4.10}$$

The "bad terms" vanish if $\Lambda_j(u)=0$ for all $j=1,\dots,n$. Clearly, these conditions are equivalent to the Bethe equations.

4.2 Models of general form with trigonometric R-matrix

The algebraic method described above not only provides an elegant method to solve the 6-vertex model but also allows one to extend the family of integrable models.

4.2.1 Inhomogeneous models

To begin with, we represent the R-matrix on the j-th site in the form

$$\mathsf{L}_j(u) = \begin{pmatrix} \sinh\Big(u+\frac{\eta}{2}\,\sigma_z^{(j)}\Big) & \sinh\eta\sigma_-^{(j)} \\ \sinh\eta\sigma_+^{(j)} & \sinh\Big(u-\frac{\eta}{2}\,\sigma_z^{(j)}\Big) \end{pmatrix} \tag{4.11}$$

and call it *the quantum L-operator* (the terminology goes back to the inverse scattering method). It is obvious that $\mathsf{L}_j(u)=\mathsf{R}_{j0}(u-\frac{\eta}{2})$ and $\mathcal{T}(u)=\mathsf{L}_1(u)\mathsf{L}_2(u)\dots\mathsf{L}_N(u)$, where we have redefined $\mathcal{T}(u)$ by a shift of the argument. The quantum L-operator can be regarded as an "elementary" quantum monodromy matrix (for the model on one site). Therefore, it is clear that the L-operator satisfies the basic intertwining relation

$$\check{\mathsf{R}}(u-v)(\mathsf{L}(u)\otimes\mathsf{L}(v)) = (\mathsf{L}(v)\otimes\mathsf{L}(u))\check{\mathsf{R}}(u-v) \tag{4.12}$$

with the R-matrix

$$\check{\mathsf{R}}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

The elements of the L-operator act to the vector $|+\rangle$ as follows:

$$\mathsf{L}(u)\,|+\rangle = \begin{pmatrix} \sinh\Big(u+\frac{\eta}{2}\Big) & \star \\ 0 & \sinh\Big(u-\frac{\eta}{2}\Big) \end{pmatrix}.|+\rangle \tag{4.13}$$

The star stands for an inessential for us nonzero element.

The generalization which allows one to introduce a wide class of models solved by the same method is based on the remark that the intertwining relation (4.1) for quantum monodromy matrices remains valid if to make arbitrary site-depending shifts of the spectral parameter in each L-operator in the product along the chain:

$$\mathcal{T}(u) = \mathsf{L}_1(u-\xi_1)\,\mathsf{L}_2(u-\xi_2)\,\ldots\,\mathsf{L}_N(u-\xi_N). \tag{4.14}$$

(Clearly, the intertwining relation (4.1) for $\mathcal{T}(u)$, $\mathcal{T}(v)$ and commutativity of the traces holds only if the parametars ξ_i are the same for the both matrices.) Such $\mathcal{T}$-matrix is called the quantum monodromy matrix of an inhomogeneous chain while the quantities ξ_i are called inhomogeneities at the sites. Together with η, they are parameters of the model. We have thus constructed a large family of models whose transfer matrices allow for diagonalization by means of the same algebraic Bethe ansatz method. From mathematical point of view, the inhomogeneous models are probably even better then the homogeneous 6-vertex model or the XXZ chain (because their parameters are "in general position"). From physical point of view, however, they are not so good because usually they do not have a local Hamiltonian which could be included into the commutative family of operators.

An important condition of applicability of the algebraic Bethe ansatz is the existence of a vacuum vector $|\Omega\rangle$ which is an eigenvector for $A(u), D(u)$ and such that $C(u)\,|\Omega\rangle = 0$. In other words, we have:

$$\mathcal{T}(u)\,|\Omega\rangle = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} |\Omega\rangle = \begin{pmatrix} \mathsf{a}(u) & \star \\ 0 & \mathsf{d}(u) \end{pmatrix} |\Omega\rangle$$

with some functions $\mathsf{a}(u)$, $\mathsf{d}(u)$. The property (4.13) means that in our generalized model the vacuum vector is $|\Omega\rangle = |+++\ldots+\rangle$, then

$$\mathsf{a}(u) = \prod_{j=1}^{N} \sinh\Bigl(u-\xi_j+\frac{\eta}{2}\Bigr), \qquad \mathsf{d}(u) = \prod_{j=1}^{N} \sinh\Bigl(u-\xi_j-\frac{\eta}{2}\Bigr).$$

We will call functions of this form *trigonometric polynomials* (of degree N).

It is possible to generalize the model even further and to take entire functions $\mathsf{a}(u)$ and $\mathsf{d}(u)$ with period $2\pi i$ (vacuum eigenvalues of the operators $A(u)$ and $D(u)$) as *functional parameters* of the model. We call such a model the generalized XXZ model. For the generalized model with a vacuum vector there exists a general algebraic procedure of diagonalization of the commutative family of operators $\mathrm{tr}\,\mathcal{T}(u) = A(u) + D(u)$.

Theorem. *The vectors*

$$|\Phi(u_1,\ldots,u_n)\rangle = \prod_{j=1}^{n} B(u_j)\;|\Omega\rangle$$

are eigenvectors of the transfer matrix ($\mathsf{T}(u)\,|\Phi\rangle = T(u)\,|\Phi\rangle$) provided that the parameters u_j satisfy the system of Bethe equations

$$\frac{\mathsf{a}(u_j)}{\mathsf{d}(u_j)} = \prod_{k=1,\neq j}^{n} \frac{\sinh(u_j-u_k+\eta)}{\sinh(u_j-u_k-\eta)}, \tag{4.15}$$

and the eigenvalue $T(u) = T(u;u_1,\ldots,u_n)$ is given by the formula

$$T(u;u_1,\ldots,u_n) = \mathsf{a}(u)\prod_{j=1}^{n} \frac{\sinh(u-u_j-\eta)}{\sinh(u-u_j)} + \mathsf{d}(u)\prod_{j=1}^{n} \frac{\sinh(u-u_j+\eta)}{\sinh(u-u_j)}. \tag{4.16}$$

The proof is basically the same as in the case of the homogeneous 6-vertex model.

4.2.2 The Baxter Q-operator and TQ-relation

Note that the system of Bethe equations is equivalent to the condition that the eigenvalue of the transfer matrix $T(u) = T(u; u_1, \ldots, u_n)$ (4.16) does not have poles at the points $u = u_i$. This remark allows one to suggest an alternative way to obtain the Bethe equations. Namely, from the fact that the transfer matrices commute for all u it follows that they can be diagonalized by an u-independent transformation and so, since all their matrix elements are trigonometric polynomials of degree N, all their eigenvalues have to be of the same form. Therefore, one should require the right hand side of (4.16) to be a regular function in the finite part of the complex plane. Vanishing of the residues at the points u_i yields the system of Bethe equations (4.15).

Let us denote

$$Q(u) = \prod_{j=1}^{n} \sinh(u - u_j),$$

then equation (4.16) can be written in the form

$$T(u)Q(u) = \mathsf{a}(u)Q(u-\eta) + \mathsf{d}(u)Q(u+\eta), \tag{4.17}$$

and the Bethe equations are

$$\frac{\mathsf{a}(u_j)}{\mathsf{d}(u_j)} = -\frac{Q(u_j+\eta)}{Q(u_j-\eta)}.$$

It appears that it is possible to construct an *operator* $\mathsf{Q}(u)$ such that a) it commutes with all transfer matrices, i.e. $[\mathsf{T}(u), \mathsf{Q}(v)] = 0$ for all u, v and b) its eigenvalues on Bethe vectors $|\Phi(u_1, \ldots, u_n)\rangle$ are equal to $Q(u)$. The relation (4.17) can be written in the operator form:

$$\mathsf{T}(u)\mathsf{Q}(u) = \mathsf{a}(u)\mathsf{Q}(u-\eta) + \mathsf{d}(u)\mathsf{Q}(u+\eta).$$

It is called the TQ-relation, and $\mathsf{Q}(u)$ is called the Baxter Q-operator.

4.2.3 Limit to the XXX type models and the algebra sl_2.

Let us renormalize $u \to \eta u$ and tend $\eta \to 0$. This is the limit to the XXX spin chain. The trigonometric R-matrix becomes rational (polynomial):

$$\mathsf{R}(u) = \begin{pmatrix} u{+}1 & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & u{+}1 \end{pmatrix} = u\mathsf{I} + \mathsf{P}. \tag{4.18}$$

Note that it is $GL(2)$-invariant in the following sense:

$$\mathbf{g} \otimes \mathbf{g}\, \mathsf{R}(u) = \mathsf{R}(u)\, \mathbf{g} \otimes \mathbf{g} \tag{4.19}$$

for any non-degenerate matrix $\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$. The L-operator acquires the form

$$\mathsf{L}(u) = \begin{pmatrix} u + \frac{1}{2}\sigma_z & \sigma_- \\ \sigma_+ & u - \frac{1}{2}\sigma_z \end{pmatrix}.$$

The generating function of integrals of motion for the XXX spin chain is constructed as $\mathsf{T}(u) = \mathrm{tr}\Big(\mathsf{L}_1(u)\dots\mathsf{L}_N(u)\Big)$. As in the XXZ-case, the model admits an integrable inhomogeneous generalization. The $GL(2)$-invariance allows one to introduce integrable models with non-periodic (twisted) boundary conditions by insertion of an arbitrary group element $\mathbf{g} \in GL(2)$ under the trace, and the more general commutative family is

$$\mathsf{T}(u) = \mathrm{tr}\Big(\mathbf{g}\,\mathsf{L}_1(u-u_1)\dots\mathsf{L}_N(u-u_N)\Big). \tag{4.20}$$

A more general L-operator which intertwines by the rational R-matrix (i.e., satisfies the $RLL = LLR$ relation) can be found in the form

$$\mathsf{L}(u) = \begin{pmatrix} u + \frac{1}{2}\mathsf{S} & \mathsf{S}_- \\ \mathsf{S}_+ & u - \frac{1}{2}\mathsf{S} \end{pmatrix}. \tag{4.21}$$

Here $\mathsf{S}, \mathsf{S}_\pm$ are some yet undetermined operators. The substitution to the $RLL = LLR$ relation gives the algebra

$$[\mathsf{S},\, \mathsf{S}_\pm] = \pm 2\mathsf{S}_\pm, \qquad [\mathsf{S}_+,\, \mathsf{S}_-] = \mathsf{S}. \tag{4.22}$$

These are commutation relations of the Lie algebra sl_2 (one can think of it as embedded into the universal enveloping algebra $U(sl_2)$). This L-operator acts in the tensor product $\mathbb{C}^2 \otimes \mathcal{V}$, where $\mathcal{V}$ is the representation space of some representation of this algebra. Accordingly, the quantum space of the model with the transfer matrix (4.20) and the L-operator (4.21) is $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{V}_i$, where $\mathcal{V}_i \cong \mathcal{V}$.

Representations of the algebra $U(sl_2)$ can be realized by differential operators in the space of functions of a complex variable z:

$$\mathsf{S}_- = \partial_z, \qquad \mathsf{S} = z\partial_z - \ell, \qquad \mathsf{S}_+ = -z^2\partial_z + 2\ell z. \tag{4.23}$$

In general this representation is irreducible and infinite-dimensional. If, however, $2\ell+1 \in \mathbb{Z}_+$, it becomes reducible and it splits off $(2\ell+1)$-dimensional irreducible representation of spin ℓ (integer or half-integer), which is realized in the space of polynomials of degree 2ℓ. In particular, at $\ell = \frac{1}{2}$ we have the representation $\mathsf{S} = \sigma_z$, $\mathsf{S}_\pm = \sigma_\pm$.

Let us give an example of the L-operator with an infinite-dimensional quantum space. Introduce bosonic creation and annihilation operators on the lattice $\psi_n^\dagger$, ψ_n with the commutation relations $[\psi_n, \psi_m^\dagger] = \Delta^{-1}\delta_{mn}$, where Δ is the lattice spacing and take the operators $\mathsf{S}, \mathsf{S}_\pm$ in the form

$$\begin{aligned} \mathsf{S}^{(n)} &= -\frac{4}{c\Delta}\left(1 + \frac{c\Delta^2}{2}\,\psi_n^\dagger\psi_n\right) \\ \mathsf{S}_+^{(n)} &= 2i\left(1 + \frac{c\Delta^2}{4}\,\psi_n^\dagger\psi_n\right)\psi_n \\ \mathsf{S}_-^{(n)} &= \frac{2i}{c}\,\psi_n^\dagger. \end{aligned} \tag{4.24}$$

Problem. Check that these operators satisfy the algebra sl_2 (4.22).

The L-operator of the lattice version of the Bose gas model (quantum nonlinear Schrodinger equation or QNLS in short) with the constant c on nth site of the lattice has the form

$$\mathsf{L}_n^{\text{QNLS}}(\lambda) = -\tfrac{1}{2}\, c\Delta\sigma_z \mathsf{L}_n(i\lambda/c)$$

$$= \begin{pmatrix} 1 - \frac{1}{2}\, i\lambda\Delta + \frac{1}{2}\, c\Delta^2\psi_n^\dagger\psi_n & -i\Delta\psi_n^\dagger \\ ic\Delta\Big(1 + \frac{1}{4}\, c\Delta^2\psi_n^\dagger\psi_n\Big)\psi_n & 1 + \frac{1}{2}\, i\lambda\Delta + \frac{1}{2}\, c\Delta^2\psi_n^\dagger\psi_n \end{pmatrix}, \tag{4.25}$$

where L_n is the L-operator (4.21) on the nth site with the operators $\mathsf{S}, \mathsf{S}_\pm$ as in (4.24). The quantum space of this L-operator is the Fock space spanned by vectors which are obtained by action of the creation operators $\psi_n^\dagger$ to the vacuum state $|0\rangle$ ($\psi_n\,|0\rangle = 0$).

In the continuum limit $\Delta \to 0$, $N \to \infty$ (here N is the number of sites in the lattice), in such a way that $L = N\Delta$ is fixed. It can be shown that the Hamiltonian of the QNLS equation (2.55) is contained in the expansion of the transfer matrix in inverse powers of λ.

4.2.4 The XXZ type models and the q-deformation of the algebra sl_2.

By analogy with the rational vase, a more general L-operator which intertwines by the trigonometric R-matrix can be found in the form

$$\mathsf{L}(u) = \begin{pmatrix} \sinh\Big(u + \frac{\eta}{2}\,\mathsf{S}\Big) & \sinh\eta\,\mathsf{S}_- \\ \sinh\eta\,\mathsf{S}_+ & \sinh\Big(u - \frac{\eta}{2}\,\mathsf{S}\Big) \end{pmatrix}. \tag{4.26}$$

The substitution into $RLL = LLR$ leads to the following algebra:

$$[\mathsf{S},\, \mathsf{S}_\pm] = \pm 2\mathsf{S}_\pm, \qquad [\mathsf{S}_+,\, \mathsf{S}_-] = \frac{\sinh(\eta\mathsf{S})}{\sinh\eta}.$$

These relations define the algebra $U_q(sl_2)$, a q-deformation of $U(sl_2)$. Set $q = e^\eta$ and introduce the generators

$$\mathsf{A} = q^{\mathsf{S}/2}, \quad \mathsf{D} = q^{-\mathsf{S}/2}, \quad \mathsf{B} = \mathsf{S}_+, \quad \mathsf{C} = \mathsf{S}_-,$$

then the defining relations are written in the form

$$\mathsf{AB} = q\mathsf{BA}, \quad \mathsf{BD} = q\mathsf{DB}, \quad \mathsf{DC} = q\mathsf{CD}, \quad \mathsf{CA} = q\mathsf{AC}, \quad [\mathsf{B},\, \mathsf{C}] = \frac{\mathsf{A}^2 - \mathsf{D}^2}{q - q^{-1}}. \tag{4.27}$$

The Casimir element (the central element) is given by

$$\Omega = \frac{q^{-1}\mathsf{A}^2 + q\mathsf{D}^2 - 2}{(q - q^{-1})^2} + \mathsf{BC} = \frac{\sinh^2(\frac{\eta}{2}(\mathsf{S} - 1))}{\sinh^2\eta} + \mathsf{S}_+\mathsf{S}_-. \tag{4.28}$$

If q is in general position (not equal to a root of unity), then the representations of the algebra $U_q(sl_2)$ are smooth deformations of the representations of $U(sl_2)$. The generators can be realized by difference operators in the space of functions of a complex variable z. Introduce q-shift operators $T_\pm f(z) = f(q^{\pm 1}z)$, then

$$
\begin{aligned}
\mathsf{A} &= q^{-\ell}T_+, \qquad \mathsf{D} = q^{\ell}T_-, \\
\mathsf{B} &= \frac{z}{q^{-1}-q}\Big(q^{-2\ell}T_+ - q^{2\ell}T_-\Big), \\
\mathsf{C} &= \frac{z^{-1}}{q^{-1}-q}\Big(T_- - T_+\Big).
\end{aligned}
\tag{4.29}
$$

As in the case of $U(sl_2)$, at $2\ell+1 \in \mathbb{Z}_+$ this representation becomes reducible and it splits off $(2\ell+1)$-dimensional irreducible representation of spin ℓ realized in the space of polynomials of degree 2ℓ. At $\ell = \frac{1}{2}$ we have two-dimensional representation $\mathsf{S} = \sigma_z$, $\mathsf{S}_\pm = \sigma_\pm$.

