First steps of énumérative geometry

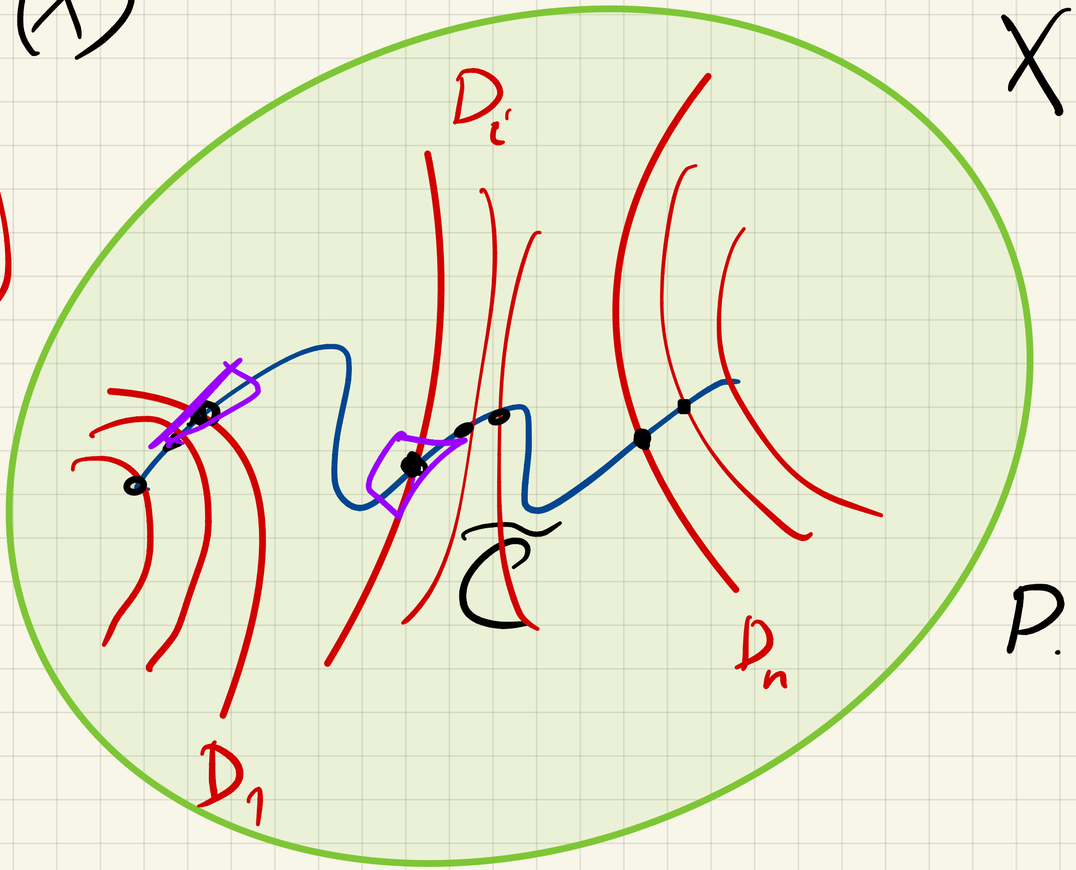
Geometry from enumeration

$\beta = [\tilde{C}] \in H_2(X)$

$\#\big(\tilde{C},$ passing through $D_1, \dots, D_n\big)$

$= N_{\beta, g}(\omega_1, \dots, \omega_n)$

$g = \text{genus}(C)$



$P.D.[D_i] \in H^*(X)$

$\cup$

ω_i

$\phi : (C, x_1, \dots, x_n) \longrightarrow X$

$\phi(x_i) \in D_i$

$\phi(C) = \tilde{C} \subset X$

$+$ tangency conditions => gravitational descendents

$\tau_k(\omega_i)$

$$\parallel \quad \exp \mathcal{F}(T_i, \hbar, g)$$

$$Z(T_i, \hbar, q) = \sum_{\substack{g \geq 0 \\ \beta \in H_2(X)}} q^{\beta} \hbar^{2g-2}$$

$$q \in (\mathbb{C}^{\times})^{b_2}$$

$$\sum \prod_i \frac{T_i^{n_i}}{n_i!} N_{\beta, g} (\underbrace{\omega_1, \ldots, \omega_1}_{n_1}, \ldots, \underbrace{\omega_k \ldots \omega_k}_{n_k})$$

