Second steps of énumérative geometry

$(Z^1 : \dots : Z^N)$

$$Z^I(z,\bar z) = e^{\chi(z,\bar z)}\, p^I(z)$$
$$\partial_{\bar z}\chi + A_{\bar z} = 0 \quad \text{locally}$$

framed

$$QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1}) \subset QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1}) \approx \mathbb{P}^{N-1+N\beta}$$

$\mathbb{P}^1 = \Sigma$

$F_{z\bar z} \to 0$, $z \to \infty$

complex srructure

$$(A_z, A_{\bar z}; Z^1, \dots, Z^N) \longleftrightarrow \left\{ \begin{array}{l} \nabla_{\bar z} Z^I = 0 \\ F_{z\bar z} + \left(\sum_{I=1}^N |Z^I|^2 - r\right)\mathrm{vol}_\Sigma = 0 \end{array} \right\} \Big/ \mathbb{C}^\times$$

$+\Delta\chi + e^{i\chi}(\dots)$

symplectic structure

ω

хотя бы одно сечение отлично от 0 хотя бы в одной точке $(z,\bar z)$ $Z^I(z,\bar z) \neq 0$

$\exists I$

$\mathbb{C}^\times$ — gauge transformations

$$e^\chi \qquad A_{\bar z} \to A_{\bar z} + \partial_{\bar z}\chi, \quad A_z \to A_z - \partial_z \chi, \quad Z^I \to e^\chi Z^I$$