4.2.5 The trigonometric R-matrix and the quantized algebra of functions on the group $GL(2)$

Instead of symmetric R-matrix (3.9) one can work with the asymmetric one

$$
\bar{\mathsf{R}}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & e^{u}\sinh\eta & 0 \\ 0 & e^{-u}\sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}. \tag{4.30}
$$

Problem. Prove that this R-matrix satisfies the Yang-Baxter equation.

It turns out that there exists a solution to the intertwining relation

$$
\bar{\mathsf{R}}_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\bar{\mathsf{R}}_{12}(u-v),
$$

which does not depend on the spectral parameter. It has the form $\mathcal{T}(u) = \hat{\mathbf{g}}$, where $\hat{\mathbf{g}}$ is the matrix with non-commutative matrix elements

$$
\hat{\mathbf{g}} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix},
$$

satisfying the algebra

$$
\hat{a}\hat{b} = q\hat{b}\hat{a}, \quad \hat{a}\hat{c} = q\hat{c}\hat{a}, \quad \hat{b}\hat{d} = q\hat{d}\hat{b}, \quad \hat{c}\hat{d} = q\hat{d}\hat{c}, \quad [\hat{b},\hat{c}] = 0, \quad [\hat{a},\hat{d}] = (q-q^{-1})\hat{b}\hat{c}. \tag{4.31}
$$

Here $q = e^{\eta}$, and the intertwining relation acquires the form

$$
\bar{\mathsf{R}}_{12}(u-v)\hat{\mathbf{g}}_1\hat{\mathbf{g}}_2 = \hat{\mathbf{g}}_2\hat{\mathbf{g}}_1\bar{\mathsf{R}}_{12}(u-v). \tag{4.32}
$$

This relation generalizes the property of the $GL(2)$-invariance (4.19) of the rational R-matrix. The algebra with generators and relations (4.31) is known as $GL_q(2)$, the quantized algebra of functions on the group $GL(2)$.

4.3 Algebraic Bethe ansatz for the 8-vertex model

The R-matrix (3.27) can be also represented in the form of the L-operator

$$\mathsf{L}(u) = \begin{pmatrix} W_0(u)\sigma_0 + W_3(u)\sigma_3 & W_1(u)\sigma_1 - iW_2(u)\sigma_2 \\ \\ W_1(u)\sigma_1 + iW_2(u)\sigma_2 & W_0(u)\sigma_0 - W_3(u)\sigma_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathsf{a}}(u) & \hat{\mathsf{b}}(u) \\ \\ \hat{\mathsf{c}}(u) & \hat{\mathsf{d}}(u) \end{pmatrix} \qquad (4.33)$$

which is the 2×2 matrix whose matrix elements are operators in $\mathbb{C}^2$. Clearly, it is the same R-matrix (3.27) written as a block matrix. Usually in the literature the L-operator differs from the R-matrix by a shift of the spectral parameter $u \to u - \eta/2$ but we do not make this shift here. The Yang-Baxter equation for R is the $RLL = LLR$ relation for L

$$\mathsf{R}_{12}(u-v)\mathsf{L}_1(u)\mathsf{L}_2(v) = \mathsf{L}_2(v)\mathsf{L}_1(u)\mathsf{R}_{12}(u-v), \qquad (4.34)$$

where $\mathsf{L}_1(u) = \mathsf{L}(u) \otimes 1$, $\mathsf{L}_2(v) = 1 \otimes \mathsf{L}(v)$.

The quantum monodromy matrix of the inhomogeneous 8-vertex model is

$$\mathcal{T}(u) = \mathsf{L}_1(u-\xi_1)\mathsf{L}_2(u-\xi_2)\ldots\mathsf{L}_N(u-\xi_N) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \qquad (4.35)$$

where ξ_i are inhomogeneity parameters. Here $\mathsf{L}_j(u)$ is given by the formula (4.33), where the sigma-matrices $\sigma_a^{(j)}$ act in the j-th copy of $\mathbb{C}^2$ associated with the j-th site of the lattice. We consider the case of even $N = 2n$, otherwise solvability of the model is problematic. Equations (3.29) imply the following properties of the quantum monodromy matrix:

$$\begin{aligned} \mathcal{T}(u+1) &= \sigma_3\mathcal{T}(u)\sigma_3 = \begin{pmatrix} A(u) & -B(u) \\ -C(u) & D(u) \end{pmatrix} \\ \mathcal{T}(u+\tau) &= e^{-\pi i c(u)}\sigma_1\mathcal{T}(u)\sigma_1 = e^{-\pi i c(u)} \begin{pmatrix} D(u) & C(u) \\ B(u) & A(u) \end{pmatrix}, \end{aligned} \qquad (4.36)$$

where

$$c(u) = N(2u + \eta + \tau) - 2\sum_{k=1}^{N} \xi_k. \qquad (4.37)$$

The quantum monodromy matrix satisfies the $RTT = TTR$ relation (4.2) (or (4.1)). This relation implies that the transfer matrices

$$\begin{aligned} \mathsf{T}(u) &= \mathrm{tr}_0 \Big(\mathsf{R}_{01}(u-\xi_1)\mathsf{R}_{02}(u-\xi_2)\ldots\mathsf{R}_{0N}(u-\xi_N)\Big) \\ &= \mathrm{tr}\Big(\mathsf{L}_1(u-\xi_1)\mathsf{L}_2(u-\xi_2)\ldots\mathsf{L}_N(u-\xi_N)\Big) = \mathrm{tr}\,\mathcal{T}(u) = A(u) + D(u) \end{aligned} \qquad (4.38)$$

commute for different values of the spectral parameter u. The transfer matrix is an operator in the space $\mathcal{H} = \bigotimes_{i=1}^{N} V_i$, $V_i \cong \mathbb{C}^2$.

4.3.1 Vacuum vectors

The L-operator of the 8-vertex model does not have a vacuum vector, i.e. a vector annihilated by the operator $\hat{\mathsf{c}}(u)$, because the matrix $\hat{\mathsf{c}}(u)$ is non-degenerate for almost all u. This fact makes it impossible to apply directly the algebraic Bethe ansatz method used for the solution of the 6-vertex model. The starting point of the algebraic Bethe ansatz for the 8-vertex model is the rule of the action of the R-matrix (3.27) to some special vectors. Note that when one acts by the R-matrix to the tensor product of two vectors, one in general obtains a linear combination of pure tensor products. The situation when one gets just one pure tensor product term,

$$\mathsf{R}(u)\begin{pmatrix} x \\ 1 \end{pmatrix}\otimes\begin{pmatrix} x' \\ 1 \end{pmatrix} = \rho\begin{pmatrix} y \\ 1 \end{pmatrix}\otimes\begin{pmatrix} y' \\ 1 \end{pmatrix}, \tag{4.39}$$

is exceptional. This property was called by Baxter "passing of a pair of vectors through the vertex" and it played a very important role in his solution of the 8-vertex model. As we will see below, the vectors that satisfy this property are parametrized by points of an elliptic curve. Indeed, applying (from the left) to the both sides of relation (4.39) the covector $(-1, y)$ in the first space, we conclude that the operator

$$(-1, y)\begin{pmatrix} \hat{\mathsf{a}}(u) & \hat{\mathsf{b}}(u) \\ \hat{\mathsf{c}}(u) & \hat{\mathsf{d}}(u) \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = -x\hat{\mathsf{a}} - \hat{\mathsf{b}} + xy\hat{\mathsf{c}} + y\hat{\mathsf{d}}$$

has an eigenvector with zero eigenvalue (a vacuum vector). This means that the determinant of this operator (a 2×2 matrix) vanishes:

$$\det\begin{pmatrix} yb - xa & xyc - d \\ xyd - c & ya - xb \end{pmatrix} = 0,$$

or

$$\Gamma(x^2y^2 + 1) + 2\Delta xy - x^2 - y^2 = 0, \tag{4.40}$$

where Γ and Δ are given by (3.22). We see that the components of the vectors x, y should lie on the elliptic curve (4.40). For what follows it is extremely convenient to introduce a uniformization parameter on this elliptic curve.

Let us introduce the family of vectors

$$|\phi(s)\rangle = \begin{pmatrix} \theta_1(s|2\tau) \\ \theta_4(s|2\tau) \end{pmatrix}, \tag{4.41}$$

where s is a complex parameter uniformizing the elliptic curve. The covector orthogonal to $|\phi(s)\rangle$ is

$$\langle\phi^\perp(s)| = \Big(-\theta_4(s|2\tau), \theta_1(s|2\tau)\Big) = ie^{-\pi i(s+\frac{\tau}{2})}\langle\phi(s+\tau+1)|$$

and the scalar product $\langle\phi^\perp(t)|\phi(s)\rangle$ is given by

$$\begin{aligned}\langle\phi^\perp(t)|\phi(s)\rangle &= \theta_1\left(\tfrac{1}{2}(t-s)\Big|\tau\right)\theta_2\left(\tfrac{1}{2}(t+s)\Big|\tau\right) \\ &= 2\frac{\theta_1(\frac{1}{2}(t-s)|2\tau)\theta_4(\frac{1}{2}(t-s)|2\tau)\theta_2(\frac{1}{2}(t+s)|2\tau)\theta_3(\frac{1}{2}(t+s)|2\tau)}{\theta_2(0|2\tau)\theta_3(0|2\tau)}.\end{aligned} \tag{4.42}$$

Using the identities for the theta-functions, one can prove the following important identity:

$$\mathsf{R}(u)\,|\phi(s{+}\eta)\rangle \otimes |\phi(s-u)\rangle = \theta_1(u+\eta|\tau)\;|\phi(s)\rangle \otimes |\phi(s{-}u{+}\eta)\rangle \tag{4.43}$$

or, indicating explicitly the spaces where the vectors live:

$$\mathsf{R}_{12}(u)\,|\phi(s{+}\eta)\rangle_1\,|\phi(s-u)\rangle_2 = \theta_1(u+\eta|\tau)\;|\phi(s)\rangle_1\,|\phi(s{-}u{+}\eta)\rangle_2\,. \tag{4.44}$$

This is nothing else than relation (4.39) written in the uniformizing parameter of the elliptic curve. Here the first space is associated with the horizontal leg of the R-matrix and the second space is associated with the vertical leg.

There are other useful versions of identity (4.44). Changing $u \to -u$, $\eta \to -\eta$ in (4.44) and using (3.32), we write it in the form

$$\mathsf{R}_{12}(u)\,|\phi(s{-}\eta)\rangle_1\,|\phi(s+u)\rangle_2 = \theta_1(u+\eta|\tau)\;|\phi(s)\rangle_1\,|\phi(s{+}u{-}\eta)\rangle_2\,. \tag{4.45}$$

Shifting $s \to s+\tau+1$ and transposing in the both spaces, we also get the transposed version of equation (4.44):

$$\langle\phi^{\perp}(s+\eta)|_1\langle\phi^{\perp}(s-u)|_2\mathsf{R}_{12}(u) = \theta_1(u+\eta|\tau)\,\langle\phi^{\perp}(s)|_1\langle\phi^{\perp}(s-u+\eta)|_2. \tag{4.46}$$

Shifting $u \to u-\xi$ and then $s \to s+u$ in (4.43) and taking the scalar product of both sides with the covector $\langle\phi^{\perp}(s+u)|$, we get:

$$\langle\phi^{\perp}(s+u)|_1\mathsf{R}_{12}(u-\xi)|\phi(s{+}u{+}\eta)\rangle_1\;|\phi(s+\xi)\rangle_2 = 0, \tag{4.47}$$

where ξ is an additional arbitrary parameter. Here the operator

$$\langle\phi^{\perp}(s+u)|_1\mathsf{R}_{12}(u-\xi)|\phi(s{+}u{+}\eta)\rangle_1$$

acts in the vertical space (the space number 2). Taking the scalar product of (4.43) with the covector $\langle\phi^{\perp}(t)|$, we obtain, after the shifts $u \to u-\xi$, $s \to s+u$, $t \to t-u$:

$$\frac{\langle\phi^{\perp}(t-u)|_1\mathsf{R}_{12}(u-\xi)|\phi(s{+}u{+}\eta)\rangle_1}{\langle\phi^{\perp}(t-u)|\phi(s+u)\rangle}\;|\phi(s+\xi)\rangle_2 = \theta_1(u{-}\xi{+}\eta|\tau)\;|\phi(s{+}\xi{+}\eta\rangle_2\,. \tag{4.48}$$

Shifting the arguments in (4.44) and changing $\eta \to -\eta$, using the property (3.32) and transposing in the first space, we can obtain the following important corollary:

$$\langle\phi^{\perp}(s)|_1\mathsf{R}_{12}(u)|\phi(s-u)\rangle_2 = \theta_1(u|\tau)\langle\phi^{\perp}(s+\eta)|_1|\phi(s-u-\eta)\rangle_2 \tag{4.49}$$

or, what is the same but with an additional parameter ξ,

$$\langle\phi^{\perp}(s+u)|_1\mathsf{R}_{12}(u-\xi)|\phi(s+\xi)\rangle_2 = \theta_1(u-\xi|\tau)\langle\phi^{\perp}(s+u+\eta)|_1|\phi(s+\xi-\eta)\rangle_2. \tag{4.50}$$

Here $\langle\ldots|_1\,|\ldots\rangle_2$ is not a scalar product but the tensor product of the vector and covector (which live in difference spaces). Taking the scalar product with the vector $|\phi(t-u+\eta)\rangle_1$ in the first space, we can write this identity in the following form:

$$\frac{\langle\phi^{\perp}(s+u)|_1\mathsf{R}_{12}(u-\xi)|\phi(t-u+\eta)\rangle_1}{\langle\phi^{\perp}(s+u+\eta)|\phi(t-u+\eta)\rangle}\;|\phi(s+\xi)\rangle_2 = \theta_1(u-\xi|\tau)\,|\phi(s+\xi-\eta)\rangle_2\,. \tag{4.51}$$

Let us now give a more general identity for the intertwining vectors which can be proved basically in the same way as (4.44):

$$\begin{aligned}&\mathsf{R}_{12}(u)\,|\phi(s+\eta)\rangle_1\,|\phi(t-u)\rangle_2\\ &=\frac{\theta_1(\eta|\tau)\theta_2(\frac{1}{2}(s+t)-u|\tau)}{\theta_2(\frac{1}{2}(s+t)|\tau)}\,|\phi(t)\rangle_1\,|\phi(s+u+\eta)\rangle_2\\ &+\frac{\theta_1(u|\tau)\theta_2(\frac{1}{2}(s+t)+\eta|\tau)}{\theta_2(\frac{1}{2}(s+t)|\tau)}\,|\phi(s)\rangle_1\,|\phi(t-u-\eta)\rangle_2.\end{aligned} \tag{4.52}$$

It provides a rule of how the R-matrix acts to tensor products of two arbitrary vectors. At $t=s$ (4.52) coincides with (4.44) (this can be seen after using an identity for the theta-functions). Substituting $u\to -u$, $\eta\to -\eta$, we also obtain:

$$\begin{aligned}&\mathsf{R}_{12}(u)\,|\phi(s-\eta)\rangle_1\,|\phi(t+u)\rangle_2\\ &=\frac{\theta_1(\eta|\tau)\theta_2(\frac{1}{2}(s+t)+u|\tau)}{\theta_2(\frac{1}{2}(s+t)|\tau)}\,|\phi(t)\rangle_1\,|\phi(s-u-\eta)\rangle_2\\ &+\frac{\theta_1(u|\tau)\theta_2(\frac{1}{2}(s+t)-\eta|\tau)}{\theta_2(\frac{1}{2}(s+t)|\tau)}\,|\phi(s)\rangle_1\,|\phi(t+u+\eta)\rangle_2.\end{aligned} \tag{4.53}$$

The next step is to consider a gauge transformation of the L-operator

$$\mathsf{L}'_k(u,\xi_k)=M^{-1}_{k+l-1}(u)\mathsf{L}_k(u-\xi_k)M_{k+l}(u)=\begin{pmatrix}\hat{\mathsf{a}}'_k(u) & \hat{\mathsf{b}}'_k(u)\\ \hat{\mathsf{c}}'_k(u) & \hat{\mathsf{d}}'_k(u)\end{pmatrix}, \tag{4.54}$$

where $l\in\mathbb{Z}$ is an integer parameter. The matrix $M_k(u)$ is given by

$$M_k(u)=\begin{pmatrix}\theta_1(s_k+u|2\tau) & \gamma_k\theta_1(t_k-u|2\tau)\\ \theta_4(s_k+u|2\tau) & \gamma_k\theta_4(t_k-u|2\tau)\end{pmatrix}, \tag{4.55}$$

where $s_k=s+k\eta$, $t_k=t+k\eta$, $s,t\in\mathbb{C}$ are arbitrary parameters and

$$\gamma_k=\frac{1}{\theta_2(\tau_k|2\tau)\theta_3(\tau_k|2\tau)},\qquad \tau_k=\frac{1}{2}(s_k+t_k). \tag{4.56}$$

Note that the columns of this matrix are the intertwining vectors. The inverse matrix is

$$M_k^{-1}(u)=\frac{1}{\det M_k(u)}\begin{pmatrix}\gamma_k\theta_4(t_k-u|2\tau) & -\gamma_k\theta_1(t_k-u|2\tau)\\ -\theta_4(s_k+u|2\tau) & \theta_1(s_k+u|2\tau)\end{pmatrix}, \tag{4.57}$$

where

$$\begin{aligned}\det M_k(u)=&\;-\gamma_k\langle\phi^{\perp}(t_k-u)|\phi(s_k+u)\rangle\\ &=\gamma_k\theta_1(\tfrac{1}{2}(s-t)+u|\tau)\theta_2(\tau_k|\tau)\\ &=2\frac{\theta_1(\frac{1}{2}(s-t)+u|2\tau)\theta_4(\frac{1}{2}(s-t)+u|2\tau)}{\theta_2(0|2\tau)\theta_3(0|2\tau)}\equiv\mu(u).\end{aligned} \tag{4.58}$$

It is important that $\det M_k(u) = \mu(u)$ does not depend on k.