C : не связные

$\hbar \to 0 \qquad \mathcal{F} \simeq \hbar^{-2} \mathcal{F}_0$

$\mathcal{F}_0$ defines the dual (mirror) geometry

$$\tau_I = \int_{\Gamma_I} \Omega \qquad X^\vee \supset \Gamma_I$$

$$\frac{\partial \mathcal{F}_0}{\partial \tau_I} //$$

Special Kähler geometry

HK is related to rigid special Kähler geometry

$$\underset{(z,\bar z)}{\Sigma} \longrightarrow X = \mathbb{CP}^{N-1} = (\mathbb{C}^N \setminus 0)/\mathbb{C}^\times$$

$$= \mathbb{C}^N /\!/ U(1)$$

$$(Z^1, \dots, Z^N) \in \mathbb{C}^N$$

$$\bar\partial \phi^i = 0$$

$$\Longleftrightarrow$$

$$\boxed{\nabla_{\bar z} Z^I = 0}$$

$$I = 1, \dots, N$$

$$(Z^I) \sim (t Z^I)$$

$$\partial_{\bar z} Z^I + A_{\bar z} Z^I = 0$$

$$A_{\bar z} \sim A_{\bar z} + t^{-1} \partial_{\bar z} t$$

$$\Downarrow$$

$$\phi^i = Z^i / Z^N \Rightarrow \partial_{\bar z} \phi^i = 0$$

$$t(z,\bar z) \in \mathbb{C}^\times \supset U(1)$$

$\mathcal{A}_P \times \underline{\mathrm{Maps}(\Sigma, \mathbb{C}^N)} = \mathcal{X}$ – infinite-dimensional symplectic (Kahler) manifold

$$\omega^{(1,1)} = \int_\Sigma \sum_{I=1}^{N} \delta Z^I \wedge \delta \bar{Z}^I \, \mathrm{vol}_\Sigma + \int_\Sigma \delta A \wedge \delta A$$

$Z^I(z, \bar{z})$

P – principal U(1) bundle over Σ

Σ

$\omega^{(1,1)}\big|_S = \omega_S$

$$S \subset \mathcal{X}$$

$$= \{\nabla_{\bar{z}} Z^I = 0\}$$

holomorphic equations

$S /\!/ \mathcal{G}_{U(1)} =$ QMaps(Σ, X)

partial compactification of the space of holomorphic maps

$A_{\bar{z}}, \; Z^I(z, \bar{z})$

Fayet-Iliopoulis term

$$\mu = F_{z\bar{z}} + \mathrm{vol}_\Sigma \left(\sum_{I=1}^{N} |Z^I|^2 - \xi \right)$$

$\xi \to \xi(z,\bar{z})$?

$\mathcal{N}=(2,2)$

susy

$d=2$

$\xi = \mathrm{const}$

$$\left\{ \begin{array}{l} \mu = 0 \\ \nabla_{\bar{z}} Z^I = 0 \end{array} \right. \Big/ \text{gauge transformations}$$

S^{2N-1}

$\mathrm{vol}_\Sigma \to \infty$

$\sum_{I=1}^{N} |Z^I|^2 \approx \xi$ almost everywhere on Σ

$/U(1)$

$$A = A_z dz + A_{\bar{z}} d\bar{z}$$

$$\frac{1}{2\pi i}\int_{\Sigma} F = \beta \in \mathbb{Z}$$

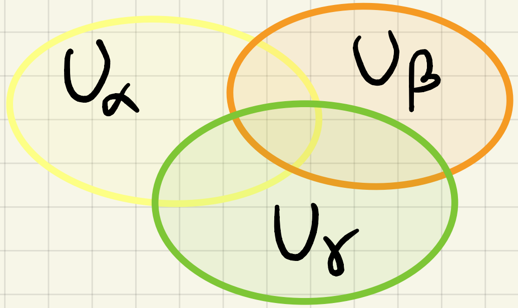
$$(n_{\alpha\beta\gamma} \in \mathbb{Z})$$

$$c_1(P) \in H^2(\Sigma, \mathbb{Z})$$

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow U(1)$$

$$g_{\alpha\beta} = e^{i\varphi_{\alpha\beta}}$$

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \in U(1)$$



$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} + \varphi_{\gamma\alpha} = 2\pi\, n_{\alpha\beta\gamma}$$

$$X = X^*$$

$$(A_{\bar z}, Z^1, \dots, Z^N) \mapsto (A_{\bar z} + \partial_{\bar z} X, e^X Z^1, \dots, e^X Z^N)$$

$$0 = F_{z\bar z} + \partial\bar\partial X + \mathrm{vol}_\Sigma \left(e^{2X} \underbrace{\sum_I |Z^I|^2}_{\neq 0} - \xi\right)$$