разрешение особенностей

$PGL(2) \times Aut(X)$

$QMaps_\beta(\mathbb{P}^1, X)$

math trick

$PGL(2) \times Aut(X)$

$\mathcal{M}_{0,0}(\mathbb{P}^1 \times X)$

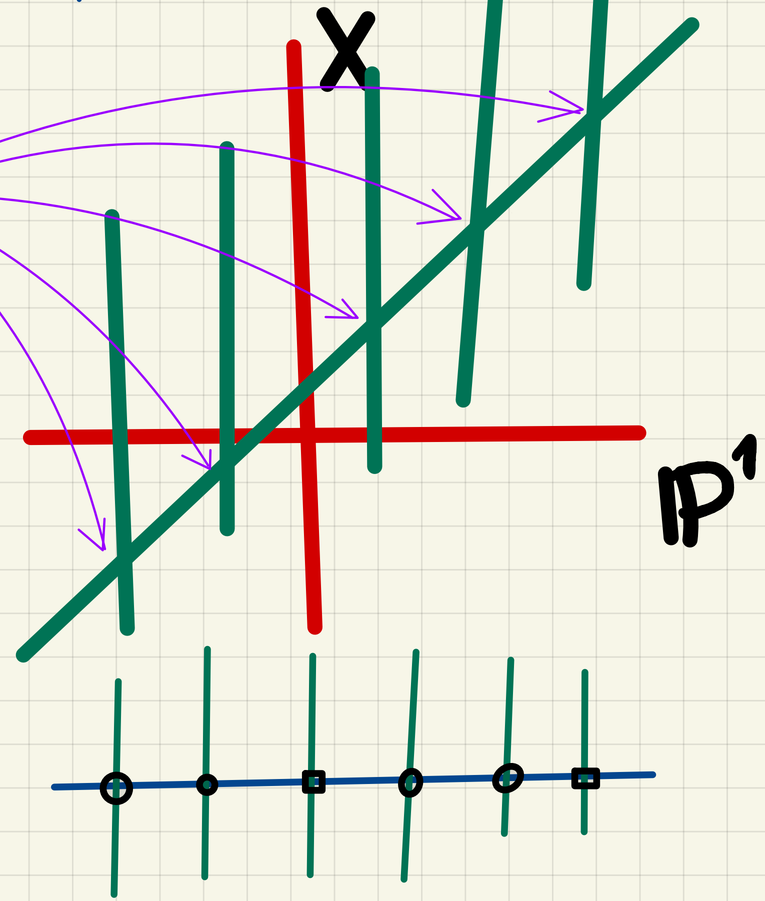
genus zero curves

no punctures

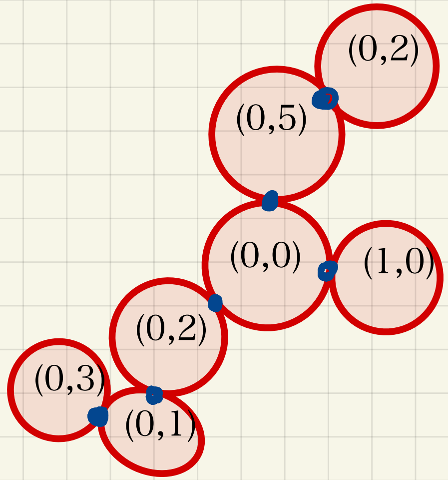
common zeroes of all $P^I(z)$

$\beta = \sum_i \beta_i$

$n = \sum_i n_i = 1$



$\mathbb{P}^1 \times X$



maps without infinitesimal automorphisms

$$QMaps_{\beta}(\mathbb{P}^1, X) \dashleftarrow \mathcal{M}_{0,0}(\mathbb{P}^1 \times X, (1,\beta)) \rightsquigarrow \mathcal{M}_{g,0}(\Sigma \times X, (1,\beta))$$

[о б о б щ е н и е]