The gauge-transformed L-operator (4.54) has a local u-independent vacuum vector

$$\left|\omega_k^l\right\rangle = \begin{pmatrix} \theta_1(s_{k+l-1}+\xi_k|2\tau) \\ \theta_4(s_{k+l-1}+\xi_k|2\tau) \end{pmatrix} = |\phi(s_{k+l-1}+\xi_k)\rangle_k \in V_k \tag{4.59}$$

which is annihilated by the left lower element $\hat{\mathsf{c}}'_k(u)$:

$$\hat{\mathsf{c}}'_k(u)\left|\omega_k^l\right\rangle = 0 \tag{4.60}$$

(recall that $\hat{\mathsf{c}}'_k(u)$ depends also on s and l). This directly follows from equation (4.47) (one should put $s = s_{k+l-1}$ in the latter). In their turn, equations (4.48) and (4.51) (where one should put $s = s_{k+l-1}$, $t = t_{k+l-1}$) tell us how the operators $\mathsf{a}'_k(u)$, $\mathsf{d}'_k(u)$ act to the vacuum vector:

$$\begin{aligned} \hat{\mathsf{a}}'_k(u)\left|\omega_k^l\right\rangle &= \theta_1(u-\xi_k+\eta|\tau)\left|\omega_k^{l+1}\right\rangle, \\ \hat{\mathsf{d}}'_k(u)\left|\omega_k^l\right\rangle &= \theta_1(u-\xi_k|\tau)\left|\omega_k^{l-1}\right\rangle. \end{aligned} \tag{4.61}$$

Unlike the situation in the 6-vertex model, the vacuum vector is not an eigenvector for these operators but transforms in a simple way.

The gauge-transformed quantum monodromy matrix is

$$\begin{aligned} \mathcal{T}'(u) &= \mathsf{L}'_1(u-\xi_1)\mathsf{L}'_2(u-\xi_2)\dots\mathsf{L}'_N(u-\xi_N) \\ &= M_l^{-1}(u)\mathcal{T}(u)M_{N+l}(u) = \begin{pmatrix} A^l(u) & B^l(u) \\ C^l(u) & D^l(u) \end{pmatrix}. \end{aligned}$$

The global vacuum vectors are defined as

$$\left|\Omega^l\right\rangle = \left|\omega_1^l\right\rangle \otimes \left|\omega_2^l\right\rangle \otimes \dots \otimes \left|\omega_N^l\right\rangle.$$

According to (4.60), (4.61), the action of the operators $A^l(u)$, $D^l(u)$ and $C^l(u)$ on the global vacuum vector is given by

$$\begin{aligned} C^l(u)\left|\Omega^l\right\rangle &= 0, \\ A^l(u)\left|\Omega^l\right\rangle &= \prod_{i=1}^N \theta_1(u-\xi_i+\eta|\tau)\left|\Omega^{l+1}\right\rangle, \\ D^l(u)\left|\Omega^l\right\rangle &= \prod_{i=1}^N \theta_1(u-\xi_i|\tau)\left|\Omega^{l-1}\right\rangle. \end{aligned} \tag{4.62}$$

4.3.2 The permutation relations

Let us introduce the generalized monodromy matrices

$$\mathcal{T}_{k,l}(u) = M_k^{-1}(u)\mathcal{T}(u)M_l(u) = \begin{pmatrix} A_{k,l}(u) & B_{k,l}(u) \\ C_{k,l}(u) & D_{k,l}(u) \end{pmatrix}. \tag{4.63}$$

Note that in this new notation $\mathcal{T}'(u) = \mathcal{T}_{l,l+N}(u)$. We have:

$$\begin{aligned} A_{k,l}(u) &= \frac{\langle \phi^{\perp}(t_k - u)|\mathcal{T}(u)|\phi(s_l + u)\rangle}{\langle \phi^{\perp}(t_k - u)|\phi(s_k + u)\rangle}, \\ B_{k,l}(u) &= \gamma_l \frac{\langle \phi^{\perp}(t_k - u)|\mathcal{T}(u)|\phi(t_l - u)\rangle}{\langle \phi^{\perp}(t_k - u)|\phi(s_k + u)\rangle}, \\ C_{k,l}(u) &= -\frac{1}{\gamma_k} \frac{\langle \phi^{\perp}(s_k + u)|\mathcal{T}(u)|\phi(s_l + u)\rangle}{\langle \phi^{\perp}(t_k - u)|\phi(s_k + u)\rangle}, \\ D_{k,l}(u) &= -\frac{\gamma_l}{\gamma_k} \frac{\langle \phi^{\perp}(s_k + u)|\mathcal{T}(u)|\phi(t_l - u)\rangle}{\langle \phi^{\perp}(t_k - u)|\phi(s_k + u)\rangle}. \end{aligned} \tag{4.64}$$

It follows from equations (4.36) that the generalized monodromy matrix has the following properties:

$$\begin{aligned} \mathcal{T}_{k,l}(u+1) &= \mathcal{T}_{k,l}(u), \\ \mathcal{T}_{k,l}(u+\tau) &= e^{-\pi i c(u)} \begin{pmatrix} e^{\pi i s_k} & 0 \\ 0 & -e^{-\pi i t_k} \end{pmatrix} \mathcal{T}_{k,l}(u) \begin{pmatrix} e^{-\pi i s_l} & 0 \\ 0 & -e^{\pi i t_l} \end{pmatrix} \\ &= e^{-\pi i c(u)} \begin{pmatrix} e^{\pi i (s_k - s_l)} A_{k,l}(u) & -e^{\pi i (s_k + t_l)} B_{k,l}(u) \\ -e^{-\pi i (s_l + t_k)} C_{k,l}(u) & e^{\pi i (t_l - t_k)} D_{k,l}(u) \end{pmatrix}, \end{aligned} \tag{4.65}$$

where $c(u)$ is defined in (4.37).

For the calculations below it is convenient to introduce the auxiliary notation for the vectors

$$X^l(u) = |\phi(s_l + u)\rangle, \quad Y^l(u) = |\phi(t_l - u)\rangle,$$
$$\tilde{X}^k(u) = \langle \phi^{\perp}(s_k + u)|, \quad \tilde{Y}^k(u) = \langle \phi^{\perp}(t_k - u)|,$$

then

$$\begin{aligned} A_{k,l}(u) &= -\frac{\gamma_k}{\mu(u)} \tilde{Y}^k(u)\mathcal{T}(u)X^l(u), \\ B_{k,l}(u) &= -\frac{\gamma_k \gamma_l}{\mu(u)} \tilde{Y}^k(u)\mathcal{T}(u)Y^l(u), \\ C_{k,l}(u) &= \frac{1}{\mu(u)} \tilde{X}^k(u)\mathcal{T}(u)X^l(u), \\ D_{k,l}(u) &= \frac{\gamma_l}{\mu(u)} \tilde{X}^k(u)\mathcal{T}(u)Y^l(u). \end{aligned}$$

In this notation, equations (4.44), (4.45), (4.52), (4.53) look as follows:

$$\mathsf{R}_{12}(u-v)X_1^{l+1}(u)X_2^l(v) = \theta_1(u-v+\eta|\tau)X_1^l(u)X_2^{l+1}(v), \tag{4.66}$$

$$\mathsf{R}_{12}(u-v)Y_1^{l-1}(u)Y_2^l(v) = \theta_1(u-v+\eta|\tau)Y_1^l(u)Y_2^{l-1}(v), \tag{4.67}$$

$$\mathsf{R}_{12}(u-v)Y_1^{l+1}(u)X_2^l(v) = f_l^+(u-v)Y_1^l(u)X_2^{l-1}(v) + g_l(v-u)X_1^l(u)Y_2^{l+1}(v), \quad (4.68)$$

$$\mathsf{R}_{12}(u-v)X_1^k(u)Y_2^{k-1}(v) = f_k^-(u-v)X_1^{k+1}(u)Y_2^k(v) + g_k(u-v)Y_1^{k-1}(u)X_2^k(v), \quad (4.69)$$

$$\mathsf{R}_{12}(u-v)Y_1^l(u)X_2^{l+1}(v) = f_l^+(u-v)Y_1^{l-1}(u)X_2^l(v) + g_l(v-u)X_1^{l+1}(u)Y_2^l(v), \quad (4.70)$$

$$\mathsf{R}_{12}(u-v)X_1^{k-1}(u)Y_2^k(v) = f_k^-(u-v)X_1^k(u)Y_2^{k+1}(v) + g_k(u-v)Y_1^k(u)X_2^{k-1}(v), \quad (4.71)$$

where

$$f_k^{\pm}(u) = \frac{\theta_1(u|\tau)\theta_2(\tau_{k\pm1}|\tau)}{\theta_2(\tau_k|\tau)}, \quad g_k(u) = \frac{\theta_1(\eta|\tau)\theta_2(\tau_k+u|\tau)}{\theta_2(\tau_k|\tau)}.$$

Similar relations hold for the covectors $\tilde{X}^l(u)$, $\tilde{Y}^l(u)$; they are obtained by transposition in both spaces.

Multiplying both sides of the $RTT = TTR$ relation (4.2) by the vectors $Y_1^{l-1}(u)Y_2^l(v)$ from the right and $\tilde{Y}_1^{k-1}(u)\tilde{Y}_2^k(v)$ from the left and using (4.67), one obtains the permutation relation

$$B_{k+1,l}(u)B_{k,l+1}(v) = B_{k+1,l}(v)B_{k,l+1}(u). \quad (4.72)$$

The commutation relations between A- and B-operators are obtained by multiplying both sides of (4.2) by the vectors $Y_1^{l+1}(u)X_2^l(v)$ from the right and $\tilde{Y}_1^{k-1}(u)\tilde{Y}_2^k(v)$ from the left and using the transposed version of (4.67) and (4.68):

$$\begin{aligned} &\theta_1(u-v+\eta|\tau)B_{k,l+1}(u)A_{k-1,l}(v) \\ &= \theta_1(u-v|\tau)A_{k,l-1}(v)B_{k-1,l}(u) + g_l(v-u)B_{k,l+1}(v)A_{k-1,l}(u). \end{aligned} \quad (4.73)$$

Multiplying (4.2) by the vectors $Y_1^l(u)Y_2^{l+1}(v)$ from the right and $\tilde{X}_1^k(u)\tilde{Y}_2^{k-1}(v)$ from the left and using the transposed version of (4.69), one obtains

$$\begin{aligned} &\theta_1(u-v+\eta|\tau)B_{k-1,l}(v)D_{k,l+1}(u) \\ &= \theta_1(u-v|\tau)D_{k+1,l}(u)B_{k,l+1}(v) + g_k(u-v)B_{k-1,l}(u)D_{k,l+1}(v). \end{aligned} \quad (4.74)$$

These are the operator permutation relations which we need in what follows in the generalized algebraic Bethe ansatz procedure. The other commutation relations can be obtained in a similar way.

4.3.3 The generalized algebraic Bethe ansatz

Let us consider the vector

$$|\Psi^l(u_1,\dots,u_n)\rangle = B_{l-1,l+1}(u_1)B_{l-2,l+2}(u_2)\dots B_{l-n,l+n}(u_n)|\Omega^{l-n}\rangle. \quad (4.75)$$

We recall that n is fixed and is equal to $N/2$. The commutation relation (4.72) implies that this vector is a symmetric function of the parameters $u_1,\dots,u_n$. We are going to act to this vector by the transfer matrix $\mathsf{T}(u) = A_{l,l}(u) + D_{l,l}(u)$. To this end, let us rewrite equations (4.73), (4.74) in a more convenient form suitable for moving the operators A and D to the right through B:

$$A_{k,l}(u)B_{k-1,l+1}(v) = \alpha(u-v)B_{k,l+2}(v)A_{k-1,l+1}(u) + \beta_{l+1}(u-v)B_{k,l+2}(u)A_{k-1,l+1}(v), \quad (4.76)$$

$$D_{k,l}(u)B_{k-1,l+1}(v) = \alpha(v-u)B_{k-2,l}(v)D_{k-1,l+1}(u) - \beta_{k-1}(u-v)B_{k-2,l}(u)D_{k-1,l+1}(v), \tag{4.77}$$

where

$$\alpha(u) = \frac{\theta_1(u-\eta|\tau)}{\theta_1(u|\tau)}, \quad \beta_k(u) = \frac{\theta_1(\eta|\tau)\,\theta_2(\tau_k+u|\tau)}{\theta_1(u|\tau)\,\theta_2(\tau_k|\tau)}. \tag{4.78}$$

The action of the operators $A_{l,l}(u)$, $D_{l,l}(u)$ to the vector (4.75) can be found, with the help of the standard algebraic Bethe ansatz argument, using the permutation relations (4.76), (4.77) and the property (4.62). The result is:

$$\begin{aligned} A_{l,l}(u)|\Psi^l(u_1,\dots,u_n)\rangle &= T_A(u)|\Psi^{l+1}(u_1,\dots,u_n)\rangle \\ &+ \sum_{j=1}^n \Lambda^l_{A,j}(u)|\Psi^{l+1}(u_1,\dots,u_{j-1},u,u_{j+1},\dots,u_n)\rangle, \\ D_{l,l}(u)|\Psi^l(u_1,\dots,u_n)\rangle &= T_D(u)|\Psi^{l-1}(u_1,\dots,u_n)\rangle \\ &+ \sum_{j=1}^n \Lambda^l_{D,j}(u)|\Psi^{l-1}(u_1,\dots,u_{j-1},u,u_{j+1},\dots,u_n)\rangle, \end{aligned}$$

where

$$\begin{aligned} T_A(u) &= \prod_{i=1}^N \theta_1(u-\xi_i+\eta|\tau) \prod_{k=1}^n \frac{\theta_1(u-u_k-\eta|\tau)}{\theta_1(u-u_k|\tau)}, \\ T_D(u) &= \prod_{i=1}^N \theta_1(u-\xi_i|\tau) \prod_{k=1}^n \frac{\theta_1(u-u_k+\eta|\tau)}{\theta_1(u-u_k|\tau)}, \\ \Lambda^l_{A,j}(u) &= \frac{\theta_1(\eta|\tau)}{\theta_1'(0|\tau)}\,\Phi\Big(u-u_j,\tau_{l+1}+\frac{1}{2}\Big) \prod_{i=1}^N \theta_1(u_j-\xi_i+\eta|\tau) \prod_{k=1,\neq j}^n \frac{\theta_1(u_j-u_k-\eta|\tau)}{\theta_1(u_j-u_k|\tau)}, \\ \Lambda^l_{D,j}(u) &= -\frac{\theta_1(\eta|\tau)}{\theta_1'(0|\tau)}\,\Phi\Big(u-u_j,\tau_{l-1}+\frac{1}{2}\Big) \prod_{i=1}^N \theta_1(u_j-\xi_i|\tau) \prod_{k=1,\neq j}^n \frac{\theta_1(u_j-u_k+\eta|\tau)}{\theta_1(u_j-u_k|\tau)}. \end{aligned}$$

In the last two formulas we have introduced the function

$$\Phi(u,v) = \frac{\theta_1'(0|\tau)\,\theta_1(u+v|\tau)}{\theta_1(u|\tau)\,\theta_1(v|\tau)}. \tag{4.79}$$

It has a simple pole at $u=0$ with residue 1.

Consider now the Fourier transform of the vector $|\Psi^l\rangle$:

$$|\Psi_\nu(u_1,\dots,u_n)\rangle = \sum_{l\in\mathbb{Z}} e^{-il\pi\eta\nu}|\Psi^l(u_1,\dots,u_n)\rangle. \tag{4.80}$$

The action of the transfer matrix $\mathsf{T}(u) = A_{l,l}(u) + D_{l,l}(u)$ on this vector is given by

$$\begin{aligned} \mathsf{T}(u)|\Psi_\nu(u_1,\dots,u_n)\rangle &= T(u)|\Psi_\nu(u_1,\dots,u_n)\rangle \\ +\sum_{l\in\mathbb{Z}}\sum_{j=1}^n e^{-il\pi\eta\nu}&\left(e^{i\pi\eta\nu}\Lambda^{l-1}_{A,j}(u) + e^{-i\pi\eta\nu}\Lambda^{l+1}_{D,j}(u)\right)|\Psi^l(u_1,\dots,u_{j-1},u,u_{j+1},\dots,u_n)\rangle, \end{aligned} \tag{4.81}$$

where

$$T(u) = e^{i\pi\eta\nu}\prod_{i=1}^{N}\theta_1(u-\xi_i+\eta|\tau)\prod_{k=1}^{n}\frac{\theta_1(u-u_k-\eta|\tau)}{\theta_1(u-u_k|\tau)}$$

$$+\ e^{-i\pi\eta\nu}\prod_{i=1}^{N}\theta_1(u-\xi_i|\tau)\prod_{k=1}^{n}\frac{\theta_1(u-u_k+\eta|\tau)}{\theta_1(u-u_k|\tau)}.$$

Note that one can rewrite (4.81) in the form

$$\begin{aligned}\mathsf{T}(u)\,|\Psi_\nu(u_1,\dots,u_n)\rangle &= T(u)\,|\Psi_\nu(u_1,\dots,u_n)\rangle \\ -\sum_{l\in\mathbb{Z}}\sum_{j=1}^{n} e^{-il\pi\eta\nu}\Phi\Big(u-u_j,\tau_l+\frac{1}{2}\Big)&\Big(\operatorname*{res}_{u=u_j}T(u)\Big)\,\Big|\Psi^l(u_1,\dots,u_{j-1},u,u_{j+1},\dots,u_n)\Big\rangle.\end{aligned} \tag{4.82}$$

In this form, it is clear that the r.h.s. is regular at $u=u_j$ as it should be.