Liouville-type equation

somewhere

где данной $(z, \bar z)$ $(Z^I(z,\bar z))$

нет evaluation

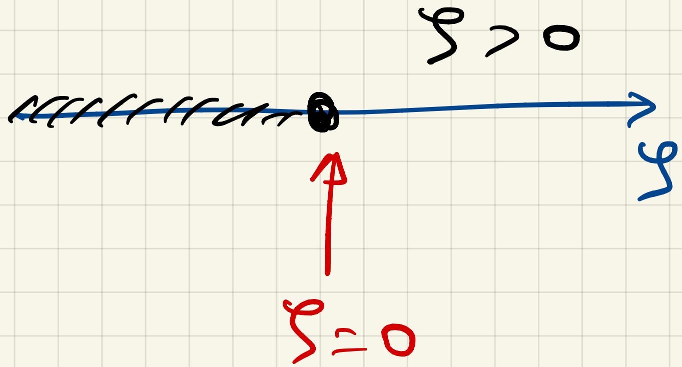
Unlike holomorphic maps

$$\Sigma \to \mathbb{CP}^{N-1}$$

$\forall (z,\bar z)$ $\exists I$, $Z^I(z,\bar z) \neq 0$ $\xrightarrow{\text{weaker condition}}$

$\mu = 0 \Rightarrow \exists (z,\bar z)$ $\exists I$ $Z^I(z,\bar z) \neq 0$

$$\mathbb{CP}^{N-1} = \mathbb{C}^N /\!\!/ U(1)$$



$$\sum_I |Z^I|^2 - \xi = 0$$

$$\frac{1}{2\pi i} F + \mathrm{vol}_\Sigma \left(\cancel{\sum |Z^I|^2} - \xi \right) = 0$$

$$\int_\Sigma \xi \, \mathrm{vol}_\Sigma = \beta$$

$$\mathrm{QMaps}^{\text{framed}}_\beta(\mathbb{P}^1, \mathbb{P}^{N-1}) = \mathbb{P}^{N(\beta+1)-1}$$

$$\mathbb{P}^1 \cong \Sigma$$

$$= \mathbb{P}^{N\beta + \underline{N-1}}$$

$$N\beta \leftarrow \int_\beta c_1(\mathbb{P}^{N-1}) \qquad N-1 \leftarrow \text{dim } \mathbb{P}^{N-1}$$

$$\nabla_{\bar z} Z^I = 0$$

$$I = 1, \dots, N$$

$$= \{(U^I_i)\}/\mathbb{C}^\times$$

$$H^0(\mathcal{O}(\beta)) \cong \mathbb{C}^{\beta+1} =$$

$$Z^I \in H^0(\mathcal{O}(\beta))$$

homogeneous polynomials in (ξ_0, ξ_1)