symplectic structure of Σ

TFT п р и н и м а е т з н а ч е н и я в $H^0(\mathcal{M}_{\text{cmplx}})$

complex structure of Σ

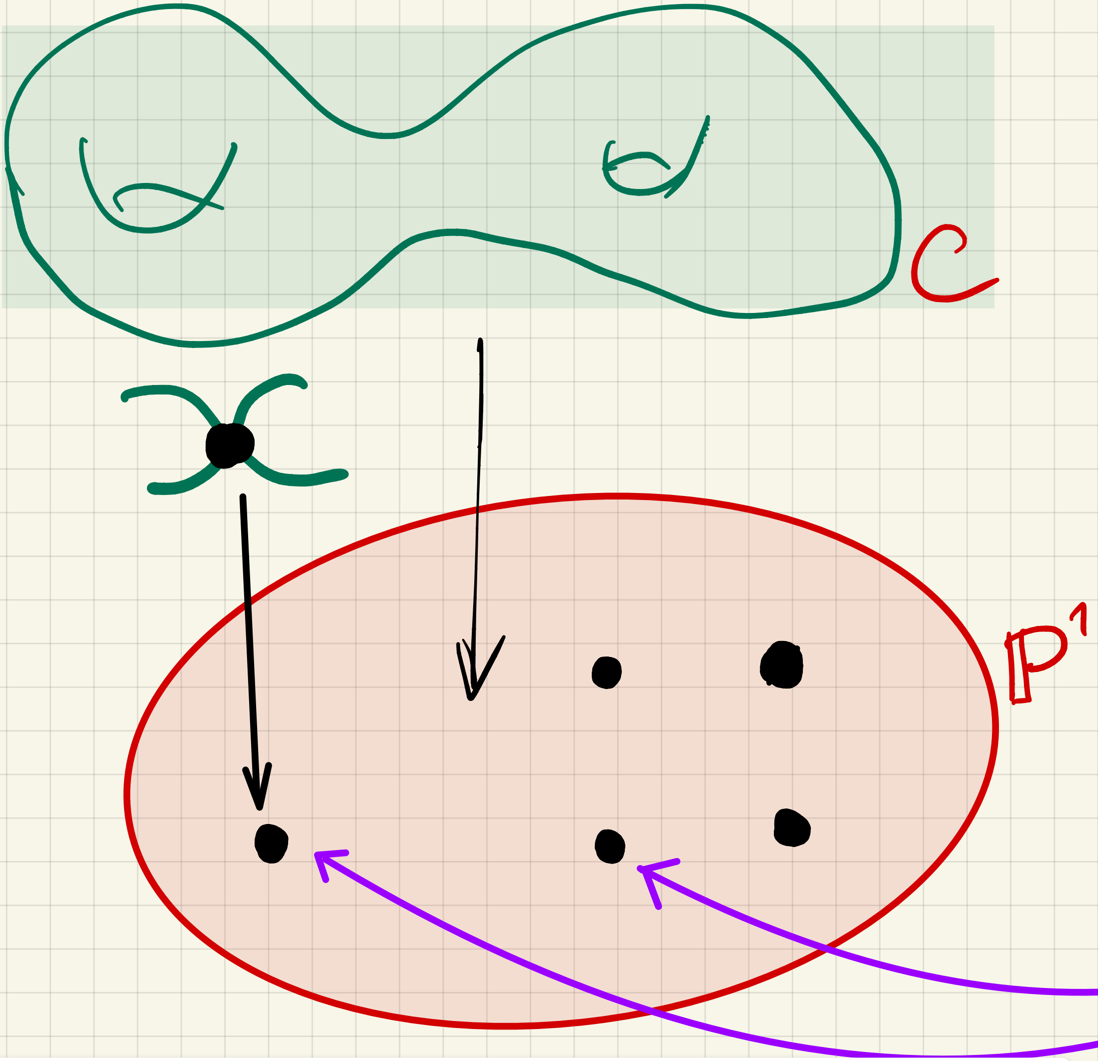
through TARGET SPACE GRAVITY

TFT + gravity п р и н и м а е т з н а ч е н и я в $H^*(\mathcal{M}_{\text{cmplx}})$

(aka top.string) ??

$$BDiff(\Sigma) = Met(\Sigma) / Diff(\Sigma) \ltimes Weyl$$

V.Knizhnik, “Analytic fields on Riemann surfaces”, Phys Lett B 180 (3) 1986



$$\int_C \mathcal{L}(\phi, \partial\phi)$$

$$\sum_{i=1}^{N} \int_{\mathbb{P}^1} \mathcal{L}(\phi_i, \partial\phi_i)$$

$+$

twist vertex operators

$$DT(\mathbb{P}^1 \times X)$$

$$\updownarrow$$

$$\left[\mathbb{P}^1 \to Hilb^{[n]}(X)\right]$$

$$\updownarrow$$

$$\coprod_g GW_g(X)$$

“Matrix string theory”, Dijkgraaf, Verlinde, Verlinde, 1997

“Proposals on nonperturbative superstrings interactions”, L.Motl 1997 earlier

$$\{(U_i^I)\}/\mathbb{C}^\times = QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1}) \longleftarrow \mathcal{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{N-1}, (1,\beta))$$

голоморфное отображение многообразий одной размерности, “разрешение особенностей»

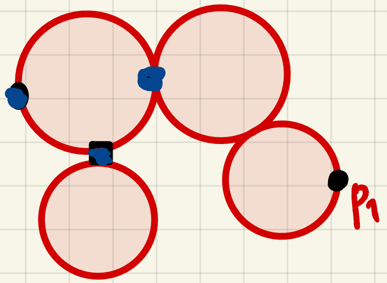
$$\vec{U}_\beta \neq 0 \qquad QMaps_\beta^{framed}(\mathbb{P}^1, \mathbb{P}^{N-1}) \xrightarrow{ev_\infty} \mathbb{P}^{N-1}$$

$$\mathcal{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^{N-1}, (1,\beta)) \xrightarrow{fgt_1} \mathcal{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{N-1}, (1,\beta))$$

$$\mathcal{M}_{0,1} \xrightarrow{ev_1} \infty \times \mathbb{P}^{N-1}$$

bi-degree

требуем, чтобы в бесконечности было настоящее отображение ∞



$$ev_\infty(U_i^I) = (U_\beta^1 : \cdots : U_\beta^N) \in \mathbb{P}^{N-1}$$

$$P^I(z) = \sum_{i=0}^{\beta} U_i^I z^i$$

$$Z^I = e^{\xi(z,\bar{z})} P^I(z)$$

$$\mathbb{C}[a_1,\dots,a_N]^{S(N)} / \langle \Sigma a_i \rangle = H^*_{PGL(N)}(\mathbb{P}^{N-1})$$

equivariant cohomology

$$PGL(2) \times PGL(N)$$

$$QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1})$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(M^I_J\right) \right) \left(U^I_i\right) = \left(\mathrm{Coeff}_{z^i} \sum_{J,k} M^I_J (az+b)^{\beta-k} (cz+d)^k U^J_k \right)$$