The eigenvalue of the transfer matrix should be a regular function of u. The conditions $\operatorname*{res}_{u=u_j}T(u)=0$ are simultaneously the conditions of cancellation of the "unwanted terms" in (4.82). These conditions have the form of the Bethe equations

$$e^{2i\pi\eta\nu}\prod_{i=1}^{N}\frac{\theta_1(u_j-\xi_i+\eta|\tau)}{\theta_1(u_j-\xi_i|\tau)} = \prod_{k=1,\neq j}^{n}\frac{\theta_1(u_j-u_k+\eta|\tau)}{\theta_1(u_j-u_k-\eta|\tau)}. \tag{4.83}$$

If these equations are satisfied, then the vector $|\Psi(u_1,\dots,u_n)\rangle$ is an eigenvector of the transfer matrix provided that ν is such that the series (4.80) converges and is non-zero. Presumably, this holds for some particular values of ν and $\nu=0$ is among them.

An important remark about the meaning of the Fourier transformation used in the construction of Bethe vectors is in order. The case when it can be given a precise meaning is the case of rational $\eta=2P/Q$ with coprime integers P,Q. In this case the vectors $|\Psi^l\rangle$ are Q-periodic in l and the infinite sum in the Fourier transformation can be substituted by a finite sum from 0 to $Q-1$.

4.4 The Sklyanin algebra

Recall that the elliptic R-matrix has the form (3.25)

$$\mathsf{R}(u) = \sum_{a=0}^{3} W_a(u)\sigma_a\otimes\sigma_a\,,\qquad W_a(u) = \frac{\theta_{5-a}(u+\frac{\eta}{2}|\tau)}{\theta_{5-a}(\frac{\eta}{2}|\tau)}\,, \tag{4.84}$$

where the index is understood modulo 4 and we omit the inessential common multiplier $\theta_1(\eta|\tau)$. Let us find the elliptic L-operator intertwined by this R-matrix in the form

$$\mathsf{L}(u) = \sum_{a=0}^{3} W_a(u)\,\sigma_a\otimes\mathsf{S}_a = \begin{pmatrix} W_0(u)\mathsf{S}_0+W_3(u)\mathsf{S}_3 & W_1(u)\mathsf{S}_1-iW_2(u)\mathsf{S}_2 \\ W_1(u)\mathsf{S}_1+iW_2(u)\mathsf{S}_2 & W_0(u)\mathsf{S}_0-W_3(u)\mathsf{S}_3\end{pmatrix}, \tag{4.85}$$

where $\mathsf{S}_0, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3$ are some yet unknown operators. The substitution into the $RLL = LLR$ relation leads to the following six quadratic relations for them:

$$\begin{aligned} &[\mathsf{S}_0, \mathsf{S}_3] = iJ_{12}[\mathsf{S}_1, \mathsf{S}_2]_+, \qquad [\mathsf{S}_1, \mathsf{S}_2] = i[\mathsf{S}_0, \mathsf{S}_3]_+, \\ &[\mathsf{S}_0, \mathsf{S}_1] = iJ_{23}[\mathsf{S}_2, \mathsf{S}_3]_+, \qquad [\mathsf{S}_2, \mathsf{S}_3] = i[\mathsf{S}_0, \mathsf{S}_1]_+, \\ &[\mathsf{S}_0, \mathsf{S}_2] = iJ_{31}[\mathsf{S}_3, \mathsf{S}_1]_+, \qquad [\mathsf{S}_3, \mathsf{S}_1] = i[\mathsf{S}_0, \mathsf{S}_2]_+, \end{aligned} \tag{4.86}$$

or, in a compact form,

$$[\mathsf{S}_0, \mathsf{S}_\alpha]_- = iJ_{\beta\gamma}[\mathsf{S}_\beta, \mathsf{S}_\gamma]_+, \qquad [\mathsf{S}_\alpha, \mathsf{S}_\beta]_- = i[\mathsf{S}_0, \mathsf{S}_\gamma]_+. \tag{4.87}$$

Here and below $[\ ,\ ]_+$ is the anticommutator and $\{\alpha, \beta, \gamma\}$ stands for any cyclic permutation of indices $\{1, 2, 3\}$. The structure constants $J_{\alpha\beta}$ have the form

$$J_{\alpha\beta} = (-1)^{\alpha-\beta}\left(\frac{\theta_1(\frac{\eta}{2}|\tau)\,\theta_{5-\gamma}(\frac{\eta}{2}|\tau)}{\theta_{5-\alpha}(\frac{\eta}{2}|\tau)\theta_{5-\beta}(\frac{\eta}{2}|\tau)}\right)^2 \tag{4.88}$$

or

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma}, \qquad J_\alpha = \frac{\theta_{5-\alpha}(0|\tau)\theta_{5-\alpha}(\eta|\tau)}{\theta^2_{5-\alpha}(\frac{\eta}{2}|\tau)}. \tag{4.89}$$

Note that the structure constants satisfy the relation

$$J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0. \tag{4.90}$$

The commutation relations (4.87) can be also written as

$$\begin{aligned} 2i\mathsf{S}_0\mathsf{S}_\alpha &= (1 - J_{\beta\gamma})\mathsf{S}_\beta\mathsf{S}_\gamma - (1 + J_{\beta\gamma})\mathsf{S}_\gamma\mathsf{S}_\beta, \\ 2i\mathsf{S}_\alpha\mathsf{S}_0 &= (1 + J_{\beta\gamma})\mathsf{S}_\beta\mathsf{S}_\gamma - (1 - J_{\beta\gamma})\mathsf{S}_\gamma\mathsf{S}_\beta. \end{aligned} \tag{4.91}$$

The algebra with these generators and relations is called the Sklyanin algebra (introduced by E.Sklyanin in 1982).

Problem. Substitute the L-operator (4.85) into the $RLL = LLR$ relation and obtain the commutation relations (4.86). (Warning: this is a hard calculation using numerous identities for the theta-functions.)

The Sklyanin algebra has two central elements (Casimir elements)

$$\Omega_1 = \mathsf{S}_0^2 + \mathsf{S}_1^2 + \mathsf{S}_2^2 + \mathsf{S}_3^2, \qquad \Omega_2 = J_1\mathsf{S}_1^2 + J_2\mathsf{S}_2^2 + J_3\mathsf{S}_3^2. \tag{4.92}$$

Problem. Using the commutation relations of the Sklyanin algebra, prove that $[\Omega_1, \mathsf{S}_a] = [\Omega_2, \mathsf{S}_a] = 0$ for all $a = 0, \ldots, 3$. (Hint: by transforming $\mathsf{S}_\alpha\mathsf{S}_0\mathsf{S}_\alpha$ and $\mathsf{S}_0\mathsf{S}_\alpha\mathsf{S}_0$ in two ways obtain linear relations between cubic monomials in the generators.)

When $\tau \to +i\infty$, the Sklyanin algebra degenerates into $U_q(gl_2)$, the q-deformation of $U(gl_2)$ with the commutation relations (4.27) and $q = e^{\pi i\eta}$. In this limit

$$J_{12} = 0, \quad J_{23} = -J_{31} = -\frac{\sin^2\frac{\pi\eta}{2}}{\cos^2\frac{\pi\eta}{2}}.$$

Redefining the generators as

$$\mathsf{A} = \cos\tfrac{\pi\eta}{2}\,\mathsf{S}_0 + i\sin\tfrac{\pi\eta}{2}\,\mathsf{S}_3,$$

$$\mathsf{D} = \cos\tfrac{\pi\eta}{2}\,\mathsf{S}_0 - i\sin\tfrac{\pi\eta}{2}\,\mathsf{S}_3,$$

$$\mathsf{B} = \mathsf{S}_1 + i\mathsf{S}_2,$$

$$\mathsf{C} = \mathsf{S}_1 - i\mathsf{S}_2,$$

one can see that $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ satisfy the algebra (4.27).

If $\eta \neq r_1 + r_2\tau$ with rational r_1, r_2, representations of the Sklyanin algebra are smooth deformations of representations of the algebra $U(gl_2)$.

5 Scalar products of Bethe vectors

Although most of the results discussed in this section remain true (with necessary modifications) also in more general cases, we for simplicity will restrict ourselves by models based on the standard rational $GL(2)$-invariant R-matrix

$$\mathsf{R}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix} = u\mathsf{I} + \eta\mathsf{P}.$$

The quantum monodromy matrix of the inhomigeneous model is of the form

$$\mathcal{T}(u) = \mathsf{R}_{01}(u-\xi_1)\mathsf{R}_{02}(u-\xi_2)\ldots\mathsf{R}_{0N}(u-\xi_N) = \begin{pmatrix} A(u) & B(u) \\ & \\ C(u) & D(u) \end{pmatrix},$$

where ξ_i are inhomogeneoties. The transfer matrix is $\mathsf{T}(u) = \mathrm{tr}_0\,\mathcal{T}(u) = A(u) + D(u)$. The operators $A(u)$ and $D(u)$ act to the vacuum vector $|\Omega\rangle = (|+\rangle)^{\otimes N}$ in the following way:

$$A(u)\,|\Omega\rangle = \phi(u+\eta)\,|\Omega\rangle\,, \quad D(u)\,|\Omega\rangle = \phi(u)\,|\Omega\rangle\,.$$

Here $\phi(u)$ is the polynomial

$$\phi(u) = \prod_{j=1}^{N}(u-\xi_j). \tag{5.1}$$

5.1 Scalar products: historical remarks

Following the algebraic approach, we consider Bethe vectors

$$|u_1, u_2, \ldots, u_m\rangle = B(u_1)B(u_2)\ldots B(u_m)\,|\Omega\rangle \in \mathcal{H} \cong (\mathbb{C}^2)^N, \tag{5.2}$$

where u_j are arbitrary parameters. For brevity, in what follows we will sometimes write $|u_1, \ldots, u_m\rangle = |\{u_i\}_m\rangle$. According to the terminology adapted in the literature, such a vector $|u_1, u_2, \ldots, u_m\rangle$ is called off-shell Bethe vector. If the parameters u_j are constrained by Bethe equations (and thus the vector $|u_1, u_2, \ldots, u_m\rangle$ is an eigenvector of the transfer matrix), it is called on-shell Bethe vector.

It is easy to see that $(B(u))^\dagger = C(u)$. Therefore, recalling the definition of the scalar product of vectors from the space $\mathcal{H}$, we can represent the scalar product of vectors of the type (5.2) in the form

$$\langle v_m, \ldots, v_1 | u_1, \ldots, u_m\rangle = \langle\Omega|\, C(v_m)\ldots C(v_1)B(u_1)\ldots B(u_m)\,|\Omega\rangle\,. \tag{5.3}$$

Clearly, the number of operators B in the scalar product (5.3) should be the same as the number of operators C, otherwise the scalar product vanishes.

Problem. Find the scalar products (5.3) at $m = 1$ and $m = 2$.

The scalar products of Bethe vectors are necessary for calculation of such important physical quantities as form-factors and correlation functions and so the calculation of scalar products is an important problem of the theory. Attempts to calculate the scalar products directly with the help of the $RTT = TTR$ relations show that this is a difficult combinatorial problem.

The history of the problem is as follows. The first result is Gaudin's hypothesis (1972) about the norms of eigenvectors of the Hamiltonian of the bose-gas with point-like interaction. This hypothesis (proved in 1982 by Korepin in the framework of the quantum inverse scattering method for a sufficiently wide class of models) states that the squared norm of eigenvectors of the transfer matrix (on-shell Bethe vectors) is given by determinant of an $m \times m$ matrix whose explicit form is restored from the form of Bethe equations. Later, in 1989 Slavnov proved a determinant formula for scalar product of two Bethe vectors one of which is on-shell and another one is an arbitrary vector of the form (5.2). The original method to obtain this result is a complicated combinatorial analysis of the structure of scalar products and application of recurrence relations for them. In 1998, Kitanine, Maillet and Terras obtained this result by a different method (which was also sufficiently involved) and showed that the matrix elements of the matrix participating in the determinant representation of scalar products are expressed through derivatives of eigenvalues of the transfer matrix. Finally, quite recently, in 2019, Belliard and Slavnov suggested a very simple method to find the scalar products, which avoids any combinatorial difficulties. This method explains why the scalar products of on-shell and off-shell Bethe vectors have determinant representations. It is so simple that it is hard to believe that nobody discovered it during 35 years. Below we closely follow the original paper by Belliard and Slavnov [12].

5.2 Action of the transfer matrix to Bethe vectors

In what follows we assume that the vector $|v_1, \ldots, v_m\rangle$ in the scalar product (5.3) is on-shell, i.e., $\{v_j\}$ satisfy the Bethe equations

$$\prod_{k=1}^{N} \frac{v_j - \xi_k + \eta}{v_j - \xi_k} = \prod_{l=1,\neq j}^{m} \frac{v_j - v_l + \eta}{v_j - v_l - \eta}, \tag{5.4}$$

while the set $\{u_i\}$ is arbitrary with the only condition that all numbers u_i are distinct. The eigenvalue $T(u)$ of the transfer matrix on the vector $|v_1, \dots, v_m\rangle$,

$$\mathsf{T}(u)\,|v_1, \dots, v_m\rangle = T(u; v_1, \dots, v_m)\,|v_1, \dots, v_m\rangle\,,$$

is given by the formula

$$T(u) = T(u; v_1, \dots, v_m) = \prod_{k=1}^{N}(u-\xi_k+\eta)\prod_{l=1}^{m}\frac{u-v_l-\eta}{u-v_l} + \prod_{k=1}^{N}(u-\xi_k)\prod_{l=1}^{m}\frac{u-v_l+\eta}{u-v_l}. \quad (5.5)$$

As we know, residues of this expression as a function of u at the points v_j vanish (these conditions are equivalent to Bethe equations). We can extend the definition of the function $T(u) = T(u; u_1, \dots, u_m)$ with the help of this formula to an arbitrary set of numbers $\{u_j\}$; then $T(u; u_1, \dots, u_m)$ in general is not an eiganvalue of the transfer matrix and residues at the points u_j are nonzero. Clearly, it is possible to represent the formula for the function $T(u; u_1, \dots, u_m)$ in the form

$$T(u; u_1, \dots, u_m) = T(u; \{u_i\}_m) = \frac{P(u; u_1, \dots, u_m)}{\prod\limits_{j=1}^{m}(u-u_j)}, \quad (5.6)$$

where $P(u; u_1, \dots, u_m)$ is a polynomial of u and a symmetric polynomial of u_j's of the form

$$P(u; u_1, \dots, u_m) = \phi(u+\eta)\prod_{l=1}^{m}(u-u_l-\eta) + \phi(u)\prod_{l=1}^{m}(u-u_l+\eta) \quad (5.7)$$

(the polynomial $\phi(u)$ is defined in (5.1)). Note that the polynomial P can be expanded in the basis of elementary symmetric polynomials $e_k^{(m)}(u_1, \dots, u_m)$:

$$P(u; u_1, \dots, u_m) = \sum_{k=0}^{m} A_k(u) e_k^{(m)}(u_1, \dots, u_m). \quad (5.8)$$

The elementary symmetric polynomials are $e_0^{(m)} = 1$, $e_1^{(m)} = \sum\limits_{j=1}^{m} u_j$, $e_2^{(m)} = \sum\limits_{i<j}^{m} u_i u_j$, …, $e_m^{(m)} = \prod\limits_{j=1}^{m} u_j$, $e_k^{(m)} = 0$ at $k > m$. The general formula is

$$e_k^{(m)}(u_1, \dots, u_m) = \frac{1}{(m-k)!}\frac{d^{m-k}}{dt^{m-k}}\prod_{i=1}^{m}(t+u_i)\bigg|_{t=0}. \quad (5.9)$$

The functions $A_k(u)$ in (5.8) can be regarded as functional parameters defining a concrete model (they replace the numerical parameters ξ_j).