$$\sum_{i=0}^{\beta} U^I_i \xi_0^i \xi_1^{\beta-i}$$

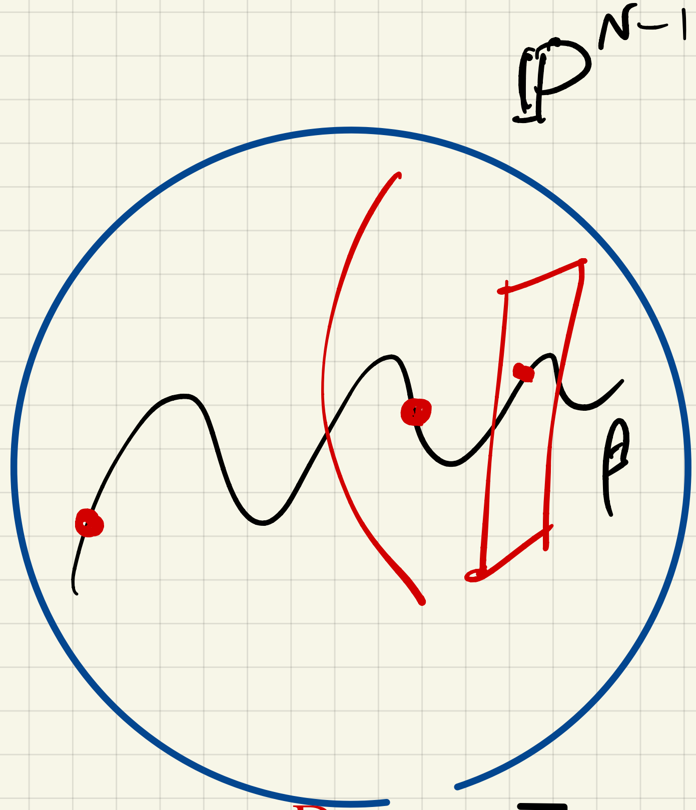
equivariant volume

symplectic form ω fron reduction

$\mu = F + \mathrm{vol}_{\Sigma} \left(\sum_{I} |Z^{I}|^{2} - \zeta \right)$

equivariant

$$\mathcal{N}_{\beta,0} = \int_{QMaps_{\beta}(\mathbb{P}^{1},\, \mathbb{P}^{N-1})} e^{\omega}$$



$$\left[\begin{array}{l} \text{multiplicities of } D_i, \\ \text{multiplicities of tangencies} \end{array} \right]$$

$$\Updownarrow$$

symmetries with respect to

$$\mathbb{P}^{1} \curvearrowleft \mathrm{PGL}(2) \times \mathrm{PGL}(N) \curvearrowright \mathbb{P}^{N-1}$$

$$A_{\bar z} = \partial_{\bar z} \chi$$

$\chi(z,\bar z)$

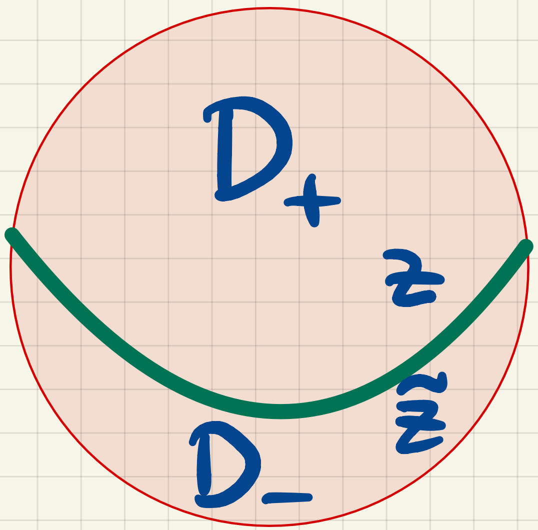
$$Z^I(z,\bar z) = e^{\chi(z,\bar z)} \underbrace{\sum_{i=0}^{\beta} U_i^I z^i}_{P_I(z)}$$

$z \in D_+$

$\tilde z \in D_-$

$z \tilde z = 1$

$$= e^{\tilde\chi(\tilde z,\bar{\tilde z})} \sum_{i=0}^{\beta} U_i^I \tilde z^{-i}$$



$P_1, \dots, P_N$

polynomials

may have common roots

$z_1 \dots z_K$

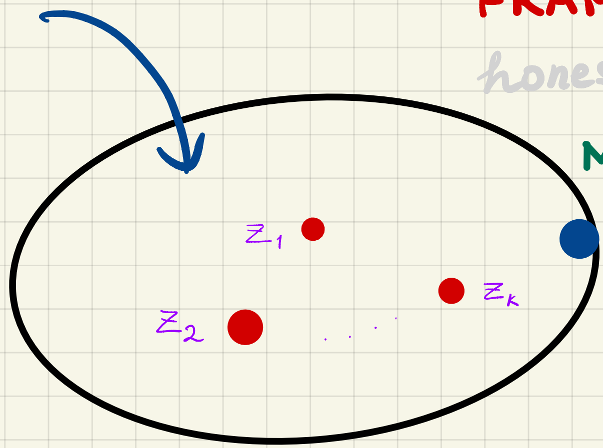
$$\mathrm{QMaps}^{\text{framed}}_\beta\left((\mathbb{P}^1_\infty),\ \mathbb{P}^{N-1}\right) \subset \mathrm{QMaps}_\beta(\mathbb{P}^1, \mathbb{P}^{N-1})$$

$$\mathrm{QMaps}^{\text{framed}}_\beta\left((\mathbb{P}^1_\infty),\ \mathbb{P}^{N-1}\right) \xrightarrow{ev} \mathbb{P}^{N-1}$$

evaluation map

$$\left(U^I_\beta\right)^N_{I=1} \neq 0$$

point-like instantons at $z_1, \dots, z_k$



FRAME:

honest

MAP

∞

$$\int^{\text{equiv}}_{\mathrm{QMaps}^{\text{framed}}_\beta} e^{\omega + \sum_i T_i\, ev^*(\omega_i)}$$

$$\omega_i \in H^*(\mathbb{P}^{N-1})$$