инфинитезимальное действие компактной подгруппы **SU(2) x PSU(N)** $\ni$ maximal torus $U(1)^N$ π_N

$$\sum_i a_i = 0$$

$$(\hbar, a) \in \mathrm{Lie}(\ ), \quad a = (a_1, \dots, a_N)$$

$\mathbb{C}$

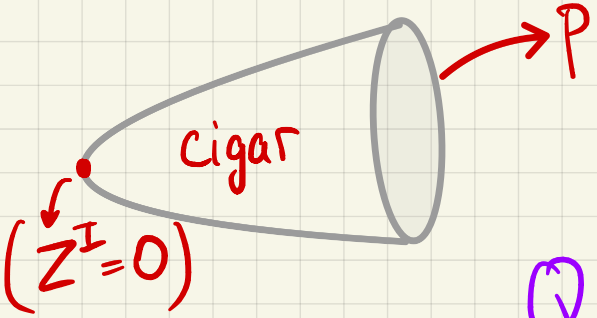
$$V_{(a,\hbar)} \in \mathrm{Vect}\left(QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1}) \right)$$

p — неподвижная точка $\mathbf{T}$-действия на $\mathbb{P}^{N-1}$

$$Z^{p}_{\beta}(a,\hbar) = \int \exp(-D\lambda)$$

$$\lambda = g(\cdot, V_{(\bar{a},\bar{\hbar})})$$

$$D = d + \iota_{V_{(a,\hbar)}}$$



$$ev_\infty^{-1}(p) \cap \text{framed } QMaps_\beta(\mathbb{P}^1, \mathbb{P}^{N-1})$$

касательное пространство
к пространству
квазиотображений

$$T_{Q_I}$$

$$\left(\delta U^J_i\right) \sim \left(\delta U^J_i + t U^J_i\right) \qquad t \in \mathbb{C}$$

at Q_I : $U^J_i = \delta^J_I \, \delta_{i,\beta} \implies \delta U^I_\beta = 0$ gauge

$$T_{Q_I} = \left\{ \delta U^J_i \;\middle|\; \begin{array}{ll} J = 1, \dots, N, & 1 \le i < \beta \\ J \neq I, & i = \beta \end{array} \right\} = \bigoplus \mathbb{C} e_\eta$$

$\eta = (J, i)$

$$\delta U^J_i \mapsto e^{(a_J - i\hbar) + (\beta\hbar - a_I)} \, \delta U^J_i$$

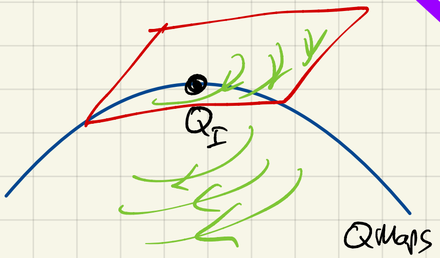
$$\parallel$$

$$\exp(\mathrm{weight}_\eta)$$

$\prod_N$ eigenspaces

$$Z^{\beta}_{I}(a,\hbar) = \prod_{J} \prod_{i=0}^{\beta-1} \left(\frac{1}{a_J - a_I + (\beta - i)\hbar} \right) \times \prod_{J \neq I} \left(\frac{1}{a_J - a_I} \right)$$

weights of torus action on $T_{Q_I}\, QMaps^{framed}_{\beta}(\mathbb{P}^1, \mathbb{P}^{N-1})$