Recalling the arguments of the algebraic Bethe ansatz method, we can write:

$$\mathsf{T}(u)\Big|\{u_i\}_m\Big\rangle = T(u; \{u_i\}_m)\Big|\{u_i\}_m\Big\rangle + \sum_{j=1}^{m}\Lambda_j(u, \{u_i\}_m)\Big|\{u_i\}_m \setminus u_j, u\Big\rangle, \quad (5.10)$$

where $\Big|\{u_i\}_m \setminus u_j, u\Big\rangle$ means that in this vector the parameter u_j is replaced by u. As it follows from the algebraic Bethe ansats, the coefficient Λ_j has the form

$$\Lambda_j(u,\{u_i\}_m) = -\frac{1}{u-u_j}\operatorname*{res}_{u=u_j} T(u;u_1,\dots,u_m)$$
$$= \frac{\eta}{u-u_j}\left(\phi(u_j+\eta)\prod_{k=1,\neq j}\frac{u_j-u_k-\eta}{u_j-u_k} - \phi(u_j)\prod_{k=1,\neq j}\frac{u_j-u_k+\eta}{u_j-u_k}\right). \tag{5.11}$$

Note that equation (5.10) with the first equality in (5.11) means that the vector-function $\mathsf{T}(u)|u_1,\dots,u_m\rangle$ does not have poles at $u=u_i$, as it must be because the transfer matrix is a polynomial in u.

5.3 Derivation of a system of linear equations for scalar products

The main idea of the method developed by Belliard and Slavnov is to show that the scalar products of Bethe vectors satisfy a system of linear equations. After the necessary preparations made above, we can proceed to the derivation of it. Let $u_1,u_2,\dots,u_{m+1} = \{u_i\}_{m+1}$ be a set of $m+1$ arbitrary parameters. Consider the matrix element of the transfer matrix

$$S_j = \Big\langle\{v_i\}_m\Big|\mathsf{T}(u_j)\Big|\{u_i\}_{m+1}\setminus u_j\Big\rangle, \quad j=1,\dots,m+1 \tag{5.12}$$

between the eigenvector $\Big\langle\{v_i\}_m\Big|$ (an on-shell Bethe vector) and an arbitrary off-shell Bethe vector $\Big|\{u_i\}_{m+1}\setminus u_j\Big\rangle$, which belongs, as $\Big|\{v_i\}_m\Big\rangle$, to the sector with m reversed spins, and which has the set of parameters with excluded u_j. We can calculate S_j in two different ways: either acting by the transfer matrix to the left ($\Big\langle\{v_i\}_m\Big|\mathsf{T}(u_j) = T(u_j;\{v_i\}_m)\Big\langle\{v_i\}_m\Big|$), or acting to the right with the help of equation (5.10), which we write here in short as

$$\mathsf{T}(u_j)\Big|\{u_i\}_{m+1}\setminus u_j\Big\rangle = \sum_{k=1}^{m+1}\Lambda_{jk}\Big|\{u_i\}_{m+1}\setminus u_k\Big\rangle.$$

Here $\Lambda_{jj} = T(u_j;\{u_i\}_{m+1}\setminus u_j)$ and $\Lambda_{jk} = \Lambda_k(u_j;\{u_i\}_{m+1}\setminus u_j)$ at $j\neq k$. Comparing equations (5.6) and (5.11), we find:

$$\Lambda_{jk} = \frac{P(u_k;\{u_i\}_{m+1}\setminus u_j)}{\prod\limits_{l=1,\neq k}^{m+1}(u_k-u_l)} = \operatorname*{res}_{u=u_k}\frac{T(u;\{u_i\}_{m+1}\setminus u_j)}{u-u_j} \quad \text{for all } j,k. \tag{5.13}$$

Consider the scalar products

$$X_j = \Big\langle\{v_i\}_m\Big|\{u_i\}_{m+1}\setminus u_j\Big\rangle, \quad j=1,\dots,m+1. \tag{5.14}$$

Calculating the matrix elements (5.12) in the two ways, as above, we get the equalities

$$\sum_{k=1}^{n+1}\Lambda_{jk}X_k = T(u_j;\{v_i\}_m)X_j,$$

which constitute a system of homogeneous linear equations for $m+1$ variables X_j:

$$\sum_{k=1}^{m+1} M_{jk} X_k = 0, \quad j = 1, \dots m+1, \tag{5.15}$$

where the matrix M of size $(m+1)\times(m+1)$ has the form

$$M_{jk} = \Lambda_{jk} - T(u_j; \{v_i\}_m)\delta_{jk}$$

$$= \frac{P(u_k; \{u_i\}_{m+1} \setminus u_j)}{\prod\limits_{l=1,\neq k}^{m+1} (u_k - u_l)} - \delta_{jk} \frac{P(u_k; \{v_i\}_m)}{\prod\limits_{l=1}^{m} (u_k - v_l)}. \tag{5.16}$$

This is the system of linear equations for the scalar products.

5.4 Solvability of the system of linear equations for scalar products

The system (5.15) has nontrivial solutions if rank of the matrix M is strictly less than $m+1$, i.e., $\det M = 0$. In this case the solutions are given by minors of the matrix M, hence the origin of the determinant representations for scalar products becomes clear.

In order to show that $\det M = 0$ we take advantage of the fact that the matrix M can be multiplied from the left by any non-degenerate matrix V: $M \to \tilde{M} = VM$; and the solution remains the same. Let us extend the set $\{v_i\}_m = \{v_1, \dots, v_m\}$ to the set of $m+1$ elements $\{v_i\}_{m+1} = \{v_1, \dots, v_m, v\}$, where $v_{m+1} = v$ is a free parameter and choose the matrix V in the form

$$V_{ij} = \frac{u_j - v_j}{u_j - v_i} \prod_{l=1,\neq j}^{m+1} \frac{u_j - v_l}{u_j - u_l}, \quad i, j = 1, \dots, m+1. \tag{5.17}$$

It is the Cauchy matrix $C_{ij} = 1/(v_i - u_j)$ multiplied from the right by a diagonal matrix. Using the well known expression for determinant of the Cauchy matrix, we find:

$$\det V = \prod_{i<j} \frac{v_i - v_j}{u_i - u_j} = \frac{\Delta(\{v\})}{\Delta(\{u\})},$$

where Δ is the Vandermonde determinant, so the matrix V is non-degenerate.

For the matrix $\tilde{M} = VM$ we get, using (5.16):

$$\tilde{M}_{ik} = \Big(\prod_{l=1,\neq k}^{m+1} (u_k - u_l) \Big)^{-1} \left(\sum_{j=1}^{m+1} \frac{u_j - v_j}{u_j - v_i} \prod_{s=1,\neq j}^{m+1} \frac{u_j - v_s}{u_j - u_s} P(u_k; \{u_n\}_{m+1} \setminus u_j) \right.$$
$$\left. - \frac{u_k - v}{u_k - v_i} P(u_k; \{v_n\}_m) \right).$$

Let us use the expansion of the polynomial P in the basis of the elementary symmetric functions (5.8) and the representation (5.9) for them. We have the identity

$$\sum_{j=1}^{m+1} \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_j - v_s)}{\prod\limits_{r=1,\neq j}^{m+1}(u_j - u_r)} P(u_k; \{u_n\}_{m+1} \setminus u_j) = P(u_k; \{v_n\}_{m+1} \setminus v_i).$$

Indeed, since

$$0 = \frac{1}{2\pi i} \oint\limits_{|u|=R\to\infty} \frac{\prod\limits_{s=1,\neq i}^{m+1}(u - v_s)}{\prod\limits_{r=1}^{m+1}(u - u_r)} \frac{du}{t+u} = \sum_{j=1}^{m+1} \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_j - v_s)}{\prod\limits_{r=1,\neq j}^{m+1}(u_j - u_r)} \frac{1}{t+u_j} - \frac{\prod\limits_{s=1,\neq i}^{m+1}(t + v_s)}{\prod\limits_{r=1}^{m+1}(t + u_r)}$$

(the vanishing integral in the left hand side is represented as sum of residues), we have:

$$\sum_{j=1}^{m+1} \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_j - v_s)}{\prod\limits_{r=1,\neq j}^{m+1}(u_j - u_r)} \prod_{l=1,\neq j}^{m+1}(t + u_l) = \prod_{s=1,\neq i}^{m+1}(t + v_s),$$

and, therefore,

$$\sum_{j=1}^{m+1} \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_j - v_s)}{\prod\limits_{r=1,\neq j}^{m+1}(u_j - u_r)} e_k^{(m)}(\{u_n\}_{m+1} \setminus u_j) = e_k^{(m)}(\{v_n\}_{m+1} \setminus v_i), \quad k = 0, 1, \ldots, m.$$

For the matrix $\tilde{M}$ we thus obtain the expression

$$\begin{aligned}\tilde{M}_{ik} &= \Big(\prod_{l=1,\neq k}^{m+1}(u_k - u_l)\Big)^{-1} \Big(P(u_k; \{v_n\}_{m+1} \setminus v_i) - \frac{u_k - v}{u_k - v_i} P(u_k; \{v_n\}_m)\Big) \\ &= \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_k - v_s)}{\prod\limits_{l=1,\neq k}^{m+1}(u_k - u_l)} \Big(T(u_k; \{v_n\}_{m+1} \setminus v_i) - T(u_k; \{v_n\}_m)\Big)\end{aligned} \tag{5.18}$$

from which it is seen that the last ($(m+1)$-th) row of the matrix $\tilde{M}$ consists of zeros. Therefore, $\det M = \det \tilde{M} = 0$ and the system (5.15) has nontrivial solutions.

Let us note that in [12] a more general transformation of the matrix M was considered: $M \to \tilde{M} = WM$, where

$$W_{ij} = \frac{u_j - w_j}{u_j - w_i} \prod_{l=1,\neq j}^{m+1} \frac{u_j - w_l}{u_j - u_l}, \quad i, j = 1, \ldots, m+1. \tag{5.19}$$

Here $\{w_i\}_{m+1}$ is an arbitrary set of parameters; the matrix V is a particular case of W at $w_i = v_i$, $i = 1 \ldots, m$, $w_{m+1} = v$. Performing the same calculations as above, we get

$$\tilde{M}_{ik} = \frac{\prod\limits_{s=1,\neq i}^{m+1}(u_k - w_s)}{\prod\limits_{l=1,\neq k}^{m+1}(u_k - u_l)} \Big(T(u_k; \{w_n\}_{m+1} \setminus w_i) - T(u_k; \{v_n\}_m)\Big). \tag{5.20}$$

The expression (5.18) can be transformed further. First we note, performing a simple direct calculation with the help of equation (5.5), that

$$T(u;\{v_n\}_{m+1}\setminus v_i)-T(u;\{v_n\}_m)$$
$$=\frac{\eta(v-v_i)}{(v-u)(v_i-u)}\left(-\phi(u+\eta)\prod_{l=1,\neq i}^{m}\frac{u-v_l-\eta}{u-v_l}+\phi(u)\prod_{l=1,\neq i}^{m}\frac{u-v_l+\eta}{u-v_l}\right)$$
$$=\ (v-v_i)\,\frac{u-v_i}{u-v}\,\frac{\partial T(u;\{v_n\}_m)}{\partial v_i}.$$

Therefore,

$$\tilde{M}_{ik}=(v-v_i)\,\frac{\prod\limits_{s=1}^{m}(u_k-v_s)}{\prod\limits_{l=1,\neq k}^{m+1}(u_k-u_l)}\,T_{ik}, \tag{5.21}$$

where the matrix $T_{ik}=\partial T(u_k)/\partial v_i$ has the form

$$T_{ik}=\frac{\eta}{u_k-v_i}\left(-\frac{\phi(u_k+\eta)}{u_k-v_i-\eta}\prod_{l=1}^{m}\frac{u_k-v_l-\eta}{u_k-v_l}+\frac{\phi(u_k)}{u_k-v_i+\eta}\prod_{l=1}^{m}\frac{u_k-v_l+\eta}{u_k-v_l}\right). \tag{5.22}$$

Let us assume that rank of the matrix $\tilde{M}$ equals m (this is the case of general position) and consider the system of m equations equivalent to (5.15):

$$\sum_{k=1}^{m+1}\frac{\partial T(u_k)}{\partial v_i}\,\tilde{X}_k=0,\quad i=1,\dots,m,$$

where

$$\tilde{X}_k=X_k\frac{\prod\limits_{s=1}^{m}(u_k-v_s)}{\prod\limits_{l=1,\neq k}^{m+1}(u_k-u_l)}.$$

Using the Cramer's rule, one can construct its solution via minors of the matrix T_{ik} as follows:

$$X_k=(-1)^k c\,\frac{\prod\limits_{l=1,\neq k}^{m+1}(u_k-u_l)}{\prod\limits_{s=1}^{m}(u_k-v_s)}\,\det_{j\neq k}\left(\frac{\partial T(u_j)}{\partial v_i}\right)_{m\times m},$$

where the multiplier c does not depend on k. It is easy to verify that we have

$$X_k\frac{\det\limits_{j\neq k}\left(\frac{1}{u_j-v_i}\right)_{m\times m}}{\det\limits_{j\neq k}\left(\frac{\partial T(u_j)}{\partial v_i}\right)_{m\times m}}=X_n\frac{\det\limits_{j\neq n}\left(\frac{1}{u_j-v_i}\right)_{m\times m}}{\det\limits_{j\neq n}\left(\frac{\partial T(u_j)}{\partial v_i}\right)_{m\times m}}$$

for all $k,n=1,\dots,m+1$. The left hand side does not depend on u_k (but possibly depends on all other variables u_j) while the right hand side does not depend on u_n. From this it follows that the left hand side does not in fact depend on all u_j's, and we can write

$$X_k=\Phi(\{v_i\}_m)\,\frac{\det\limits_{j\neq k}(\partial T(u_j)/\partial v_i)}{\det\limits_{j\neq k}\left(\frac{1}{u_j-v_i}\right)}, \tag{5.23}$$

where Φ is some yet undetermined symmetric function of $\{v_i\}$. It can be found if one calculates the scalar product for some special values of u_j and compares with equation (5.23) (see below in the next section). This calculation yields $\Phi = \prod_{l=1}^{m} \phi(v_l)$. Therefore, for the scalar products we obtain:

$$
\begin{aligned}
\langle\Omega| C(v_m)\dots C(v_1)B(u_1)\dots B(u_m)|\Omega\rangle &= \frac{\det_{1\le i,j\le m} T_{ij}}{\det_{1\le i,j\le m}\left(\frac{1}{u_i-v_j}\right)} \prod_{l=1}^{m}\phi(v_l) \\
&= \frac{\prod_{r,s=1}^{m}(u_r-v_s)}{\prod_{k<k'}(u_k-u_{k'})(v_{k'}-v_k)} \prod_{l=1}^{m}\prod_{a=1}^{N}(v_l-\xi_a) \det_{1\le i,j\le m}\left(\frac{\partial T(u_j)}{\partial v_i}\right).
\end{aligned}
\tag{5.24}
$$

Let us recall that this representation is valid only under the condition that the parameters v_i satisfy the Bethe equations (5.4).

5.5 Scalar products and partition function of the 6-vertex model with domain wall boundary conditions

In this section we present, without derivation, some important results related to the theory of scalar products.

A combinatorial analysis which uses the commutation relations for the operators A, B, C, D allows one to obtain the following formula for scalar products of Bethe vectors of general form:

$$
\begin{aligned}
&\langle\Omega| \prod_{\alpha=1}^{m} C(v_\alpha) \prod_{\beta=1}^{m} B(u_\beta)|\Omega\rangle \\
&= \sum_{k=0}^{m} \sum_{\substack{\{v\}_m=\{v_{\rm I}\}_{m-k}\cup\{v_{\rm II}\}_k \\ \{u\}_m=\{u_{\rm I}\}_{m-k}\cup\{u_{\rm II}\}_k}} \prod_{i\in\{v_{\rm II}\}}\phi(v_i) \prod_{i\in\{u_{\rm I}\}}\phi(u_i) \prod_{i\in\{u_{\rm II}\}}\phi(u_i+\eta) \prod_{i\in\{v_{\rm I}\}}\phi(u_i+\eta) \\
&\times K_k\Big(\{v_{\rm II}\}_k\Big|\{u_{\rm II}\}_k\Big) K_{m-k}\Big(\{u_{\rm I}\}_{m-k}\Big|\{v_{\rm I}\}_{m-k}\Big) \prod_{\substack{i\in\{v_{\rm II}\}_k \\ j\in\{v_{\rm I}\}_{m-k}}} f(v_i,v_j) \prod_{\substack{a\in\{u_{\rm I}\}_{m-k} \\ b\in\{u_{\rm II}\}_k}} f(u_a,u_b)
\end{aligned}
\tag{5.25}
$$

(we again assume that the numbers v_α, u_β are all distinct). Here the sum goes over all partitions of the sets $\{v\}_m$ and $\{u\}_m$ to non-intersecting subsets $\{v\}_m = \{v_{\rm I}\}_{m-k}\cup\{v_{\rm II}\}_k$ and $\{u\}_m = \{u_{\rm I}\}_{m-k}\cup\{u_{\rm II}\}_k$ with the number of elements k and $m-k$, where k runs from 0 to m,

$$
f(u,v) = \frac{u-v+\eta}{u-v},
$$

and $K_m\Big(\{v\}_m\Big|\{u\}_m\Big)$ is the partition function of the rational 6-vertex model on the inhomogeneous $m\times m$ square lattice with domain wall boundary conditions. More precisely, in this model the Boltzmann weight in the vertex (i,j) (the intersection of ith vertical line and jth horizontal line counting from the left bottom angle) is given by the R-matrix

$$\tilde{\mathsf{R}}(v_j - u_i) = \frac{\mathsf{R}(v_j - u_i)}{v_j - u_i}.$$

The boundary conditions are such that the arrows in the bottom row look up, in the upper row look down, in the leftmost column look to the left and in the rightmost column look to the right. For such partition function the following determinant representation (found by Izergin in 1987) is known:

$$K_m\Big(\{v\}_m\Big|\{u\}_m\Big) = \frac{\prod\limits_{k,l=1}^{m}(v_k - u_l + \eta)}{\prod\limits_{k<l}(v_l - v_k)(u_k - u_l)} \det_{m\times m}\left(\frac{\eta}{(v_i - u_j)(v_i - u_j + \eta)}\right). \tag{5.26}$$

At present, equation (5.25) together with (5.26) provides the most complete description of scalar products of general Bethe vectors.

Now, let us put $u_i = \xi_i$ for $i = 1, \ldots, m$, then $\phi(u_i) = \phi(\xi_i) = 0$. In the sum (5.25) only one nonzero term remains which corresponds to empty sets $\{u_{\rm I}\}$ and $\{v_{\rm I}\}$ ($k = m$), and the formula becomes

$$\langle\Omega|\prod_{\alpha=1}^{m} C(v_\alpha)\prod_{\beta=1}^{m} B(\xi_\beta)\,|\Omega\rangle = \prod_{i=1}^{m}\phi(v_i)\prod_{j=1}^{m}\phi(\xi_j+\eta)\,K_m\Big(\{v\}_m\Big|\{\xi\}_m\Big). \tag{5.27}$$

Comparing with (5.23) in the case $\phi(u_i) = \phi(\xi_i) = 0$, we get $\Phi = \prod\limits_{l=1}^{m}\phi(v_l)$, so that (5.24) in this case coincides with (5.27).

5.6 Orthogonality of on-shell Bethe vectors and their norm

At last, let us discuss an important particular case when both vectors in the scalar product are on-shell. If they are different, the standard simple argument shows that they must be orthogonal. Indeed, consider the matrix element

$$\langle\Omega|\prod_{\alpha=1}^{m} C(v_\alpha)\mathsf{T}(u)\prod_{\beta=1}^{m} B(u_\beta)\,|\Omega\rangle.$$

Acting by the transfer matrix to the left and to the right, we get:

$$\Big(T(u; u_1, \ldots, u_m) - T(u; v_1, \ldots, v_m)\Big)\,\langle\Omega|\prod_{\alpha=1}^{m} C(v_\alpha)\prod_{\beta=1}^{m} B(u_\beta)\,|\Omega\rangle = 0,$$

and if $T(u; u_1, \ldots, u_m) \neq T(u; v_1, \ldots, v_m)$, the scalar product is equal to zero. Let us show how this follows from formula (5.24).

We should show that if the parameters $\{u_i\}_m$ satisfy the Bethe equations

$$\frac{\phi(u_j+\eta)}{\phi(u_j)} = \prod_{l=1,\neq j}^{m}\frac{u_j - u_l + \eta}{u_j - u_l - \eta}$$

and do not coincide with $\{v_i\}_m$ (for simplicity, we assume that no one of u_i coincides with no one of v_i), the matrix T_{ij} in (5.24) becomes degenerate: $\det T_{ij} = 0$. Substituting the Bethe equations for u_i in it, we have:

$$T_{ik} = \eta\phi(u_k) \prod_{s=1}^{m} \frac{u_k - v_s + \eta}{u_k - v_s} \tilde{T}_{ik},$$

where

$$\tilde{T}_{ik} = \frac{1}{(u_k - v_i)(u_k - v_i + \eta)} + \frac{y_k}{(u_k - v_i)(u_k - v_i - \eta)} \tag{5.28}$$

with

$$y_k = \prod_{l=1}^{m} \frac{(u_k - u_l + \eta)(u_k - v_l - \eta)}{(u_k - u_l - \eta)(u_k - v_l + \eta)}.$$

Obviously, we should show that $\det \tilde{T}_{ik} = 0$. We will show that, indeed, the matrix $\tilde{T}_{ik}$ has a row-eigenvector with zero eigenvalue, i.e., its rows are linearly dependent. Set

$$x_j = \frac{\prod\limits_{l=1}^{m} (v_j - u_l)}{\prod\limits_{s=1, \neq j}^{m} (v_j - v_s)};$$

we claim that $\sum\limits_{j=1}^{m} x_j \tilde{T}_{jk} = 0$ (see [11], page 111). We have:

$$\sum_{j=1}^{m} x_j \tilde{T}_{jk} = U_k^+ + y_k U_k^-, \quad U_k^{\pm} = \sum_{j=1}^{m} \frac{x_j}{(u_k - v_j)(u_k - v_j \pm \eta)}. \tag{5.29}$$

Consider the vanishing contour integral

$$0 = \frac{1}{2\pi i} \oint_{|z|=R\to\infty} \frac{\eta\, dz}{(u_k - z)(u_k - z \pm \eta)} \prod_{s=1}^{m} \frac{z - u_s}{z - v_s}.$$

Using the residue calculus, we arrive at the identity

$$\pm \prod_{s=1}^{m} \frac{u_k - u_s \pm \eta}{u_k - v_s \pm \eta} + \sum_{j=1}^{m} \frac{\eta}{(u_k - v_j)(u_k - v_j \pm \eta)} \frac{\prod\limits_{l=1}^{m} (v_j - u_l)}{\prod\limits_{s=1, \neq j}^{m} (v_j - v_s)} = 0$$

or

$$\eta U_k^{\pm} = \mp \prod_{s=1}^{m} \frac{u_k - u_s \pm \eta}{u_k - v_s \pm \eta}.$$

Plugging this into (5.29), we see that our linear combination is indeed equal to zero.

In order to find the squared norm of the Bethe vector

$$\mathcal{N}^2(v_1, \ldots, v_m) = \langle \Omega | \prod_{l=1}^{m} C(v_l) \prod_{s=1}^{m} B(v_s) \, |\Omega\rangle \,,$$

one should put $u_k = v_k + \epsilon_k$ in (5.24) and tend $\epsilon_k \to 0$ (recalling that $\{v_i\}_m$ satisfy the Bethe equations). In the off-diagonal elements of the matrix T_{ij} one can simply put

$\epsilon_k = 0$, while in the diagonal elements an indeterminacy appears (zero in the numerator and zero in the denominator) which should be resolved by l'Hopital's rule. As a result, one obtains:

$$\mathcal{N}^2(v_1, \ldots, v_m) = \eta^m \prod_{l=1}^{m} \phi^2(v_l) \prod_{r \neq s} \frac{v_r - v_s + \eta}{v_r - v_s} \det_{m\times m} t_{ik}, \tag{5.30}$$

where the matrix t_{ik} is given by

$$t_{ik} = -\delta_{ik} \left(\partial_{v_k} \log \frac{\phi(v_k + \eta)}{\phi(v_k)} + \sum_{l=1}^{m} \frac{2\eta}{(v_k - v_l)^2 - \eta^2} \right) + \frac{2\eta}{(v_k - v_i)^2 - \eta^2}. \tag{5.31}$$

Note that $t_{ik} = \partial B_k / \partial v_i$, where

$$B_k = \log \frac{\phi(v_k)}{\phi(v_k + \eta)} + \sum_{l=1, \neq k}^{m} \log \frac{v_k - v_l + \eta}{v_k - v_l - \eta} \tag{5.32}$$

is the logarithm of the left hand sides of the Bethe equations in the form

$$\frac{\phi(v_k)}{\phi(v_k + \eta)} \prod_{l=1, \neq k}^{m} \frac{v_k - v_l + \eta}{v_k - v_l - \eta} = 1$$

(the Bethe equations state that $B_k = 2\pi i q_k$ with integer q_k) and, therefore, t_{ik} coincides with the Hessian of the Yang function at the minimum.

6 Generalized spin chains, master T-operator and quantum-classical duality

6.1 $GL(n)$-invariant R-matrices and generalized spin chains

So far we considered R-matrices of size 4×4. It turns out that there are solutions of the Yang-Baxter equation of size $n^2 \times n^2$ with $n \geq 2$ for any $n \geq 2$. They are linear operators in the space $\mathbb{C}^n \otimes \mathbb{C}^n$. For simplicity we will restrict ourselves by considering R-matrices with rational dependence on the spectral parameter (but their trigonometric and elliptic generalizations do exist). These R-matrices have the form

$$\mathsf{R}(x) = x\mathsf{I} + \eta \mathsf{P}, \tag{6.1}$$

where P is the permutation operator in $\mathbb{C}^n \otimes \mathbb{C}^n$, and η is a parameter (in the R-matrix (4.18) $\eta = 1$). In this section the spectral parameter is denoted by x. It can be checked that this R-matrix is $GL(n)$-invariant:

$$\mathbf{g} \otimes \mathbf{g}\, \mathsf{R}(x) = \mathsf{R}(x)\, \mathbf{g} \otimes \mathbf{g} \qquad \text{or} \qquad \mathbf{g}_1 \mathbf{g}_2 \mathsf{R}_{12}(x) = \mathsf{R}_{12}(x) \mathbf{g}_1 \mathbf{g}_2 \tag{6.2}$$

for any matrix $\mathbf{g} \in GL(n)$. The permutation operator P is expressed through the elementary $n \times n$ matrices e_{ab} (with 1 at the place ab and 0 otherwise) as follows:

$$\mathsf{P} = \sum_{a,b} e_{ab} \otimes e_{ba}.$$

In this section we discuss integrable systems based on the $GL(n)$-invariant R-matrices. Such integrable systems are called generalized spin chains (or vertex models). In the way similar to the case of spin chains based on $GL(2)$, one can construct a family of commuting transfer matrices for them. For example, one can consider inhomogeneous $GL(n)$ spin chain with the transfer matrix

$$\mathsf{T}(x) = \mathrm{tr}_0\Big(\mathsf{R}_{01}(x-x_1)\mathsf{R}_{02}(x-x_2)\ldots\mathsf{R}_{0N}(x-x_N)\Big),$$

where each R-matrix has the form $\mathsf{R}_{ij}(x) = x\mathsf{I} + \eta\mathsf{P}_{ij}$ (see (6.1)). The Yang-Baxter equation for $\mathsf{R}_{ij}(x)$ guarantees that

$$[\mathsf{T}(x),\, \mathsf{T}(x')] = 0.$$

One can also consider the chain with quasiperiodic (twisted) boundary conditions inserting under the trace a group element $\mathbf{g} \in GL(n)$ (twist), which for simplicity we assume to be diagonal ($\mathbf{g} = \mathrm{diag}(g_1, g_2, \ldots, g_n)$):

$$\mathsf{T}(x) = \mathrm{tr}_0\Big(\mathsf{R}_{01}(x-x_1)\mathsf{R}_{02}(x-x_2)\ldots\mathsf{R}_{0N}(x-x_N)\,\mathbf{g}_0\Big) \tag{6.3}$$

Here $\mathbf{g}_0$ means that $\mathbf{g}$ acts in the auxiliary space (number 0). The $GL(n)$-invariance (6.2) implies that these transfer matrices commute at different values of the spectral parameter. In the homogeneous chain with periodic boundary conditions (at $x_j = 0$, $\mathbf{g} = \mathsf{I}$) there exists a local Hamiltonian which is the logarithmic derivative of $\mathsf{T}(x)$ at 0. As in the $GL(2)$ case it is proportional to the sum $\sum_j \mathsf{P}_{j,j+1}$ of permutation operators of neighboring sites. In inhomogeneous chains local Hamiltonians commuting with the transfer matrix in general do not exist.

Matrix elements of the transfer matrix (6.3) are polynomials in x of degree N. Let us normalize the transfer matrix in a different way, dividing it by the polynomial $\prod\limits_{j=1}^{N}(x-x_j)$:

$$\mathbf{T}(x) = \frac{\mathsf{T}(x)}{\prod\limits_{j=1}^{N}(x-x_j)}.$$

Obviously, the transfer matrix $\mathbf{T}(x)$ is given by

$$\mathbf{T}(x) = \mathrm{tr}_0\Big(\tilde{\mathsf{R}}_{01}(x-x_1)\tilde{\mathsf{R}}_{02}(x-x_2)\ldots\tilde{\mathsf{R}}_{0N}(x-x_N)\,\mathbf{g}_0\Big), \tag{6.4}$$

where

$$\tilde{\mathsf{R}}(x) = \mathsf{I} + \frac{\eta}{x}\,\mathsf{P}$$

is the R-matrix which differs from the $\mathsf{R}(x)$ by a scalar factor.

The transfer matrix $\mathbf{T}(x)$ has simple poles at the points x_j. One can introduce Hamiltonians $\mathbf{H}_j$ of the inhomogeneous spin chain as residues at the poles:

$$\mathbf{T}(x) = \mathrm{tr}\,\mathbf{g} + \sum_{j=1}^{N}\frac{\eta\mathbf{H}_j}{x-x_j}. \tag{6.5}$$

These operators commute with each other. However, they are non-local. Their explicit form is as follows:

$$\mathbf{H}_i = \tilde{\mathsf{R}}_{i\,i-1}(x_i - x_{i-1}) \ldots \tilde{\mathsf{R}}_{i1}(x_i - x_1)\mathbf{g}_i \tilde{\mathsf{R}}_{iN}(x_i - x_N) \ldots \tilde{\mathsf{R}}_{i\,i+1}(x_i - x_{i+1}).$$

Comparing the expansions as $x \to \infty$ of (6.5) and

$$\begin{aligned} \mathbf{T}(x) &= \mathrm{tr}_0 \left[\Big(\mathsf{I} + \frac{\eta \mathsf{P}_{01}}{x - x_1}\Big) \ldots \Big(\mathsf{I} + \frac{\eta \mathsf{P}_{0N}}{x - x_N}\Big)\, \mathbf{g}_0 \right] \\ &= \mathrm{tr}\, \mathbf{g} \cdot \mathsf{I} + \frac{\eta}{x} \sum_{i=1}^{N} \mathrm{tr}_0 \Big(\mathsf{P}_{0i}\mathbf{g}_0\Big) + \ldots = \mathrm{tr}\, \mathbf{g} \cdot \mathsf{I} + \frac{\eta}{x} \sum_{i=1}^{N} \mathbf{g}_i + \ldots, \end{aligned}$$

we get the following "sum rule":

$$\sum_{i=1}^{N} \mathbf{H}_i = \sum_{i=1}^{N} \mathbf{g}_i.$$

Let us mention the limit of this construction as $\eta \to 0$. In this limit the generalized magnet becomes the Gaudin model. Set $\mathbf{g} = e^{\eta \mathbf{h}}$, then in the limit $\eta \to 0$ we have $\mathbf{H}_i = 1 + \eta \mathbf{H}_i^G + O(\eta^2)$, where

$$\mathbf{H}_i^G = \mathbf{h}_i + \sum_{j \neq i} \frac{\mathsf{P}_{ij}}{x_i - x_j}$$

are commuting Gaudin Hamiltonians of the Gaudin model.

The operators

$$\mathbf{M}_a = \sum_{j=1}^{N} e_{aa}^{(j)}, \qquad a = 1, \ldots, n \tag{6.6}$$

commute with the transfer matrix and with themselves. Therefore, one can find eigenvectors of the transfer matrix which are simultaneously eigenvectors of the operators $\mathbf{M}_a$ with eigenvalues M_a.

Let us present the result of diagonalization of the transfer matrix $\mathbf{T}(x)$. We give it here without derivation (see [16] for details). The eigenvalues $T(x)$ of $\mathbf{T}(x)$ are given by

$$T(x) = \sum_{b=1}^{n} g_b \prod_{\gamma=1}^{N_{b-1}} \frac{x - v_\gamma^{(b-1)} + \eta}{x - v_\gamma^{(b-1)}} \prod_{\beta=1}^{N_b} \frac{x - v_\beta^{(b)} - \eta}{x - v_\beta^{(b)}}, \tag{6.7}$$

where $N_0 = N$, $N \geq N_1 \geq N_2 \geq \ldots \geq N_{n-1} \geq 0$ are non-negative integers, $N_n = 0$, $v_\gamma^{(0)} = x_\gamma$ and the sets of Bethe roots $\{v_\beta^{(b)}\}_{\beta=1}^{N_b}$ satisfy the system of *nested Bethe ansatz equations*

$$g_b \prod_{\gamma=1}^{N_{b-1}} \frac{v_\alpha^{(b)} - v_\gamma^{(b-1)} + \eta}{v_\alpha^{(b)} - v_\gamma^{(b-1)}} = g_{b+1} \prod_{\gamma \neq \alpha}^{N_b} \frac{v_\alpha^{(b)} - v_\gamma^{(b)} + \eta}{v_\alpha^{(b)} - v_\gamma^{(b)} - \eta} \prod_{\beta=1}^{N_{b+1}} \frac{v_\alpha^{(b)} - v_\beta^{(b+1)} - \eta}{v_\alpha^{(b)} - v_\beta^{(b+1)}}. \tag{6.8}$$

Here $b = 1, \ldots, n-1$, $\alpha = 1, \ldots, N_b$. The numbers N_a are such that $M_1 = N - N_1$, $M_a = N_{a-1} - N_a$, $a = 2, \ldots, n$, where M_a are eigenvalues of the operators $\mathbf{M}_a$. The total number of equations in the system is $N_1 + \ldots + N_{n-1}$. As it follows from (6.5), (6.7), the eigenvalues of the Hamiltonians $\mathbf{H}_i$ are given by

$$H_i = g_1 \prod_{k \neq i}^{N} \frac{x_i - x_k + \eta}{x_i - x_k} \prod_{\gamma=1}^{N_1} \frac{x_i - v_\gamma^{(1)} - \eta}{x_i - v_\gamma^{(1)}}. \tag{6.9}$$

6.2 Transfer matrices as generalized characters

For $GL(n)$-invariant models, the algebra of commuting integrals of motion is in fact larger than the one generated by the x-expansion coefficients of the transfer matrix $\mathsf{T}(x)$ (or the Hamiltonians $\mathbf{H}_j$). It appears that one can introduce more general transfer matrices commuting with $\mathsf{T}(x)$.