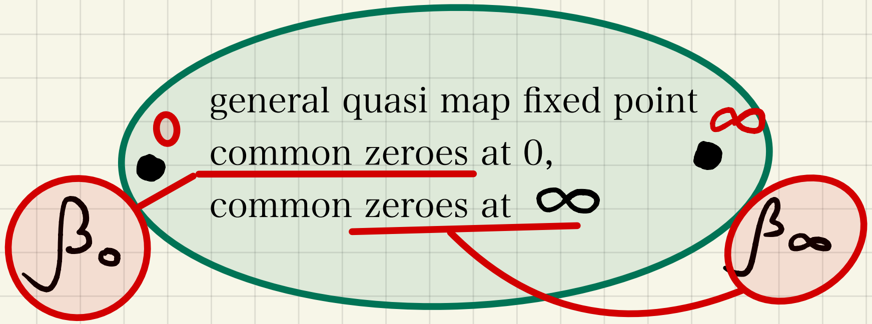


$$\frac{1}{2\pi i} \int F = \beta$$

fixed point $Q_I = \left(p^J(z) = \delta^J_I z^{\beta} \right) \in$ $U(1)_{\hbar}$ - fixed locus

$p^J(z) = \rho_J z^{\beta}$

$(\rho_1 : \dots : \rho_N) \in \mathbb{P}^{N-1}$



$$Z_I(a,\hbar) = \sum_{\beta=0}^{\infty} q^\beta Z_I^\beta(a,\hbar)$$

enumerative problem

$$\sim \frac{1}{\prod_J \Gamma\left(\beta + \frac{a_J - a_I}{\hbar}\right)}$$

$$q = e^{2\pi i t}$$

$$\beta = \frac{1}{2\pi i}\int_\Sigma F$$

$$t = \frac{\vartheta}{2\pi} + ir$$

Kahler class

more balanced system

$$\nabla_{\bar{z}} Z^I = 0$$

$$\nabla_{\bar{z}} W_I = 0$$

U(1) charges: +1 for Z, -1 for W

о б о б щ е н н ы й к о н и ф о л д

N $\mathcal{O}(-1)$ н а д $\mathbf{P}^{N-1}$

$$c_1 = 0$$

add equa'tions, e.g. Z W =0, to get $\mathbf{T}^*\mathbf{P}^{N-1}$

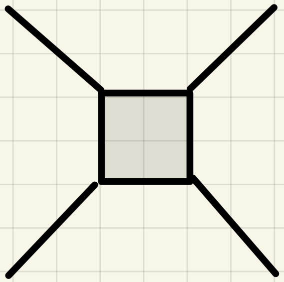
$$Z_I(a,\hbar) = \sum_{\beta=0}^{\infty} q^\beta \prod_{J=1}^{N} \frac{\Gamma\left(1+\frac{b_J - a_I}{\hbar}\right)}{\Gamma\left(1+\frac{a_J - a_I}{\hbar}\right)} \sim {}_N F_{N-1}$$

hypersurfaces in $\mathbf{P}^{N-1}$. **e.g.quintic N=5 in** $\mathbf{P}^4$

Candelas, de la Ossa, Green, Parkes' 1990 curve counting on

$\mathcal{O}(-2,-2)$ over $\mathbf{P}^1 \times \mathbf{P}^1$

$$\sum_{\beta=0}^{\infty} \int \prod_{i=1}^{N} dt_i^+ dt_i^- \, e^{-(t_i^+ + t_i^-)} \left(q \prod_{i=1}^{N} \left(\frac{t_i^+}{t_i^-} \right) \right)^{\beta} (t^x)$$



$\int$ Gelfand-Leray form Ω $\times\, e^{-W}$

$$W = \sum t^+ + t^-$$

$$q \prod t^+ = \prod t^-$$

$$\Gamma(x+\beta) = \int_0^\infty \frac{dt}{t} t^{\beta+x} e^{-t}$$

$$\frac{1}{\Gamma(y+\beta)} = (-)^\beta \frac{\sin(\pi y)}{\pi} \int_0^\infty dt\, t^{-\beta-y} e^{-t}$$