To proceed, we need some information about representations of the group $GL(n)$ and the universal enveloping algebra $U(gl_n)$ which has generators $\mathbf{e}_{ab}$ with the commutation relations

$$\mathbf{e}_{ab}\mathbf{e}_{a'b'} - \mathbf{e}_{a'b'}\mathbf{e}_{ab} = \delta_{a'b}\mathbf{e}_{ab'} - \delta_{ab'}\mathbf{e}_{a'b}. \tag{6.10}$$

Finite-dimensional irreducible representations π_λ of $U(gl_n)$ are characterized by the highest weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_{i+1} \leq \lambda_i$ are non-negative integer numbers. The set of numbers λ_i can be identified with the Young diagram λ, or, equivalently, with the partition of $|\lambda| = \sum_i \lambda_i$. Let V_λ be the representation space of π_λ. Clearly, $\pi_{(1)}(\mathbf{e}_{ab}) = e_{ab}$ and $V_{(1)} = \mathbb{C}^n$ (here (1) is the Young diagram consisting from one box, it corresponds to the vector representation). The fundamental representations correspond to one-column diagrams of height from 1 to n.

We first introduce more general $GL(n)$-invariant R-matrices. They act in the tensor product $V_\lambda \otimes \mathbb{C}^n$ and have the form

$$\mathsf{R}^\lambda(x) = x\mathsf{I} + \eta \sum_{a,b} \pi_\lambda(\mathbf{e}_{ab}) \otimes e_{ba}. \tag{6.11}$$

The $GL(n)$-invariance means that

$$\pi_\lambda(\mathbf{g}) \otimes \mathbf{g}\, \mathsf{R}^\lambda(x) = \mathsf{R}^\lambda(x)\, \pi_\lambda(\mathbf{g}) \otimes \mathbf{g}.$$

The R-matrices $\mathsf{R}^\lambda(x)$ satisfy the Yang-Baxter equation

$$\mathsf{R}^{\lambda\mu}_{12}(x-x')\mathsf{R}^\lambda_{13}(x)\mathsf{R}^\mu_{23}(x') = \mathsf{R}^\mu_{23}(x')\mathsf{R}^\lambda_{13}(x)\mathsf{R}^{\lambda\mu}_{12}(x-x'), \tag{6.12}$$

where $\mathsf{R}^{\lambda\mu}(x-x')$ is some R-matrix acting in the tensor product $V_\lambda \otimes V_\mu$. Its explicit form for arbitrary λ, μ is complicated.

It is possible to construct more general transfer matrices acting in the same quantum space $(\mathbb{C}^n)^{\otimes N}$, taking as the auxiliary space not $\mathbb{C}^n$ but the space V_λ of an irreducible representation π_λ of the algebra $U(gl_n)$. Such transfer matrix is obtained as trace in V_λ of product of the R-matrices (6.11):

$$\mathsf{T}_\lambda(x) = \mathrm{tr}_{V_\lambda}\Big(\mathsf{R}^\lambda_{01}(x-x_1)\mathsf{R}^\lambda_{02}(x-x_2)\ldots\mathsf{R}^\lambda_{0N}(x-x_N)\,\pi_\lambda(\mathbf{g}_0)\Big). \tag{6.13}$$

From the Yang-Baxter equation (6.12) and $GL(n)$-invariance it follows that the transfer matrices $\mathsf{T}_\lambda(x)$ commute for different x and λ:

$$[\mathsf{T}_\lambda(x),\, \mathsf{T}_\mu(x')] = 0.$$

In particular, if λ is an empty diagram ($\lambda = \emptyset$), we have

$$\mathsf{T}_\emptyset(x) = \prod_{i=1}^{N}(x-x_i)\cdot\mathsf{I}.$$

One can introduce normalized transfer matrices dividing by $\mathsf{T}_{\emptyset}(x)$:

$$\mathbf{T}_{\lambda}(x) = \frac{\mathsf{T}_{\lambda}(x)}{\mathsf{T}_{\emptyset}(x)}.$$

In particular, $\mathbf{T}_{(1)}(x) = \mathbf{T}(x)$ introduced in (6.4).

At $N = 0$ we have:

$$\mathbf{T}_{\lambda}^{(N=0)}(x) = \mathrm{tr}_{V_{\lambda}} \pi_{\lambda}(\mathbf{g}) = \chi_{\lambda}(\mathbf{g}), \tag{6.14}$$

where $\chi_{\lambda}(\mathbf{g})$ is the character of $\mathbf{g}$ in the representation π_{λ}. Also, we have

$$\mathbf{T}_{\lambda}(x) = \chi_{\lambda}(\mathbf{g}) \cdot \mathsf{I} + O(1/x), \qquad x \to \infty,$$

so the normalized transfer matrix can be regarded as a generalization of characters.

It is known that the characters are given by Schur polynomials s_{λ} of eigenvalues g_i of the matrix $\mathbf{g}$:

$$\chi_{\lambda}(\mathbf{g}) = s_{\lambda}(\{g_i\}) = \frac{\det_{ij}\left(g_i^{n+\lambda_j-j}\right)}{\det_{ij}\left(g_i^{n-j}\right)}.$$

Schur polynomials are symmetric functions of g_i. It is often convenient to consider Schur polynomials $s_{\lambda}(\{\xi_i\})$, where $\{\xi_i\}$ is a set of variables, as functions of the variables $t_k = \frac{1}{k}\sum\limits_i \xi_i^k$. Let us denote it as $s_{\lambda}(\mathbf{t})$, where $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$. For example, $s_{\emptyset}(\mathbf{t}) = 1$, $s_{(1)}(\mathbf{t}) = t_1$, $s_{(2)}(\mathbf{t}) = \frac{1}{2}t_1^2 + t_2$, $s_{(1^2)}(\mathbf{t}) = \frac{1}{2}t_1^2 - t_2$ and so on. For any finite diagram λ the polynomial $s_{\lambda}(\mathbf{t})$ depends only on a finite number of t_i's. Schur polynomials satisfy a number of non-trivial identities. We mention here the Cauchy-Littlewood identity

$$\sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}') = \exp\Big(\sum_{k\geq 1} k t_k t_k'\Big), \tag{6.15}$$

where the sum in the left hand side is taken over all Young diagrams including the empty one. There are also the Jacobi-Trudi identities which express the character (Schur polynomial) χ_{λ} through the characters $\chi_{(k)}$ or $\chi_{(1^k)}$ corresponding to the diagrams which are respectively a row or a column of length k:

$$\chi_{\lambda}(\mathbf{g}) = \det_{1\leq i,j\leq \lambda_1'} \chi_{(\lambda_i - i + j)}(\mathbf{g}), \tag{6.16}$$

$$\chi_{\lambda}(\mathbf{g}) = \det_{1\leq i,j\leq \lambda_1} \chi_{(1^{\lambda_i' - i + j})}(\mathbf{g}). \tag{6.17}$$

In these formulas λ' is the diagram λ transposed with respect to the main diagonal, so that $\lambda_1', \lambda_2', \ldots$ are lengths of columns of λ.

The analogy between transfer matrices and characters is supported by the fact that the transfer matrices satisfy the following identities (functional relations), which look similarly to the Jacobi-Trudi identities:

$$\mathbf{T}_{\lambda}(x) = \det_{1\leq i,j\leq \lambda_1'} \mathbf{T}_{(\lambda_i - i + j)}(x - (j-1)\eta), \tag{6.18}$$

$$\mathbf{T}_{\lambda}(x) = \det_{1\leq i,j\leq \lambda_1} \mathbf{T}_{(1^{\lambda_i' - i + j})}(x + (j-1)\eta). \tag{6.19}$$

They are called the Cherednik-Bazhanov-Reshetikhin (CBR) identities or quantum Jacobi-Trudi identities.

For transfer matrices corresponding to rectangular Young diagrams $\lambda = (s^a)$ with a rows of length s, the CBR identities are equivalent to the remarkable functional relation which has the form of 3-term difference Hirota equation known in the theory of difference soliton equations. Let us introduce the transfer matrices

$$\mathsf{T}^a_s(x) = \mathsf{T}_{(s^a)}\Big(x - \tfrac{\eta}{2}(s+a)\Big),$$

corresponding to rectangular Young diagrams, then the CBR identities imply the functional relation

$$\mathsf{T}^a_s\Big(x+\tfrac{\eta}{2}\Big)\mathsf{T}^a_s\Big(x-\tfrac{\eta}{2}\Big) - \mathsf{T}^a_{s+1}(x)\mathsf{T}^a_{s-1}(x) = \mathsf{T}^{a+1}_s(x)\mathsf{T}^{a-1}_s(x). \tag{6.20}$$

Problem. Prove the functional relation (6.20).

There is an elegant way to represent the transfer matrices $\mathsf{T}_\lambda(x)$ as special matrix derivatives of the characters $\chi_\lambda(\mathbf{g})$ with respect to the matrix $\mathbf{g}$ (which is, generally speaking, already not assumed to be diagonal). Let $f(\mathbf{g})$ be any function on the group $GL(n)$ ($\mathbf{g} \in GL(n)$). Define the matrix derivative (which we call coderivative) as follows:

$$Df(\mathbf{g}) = \sum_{a,b} e_{ab}\frac{\partial}{\partial\varepsilon} f\left(e^{\varepsilon \mathbf{e}_{ba}}\mathbf{g}\right)\Big|_{\varepsilon=0}. \tag{6.21}$$

According to this definition, if values of f belong to a space V, values of $Df(\mathbf{g})$ belong to $\mathrm{End}(\mathbb{C}^n)\otimes V$. An equivalent definition in components is

$$D^a_b = \sum_c g^a_c \frac{\partial}{\partial g^b_c},$$

where g^a_b are matrix elements of the matrix $\mathbf{g} \in GL(n)$ in the vector representation. Explicitly, we have:

$$D^a_b f(\mathbf{g}) = \frac{\partial}{\partial\varepsilon} f\left(e^{\varepsilon \mathbf{e}_{ba}}\mathbf{g}\right)\Big|_{\varepsilon=0}.$$

A direct calculation of the commutator $[D^{a_2}_{b_2}, D^{a_1}_{b_1}]$ shows that

$$[D^{a_2}_{b_2}, D^{a_1}_{b_1}] = \delta_{a_1 b_2} D^{a_2}_{b_1} - \delta_{a_2 b_1} D^{a_1}_{b_2}, \tag{6.22}$$

i.e., the operators D^a_b have the same commutation relations as the generators $\mathbf{e}_{ab}$ of the algebra $U(gl_n)$.

In the case when the coderivatives act on functions with values in the tensor product $\otimes_i V_i$ of the spaces V_i it is convenient to modify the notation by giving index i to the coderivative:

$$D_i f(\mathbf{g}) = \sum_{a,b} e^{(i)}_{ab}\frac{\partial}{\partial\varepsilon} f\left(e^{\varepsilon \mathbf{e}_{ba}}\mathbf{g}\right)\Big|_{\varepsilon=0},$$

where $e^{(i)}_{ab}$ acts non-trivially in V_i. In this notation we have, for example: $D_1 \mathrm{tr}\,\mathbf{g} = \mathbf{g}_1$, $D_2\mathbf{g}_1 = \mathsf{P}_{21}\mathbf{g}_1$, while the relation (6.22) is written in the form $[D_2, D_1] = \mathsf{P}_{12}(D_1 - D_2)$.

A careful analysis shows that the transfer matrix $\mathsf{T}_\lambda(u)$ can be expressed as

$$\mathsf{T}_\lambda(x) = (x - x_N + \eta D_N)\ldots(x - x_1 + \eta D_1)\chi_\lambda(\mathbf{g}). \tag{6.23}$$

With the help of this representation, one can prove the CBR identities.

6.3 The master T-operator as a tau-function

Let us introduce the generating function for the transfer matrices $\mathsf{T}_\lambda(x)$. It is called the master T-operator. Let $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$ be an infinite set of complex variables. The master T-operator has the form

$$\mathsf{T}(x;\mathbf{t}) = \sum_\lambda s_\lambda(\mathbf{t})\mathsf{T}_\lambda(x), \tag{6.24}$$

where the sum, as in (6.15), is taken over all Young diagrams including the empty one. As $\mathsf{T}_\lambda(x)$, it is an operator in $(\mathbb{C}^n)^{\otimes N}$. It depends on the elements g_i of the twist matrix $\mathbf{g}$ as on parameters. Clearly, the operators $\mathsf{T}(x;\mathbf{t})$ commute for all x, $\mathbf{t}$.

In terms of the coderivatives, the master T-operator can be represented in the form

$$\mathsf{T}(x;\mathbf{t}) = (x - x_N + \eta D_N)\ldots(x - x_1 + \eta D_1)\exp\Big(\sum_{k\geq 1} t_k \mathrm{tr}\,\mathbf{g}^k\Big)$$

(one should use (6.23) and the Cauchy-Littlewood identity (6.15)).

Obviously, $\mathsf{T}(x;0) = \mathsf{T}_\emptyset(x)$. Acting to $\mathsf{T}(x;\mathbf{t})$ by differential operators in t_k at $\mathbf{t} = 0$, one can reproduce all the transfer matrices $\mathsf{T}_\lambda(x)$. For example,

$$\mathsf{T}_{(1)}(x) = \partial_{t_1}\mathsf{T}(x;\mathbf{t})\Big|_{\mathbf{t}=0}, \qquad \mathsf{T}_{(2)}(x) = \frac{1}{2}\left(\partial_{t_1}^2 + \partial_{t_2}\right)\mathsf{T}(x;\mathbf{t})\Big|_{\mathbf{t}=0}. \tag{6.25}$$

The general formula is

$$\mathsf{T}_\lambda(x) = s_\lambda(\tilde{\partial})\mathsf{T}(x;\mathbf{t})\Big|_{\mathbf{t}=0},$$

where $\tilde{\partial} = \{\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots\}$. Analyzing the behavior of the master T-operator $\mathsf{T}(x;\mathbf{t})$ as a function of $\mathbf{t}$ in a neighborhood of some other points other than $\mathbf{t} = 0$, one can show that this family contains also the Baxter's Q-operators.

Let us pass to the most important property o the master T-operator, which establishes a close connection with the theory of classical integrable nonlinear partial differential equations. We will use the notation

$$\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k,$$

$$\mathbf{t} \pm [z^{-1}] = \{t_1 \pm z^{-1},\ t_2 \pm \tfrac{1}{2}z^{-2},\ t_3 \pm \tfrac{1}{3}z^{-3},\ \ldots\}.$$

It can be shown that the CBR identities are equivalent to the following *bilinear relation* for the master T-operator:

$$\boxed{\oint_C z e^{\xi(\mathbf{t}-\mathbf{t}', z)}\mathsf{T}\left(x;\mathbf{t} - [z^{-1}]\right)\mathsf{T}\left(x - \eta;\mathbf{t}' + [z^{-1}]\right)dz = 0,} \tag{6.26}$$

which is valid for all $x, \mathbf{t}, \mathbf{t}'$. The integration contour C is a big circle around ∞ which separates the singularities coming from the T-multipliers and the exponential function.

The bilinear relation (6.26) allows one to identify the master T-operator (more precisely, any one of its eigenvalues) with the tau-function of the modified Kadomtsev-Petviashvili hierarchy (mKP) known in the theory of soliton equations. Note that the spectral parameter x plays the role of "zeroth time" t_0.

Putting, for example, $\mathbf{t}' = \mathbf{t} - [z_1^{-1}] - [z_2^{-1}]$ and calculating residues in the bilinear relation, we get the 3-term Hirota equation for the mKP hierarchy:

$$\begin{aligned} z_2 \mathsf{T}\left(x+\eta; \mathbf{t} - [z_2^{-1}]\right) \mathsf{T}\left(x; \mathbf{t} - [z_1^{-1}]\right) - z_1 \mathsf{T}\left(x+\eta; \mathbf{t} - [z_1^{-1}]\right) \mathsf{T}\left(x; \mathbf{t} - [z_2^{-1}]\right) \\ = (z_1 - z_2) \mathsf{T}(x+\eta; \mathbf{t}) \mathsf{T}\left(x; \mathbf{t} - [z_1^{-1}] - [z_2^{-1}]\right). \end{aligned} \tag{6.27}$$

6.4 Connection with classical models of the Calogero-Moser type

We can say that any eigenvalue of the master T-operator as a function of the "times" $\{t_k\}$ and $t_0 = x$ is a solution of the mKP hierarchy in the bilinear form (the tau-function). The latter has many different solutions and we would like to characterize our solutions more precisely. This can be done if we take into account that $\mathsf{T}(x; \mathbf{t})$ commute and can be simultaneously diagonalized while matrix elements of these operators are polynomials in x of degree N. Therefore, the eigenvalues (we denote them by $T(x; \mathbf{t})$) are also polynomials in x of degree N, i.e., they have the form

$$T(x; \mathbf{t}) = e^{t_1 \operatorname{tr} \mathbf{g} + t_2 \operatorname{tr} \mathbf{g}^2 + \dots} \prod_{k=1}^{N} (x - x_k(\mathbf{t})) \tag{6.28}$$

(the exponential factor is restored from the limit $x \to \infty$). Roots of these polynomials depend on t_i and $x_k(0) = x_k$.

The first formula in (6.25) tells us that the eigenvalue $T(x)$ of the transfer matrix $\mathbf{T}(x)$ is

$$T(x) = \partial_{t_1} \log T(x; \mathbf{t}) \Big|_{\mathbf{t}=0}.$$

Plugging here (6.28) and comparing with (6.5), we obtain:

$$\eta H_i = -\dot{x}_i(0). \tag{6.29}$$

The dynamics of zeros of polynomial tau-functions is a well known subject in the theory of integrable nonlinear partial differential equations. From the works of Krichever and others it follows that this dynamics is described by equations of motion of integrable many-body systems of the Calogero-Moser type. In particular, the dynamics of zeros of the tau-function of the mKP hierarchy of the form (6.28) in the time t_k coincides with the dynamics of the Ruijsenaars-Schneider system of particles (a relativistic deformation of the Calogero-Moser system) with respect to the kth Hamiltonian flow. For example, the equations of motion in the time t_1 have the form

$$\ddot{x}_i = -\sum_{k \neq i} \frac{2\eta^2 \dot{x}_i \dot{x}_k}{(x_i - x_k)((x_i - x_k)^2 - \eta^2)} \tag{6.30}$$

(dot means the t_1-derivative) with the Hamiltonian

$$\mathcal{H}_1 = \sum_{i=1}^{N} e^{\eta p_i} \prod_{k \neq i} \frac{x_i - x_k + \eta}{x_i - x_k}, \qquad \{p_i, x_k\} = \delta_{ik}.$$

The system is known to be integrable: there are N independent conserved quantities in involution $\mathcal{I}_k$, $k = 1, \ldots, N$, and $\mathcal{I}_1 = \mathcal{H}_1$. The parameter η has the meaning of the inverse velocity of light. In the limit $\eta \to 0$ one reproduces the Calogero-Moser system of particles.

From this it follows a nontrivial connection between the generalized inhomogeneous quantum spin chains solvable by the algebraic Bethe ansatz and classical integrable many-body systems of the Calogero-Moser type. This connection is called quantum-classical duality. It is discussed in more detail in the next section.

6.5 Quantum-classical duality

The quantum-classical duality for integrable systems is a remarkable relation between the spectrum of a generalized inhomogeneous quantum spin chain or its limit to a model of the Gaudin type and intersection of two Lagrangian submanifolds in the $2N$-dimensional phase space of a classical relativistic N-body integrable system of the Ruijsenaars-Schneider type or its non-relativistic limit (the Calogero-Moser system). Lagrangian submanifold is a N-dimensional submanifold in the $2N$-dimensional phase space such that the restriction of the form $\omega = \sum_{i=1}^{N} dp_i \wedge dx_i$ is equal to zero. In the relation mentioned above, the first Lagrangian manifold is the N-dimensional hyperplane corresponding to fixing all coordinates x_j of the classical particles, and the second one is the level set of the N integrals of motion in involution. Their dimensions are complimentary, and thus they intersect in a finite number of points. The essence of the quantum-classical duality is that the values of the particles velocities $\dot{x}_j$ at the intersection points provide spectra of the quantum Hamiltonians of the inhomogeneous spin chain (or the Gaudin model) with the inhomogeneities x_j. Different intersection points correspond to different eigenstates of the commuting quantum Hamiltonians.

Let us describe the quantum-classical duality in more details. For this, we should recall that the classical N-body Ruijsenaars-Schneider model admits the Lax representation of the form of the Lax equation

$$\dot{L} = [L, M] \tag{6.31}$$

for $N \times N$ matrices L, M whose matrix elements are functions of x_j and $\dot{x}_j$. The matrix L is called the Lax matrix, its explicit form is

$$L_{ij} = L_{ij}\Big(\{\dot{x}_l\}_N, \{x_l\}_N\Big) = \frac{\dot{x}_i}{x_i - x_j - \eta}, \qquad i, j = 1, \ldots, N. \tag{6.32}$$

Below we do not need the explicit form of the matrix M; its existence is the only essential fact. Equations of motion (6.30) are equivalent to the matrix equation (6.31). The Lax equation implies that the time evolution of the Lax matrix $L(0) \to L(t)$ is an isospectral transformation, i.e., eigenvalues of the Lax matrix (and symmetric functions of them) are integrals of motion. It is known that they are in involution. For example, $\mathcal{H}_1 = \mathrm{tr} L$ and $\mathcal{H}_k = \frac{1}{k} \,\mathrm{tr}\, L^k$, $k \geq 2$, are higher Hamiltonians of the integrable many-body system.

Consider the Lax matrix (6.32) $L(0)$ with the substitution $\dot{x}_i(0) = -\eta H_i$, where H_i are eigenvalues of the quantum Hamiltonians $\mathbf{H}_i$ of the generalized twisted inhomogeneous spin chain given by (6.9) (see (6.29)):

$$L_{ij}(0) = L_{ij}\Big(\{-\eta H_l\}_N, \{x_l\}_N\Big) = \frac{\eta H_i}{x_j - x_i + \eta}.$$

The quantum-classical duality states that then the spectrum of L has the following very specific form:

$$\boxed{\operatorname{Spec} L\Big(\{-\eta H_i\}_N, \{x_i\}_N\Big) = \Big(\underbrace{g_1, \ldots, g_1}_{M_1}, \underbrace{g_2, \ldots, g_2}_{M_2}, \ldots, \underbrace{g_n, \ldots, g_n}_{M_n}\Big),} \tag{6.33}$$

where M_a are eigenvalues of the operators $\mathbf{M}_a$ (6.6) on the eigenstate of the transfer matrix (we recall that $\sum\limits_{a=1}^{n} M_a = N$).

Before passing to the proof of this statement, let us say a few words about its meaning. It implies, in particular, that it is possible to solve the spectral problem for the Hamiltonians of the quantum spin chain without addressing the Bethe ansatz at any step. Instead, one should solve an "inverse spectral problem" for the Lax matrix of the classical Ruijsenaars-Schneider system of particles. Namely, let $\{x_i\}_N$ be inhomogeneity parameters of the spin chain and $\mathbf{g} = \operatorname{diag}(g_1, g_2, \ldots, g_n)$ its twist matrix. Let the eigenvalues of the Lax matrix be equal to the eigenvalues g_a of the twist matrix, with some multiplicities M_a such that $\sum\limits_{a=1}^{n} M_a = N$. This fixes values of all the Ruijsenaars-Schneider Hamiltonians: $\mathcal{H}_k = \frac{1}{k}\sum\limits_a M_a g_a^k$. Then the spectrum of the non-local spin chain Hamiltonians $\mathbf{H}_j$ in the sector where eigenvalues of the operators $\mathbf{M}_a$ are equal to M_a is given by the values of H_j such that the matrix $L_{ij} = \frac{\eta H_i}{x_j - x_i + \eta}$ has the prescribed spectrum.

This kind of duality suggests an alternative way to calculate joint spectra of commuting quantum transfer matrices without any use of the coordinate or algebraic Bethe ansatz technique, which is a key tool in any exact solution of quantum integrable systems with non-trivial interaction. There is also no need in such an unavoidable intermediate step as solving Bethe equations. The spectra of *quantum* Hamiltonians appear to be encoded in algebraic properties of the Lax matrix for a very different *purely classical* model.

We now pass to the sketch of proof of (6.33). We want to prove that

$$\det\Big[L\Big(\{-\eta H_i\}_N, \{x_i\}_N\Big)\Big|_{BE} - \lambda I\Big] = \prod_{a=1}^{n} (g_a - \lambda)^{M_a}, \tag{6.34}$$

where $L\Big(\{-\eta H_i\}_N, \{x_i\}_N\Big)$ is taken on a solution to the Bethe equations (BE). Let $\{y_i\}_M = \{y_1, \ldots, y_M\}$ be a set of M auxiliary variables (we assume that $M \le N$). Let us introduce the $N \times N$ matrix $\mathcal{L} = \mathcal{L}\Big(\{x_i\}_N, \{y_i\}_M, g\Big)$ and $M \times M$ matrix $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}\Big(\{y_i\}_M, \{x_i\}_N, g\Big)$ by the following formulas:

$$\mathcal{L}_{ij}\Big(\{x_l\}_N, \{y_l\}_M, g\Big) = \frac{g\eta}{x_i - x_j + \eta} \prod_{k \neq j}^{N} \frac{x_j - x_k + \eta}{x_j - x_k} \prod_{\gamma=1}^{M} \frac{x_j - y_\gamma}{x_j - y_\gamma + \eta} \tag{6.35}$$

(here $i,j=1,\dots,N$),

$$\tilde{\mathcal{L}}_{\alpha\beta}\Big(\{y_l\}_M,\{x_l\}_N,g\Big)=\frac{g\eta}{y_\alpha-y_\beta+\eta}\prod_{\gamma\neq\beta}^{M}\frac{y_\beta-y_\gamma-\eta}{y_\beta-y_\gamma}\prod_{k=1}^{N}\frac{y_\beta-x_k}{y_\beta-x_k-\eta} \tag{6.36}$$

(here $\alpha,\beta=1,\dots,M$). The proof is based on the algebraic relation for characteristic polynomials of the matrices $\mathcal{L}$, $\tilde{\mathcal{L}}$:

$$\det_{N\times N}\bigg(\mathcal{L}\Big(\{x_i\}_N,\{y_i\}_M,g\Big)-\lambda I\bigg)=(g-\lambda)^{N-M}\det_{M\times M}\bigg(\tilde{\mathcal{L}}\Big(\{y_i\}_M,\{x_i\}_N,g\Big)-\lambda I\bigg). \tag{6.37}$$

The proof of (6.37) is rather technical. It can be found in [14]. An important ingredient of the proof is factorization of the matrices $\mathcal{L}$, $\tilde{\mathcal{L}}$ given by the formulas

$$\mathcal{L}\Big(\{x_i\}_N,\{y_i\}_M,g\Big)=gD_\eta^{-1}(\{x_i\}_N)V^t(\{x_i\}_N)C^t_{\eta,N}(V^t(\{x_i\}_N))^{-1}D_\eta(\{x_i\}_N)\mathcal{D}, \tag{6.38}$$

$$\tilde{\mathcal{L}}\Big(\{y_i\}_M,\{x_i\}_N,g\Big)=gD_0(\{y_i\}_M)V^{-1}(\{y_i\}_M)C_{-\eta,M}V(\{y_i\}_M)D_0^{-1}(\{y_i\}_M)\tilde{\mathcal{D}}. \tag{6.39}$$

Here

$$\begin{aligned}\mathcal{D}_{ij}&=\delta_{ij}\prod_{\gamma=1}^{M}\frac{x_j-y_\gamma}{x_j-y_\gamma+\eta}, \qquad i,j=1,\dots,N,\\ \tilde{\mathcal{D}}_{\alpha\beta}&=\delta_{\alpha\beta}\prod_{k=1}^{N}\frac{y_\beta-x_k}{y_\beta-x_k-\eta}, \qquad \alpha,\beta=1,\dots,M,\end{aligned} \tag{6.40}$$

$$(D_\xi(\{z_l\}_K))_{ij}=\delta_{ij}\prod_{k\neq i}^{K}(z_i-z_k+\xi), \qquad i,j=1,\dots,K, \tag{6.41}$$

$(V(\{z_l\}_K))_{ij}=z_j^{i-1}$ is the Vandermonde matrix, $C_{\eta,K}$ is the triangular matrix of the form

$$(C_{\eta,K})_{ij}=\begin{cases}\dfrac{(i-1)!\,\eta^{i-j}}{(j-1)!(i-j)!}, & j\le i,\\[2mm] 0, & j>i,\end{cases} \qquad i,j=1,\quad\dots,K \tag{6.42}$$

and upper index t means transposition. Note that $\det_{N\times N}\mathcal{D}=\det_{M\times M}\tilde{\mathcal{D}}$.

The proof of (6.34) includes $n-1$ steps and consists of successive application of (6.37) with taking into account the Bethe equations (6.8) at each step. Set

$$L^{(0)}_{ij}=L_{ij}\Big(\{-\eta H_l\}_N,\{x_l\}_N\Big)=\mathcal{L}_{ji}\Big(\{x_l-\eta\}_N,\{v^{(1)}_\gamma\}_{N_1},g_1\Big)$$

$$=\frac{\eta g_1}{x_j-x_i+\eta}\prod_{k\neq i}\frac{x_i-x_k+\eta}{x_i-x_k}\prod_{\gamma=1}^{N_1}\frac{x_i-v^{(1)}_\gamma-\eta}{x_i-v^{(1)}_\gamma}$$

and define (at the first step)

$$L^{(1)}_{\alpha\beta}=\tilde{\mathcal{L}}_{\alpha\beta}\Big(\{v^{(1)}_\gamma\}_{N_1},\{x_l-\eta\}_N,g_1\Big)$$

$$=\frac{\eta g_1}{v^{(1)}_\alpha-v^{(1)}_\beta+\eta}\prod_{\gamma\neq\beta}^{N_1}\frac{v^{(1)}_\beta-v^{(1)}_\gamma-\eta}{v^{(1)}_\beta-v^{(1)}_\gamma}\prod_{k=1}^{N}\frac{v^{(1)}_\beta-x_k+\eta}{v^{(1)}_\beta-x_k}.$$

Equation (6.37) implies that

$$\det_{N\times N}\Big(L^{(0)}-\lambda I\Big)=(g_1-\lambda)^{N-N_1}\det_{N_1\times N_1}\Big(L^{(1)}-\lambda I\Big). \tag{6.43}$$

Next, impose Bethe equations (6.8) at $b=1$, i.e.,

$$g_1\prod_{k=1}^{N}\frac{v_\beta^{(1)}-x_k+\eta}{v_\beta^{(1)}-x_k}=g_2\prod_{\gamma\neq\beta}^{N_1}\frac{v_\beta^{(1)}-v_\gamma^{(1)}+\eta}{v_\beta^{(1)}-v_\gamma^{(1)}-\eta}\prod_{\gamma'=1}^{N_2}\frac{v_\beta^{(1)}-v_{\gamma'}^{(2)}-\eta}{v_\beta^{(1)}-v_{\gamma'}^{(2)}} \tag{6.44}$$

to obtain

$$\begin{aligned}L_{\alpha\beta}^{(1)}\Big|_{\mathrm{BE}} &= \frac{\eta g_2}{v_\alpha^{(1)}-v_\beta^{(1)}+\eta}\prod_{\gamma\neq\beta}^{N_1}\frac{v_\beta^{(1)}-v_\gamma^{(1)}+\eta}{v_\beta^{(1)}-v_\gamma^{(1)}}\prod_{\gamma'=1}^{N_2}\frac{v_\beta^{(1)}-v_{\gamma'}^{(2)}-\eta}{v_\beta^{(1)}-v_{\gamma'}^{(2)}}\\ &= \mathcal{L}_{\alpha\beta}\Big(\{v_\gamma^{(1)}-\eta\}_{N_1},\{v_\gamma^{(2)}\}_{N_2},g_2\Big).\end{aligned} \tag{6.45}$$

At the second step we define

$$L_{\alpha\beta}^{(2)}=\tilde{\mathcal{L}}_{\alpha\beta}\Big(\{v_\gamma^{(2)}\}_{N_2},\{v_\gamma^{(1)}-\eta\}_{N_1},g_2\Big),\qquad \alpha,\beta=1,\dots,N_2.$$

Similarly to the previous step, we use (6.37) to obtain

$$\det_{N_1\times N_1}\Big(L^{(1)}-\lambda I\Big)=(g_2-\lambda)^{N_1-N_2}\det_{N_2\times N_2}\Big(L^{(2)}-\lambda I\Big) \tag{6.46}$$

and use the Bethe equations to conclude that

$$L^{(2)}\Big|_{\mathrm{BE}}=\mathcal{L}\Big(\{v_\gamma^{(2)}-\eta\}_{N_2},\{v_\gamma^{(3)}\}_{N_3},g_3\Big).$$

The procedure can be continued until the last step, where

$$L_{\alpha\beta}^{(n-1)}\Big|_{\mathrm{BE}}=\frac{\eta g_n}{v_\alpha^{(n-1)}-v_\beta^{(n-1)}+\eta}\prod_{\gamma\neq\beta}^{N_{n-1}}\frac{v_\beta^{(n-1)}-v_\gamma^{(n-1)}+\eta}{v_\beta^{(n-1)}-v_\gamma^{(n-1)}},$$

and, according to (6.37) at $M=0$,

$$\det_{N_{n-1}\times N_{n-1}}\Big(L^{(n-1)}-\lambda I\Big)=(g_n-\lambda)^{N_{n-1}}.$$

Therefore, taking into account that $M_a=N_{a-1}-N_a$, we have proved (6.34).

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