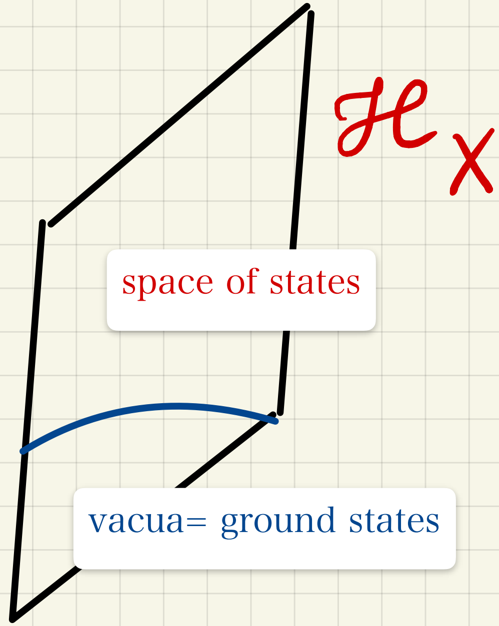
Moduli spaces of vacua, hyperkahler manifolds, Higgs bundles

Supersymmetric theories in dimensions 3,4,5,6 $\ddot{D}$
8 supercharges

QFT on $\mathbb{R}^{1,d-1|8} \times T^{D-d}$



We are interested in the moduli spaces of vacua of the theory

$$X = \mathbb{R}^{d-1|8} \times T^{D-d}$$

r=rk(G)

d	$\dim \mathcal{M}_V$
3	hyperkahler
4	special kahler
5	tropical kahler
6	empty

$$\mathcal{M}_V \times \mathcal{M}_H$$

гиперкелерово

связано с гиперкелеровой геометрией

$$\mathcal{M}_V(\mathbb{R}^3 \times S^1_R)$$

g

hyperkahler, metrics depends on R, gauge couplings, θ -angles, masses,
holonomies of flavor and topological symmetries

$$(a\,I + b\,J + c\,K)^2 = -1$$

$a^2+b^2+c^2=1 \;:\; S^2$

twistor sphere

three (two-sphere) covariantly constant complex structures

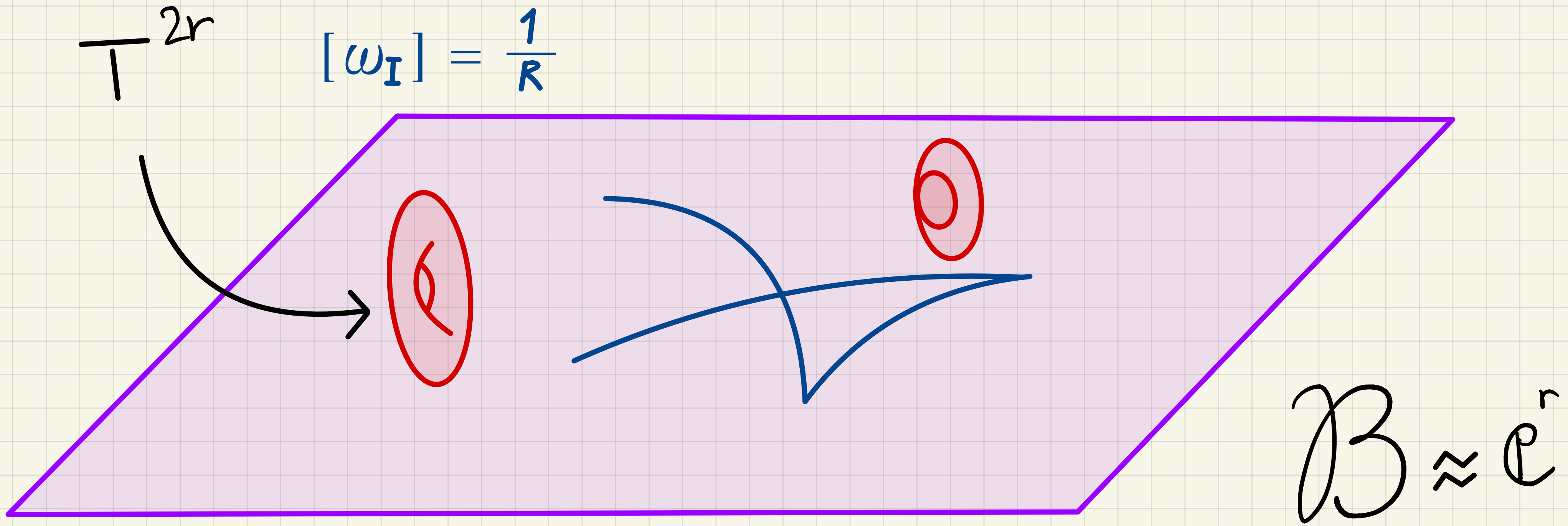
$$\nabla I = \nabla J = \nabla K$$

$$\omega_I(\,.\,,\,.\,) = g(\,I\,.\,,\,.\,)$$
$$\omega_J(\,.\,,\,.\,) = g(\,J\,.\,,\,.\,)$$
$$\omega_K(\,.\,,\,.\,) = g(\,K\,.\,,\,.\,)$$

Decompactification $R \longrightarrow \infty$: metric collapse

from hyperkähler $\mathcal{M}_V(\mathbb{R}^3 \times S^1_R)$

to special kähler base $\rightarrow \mathcal{B}$



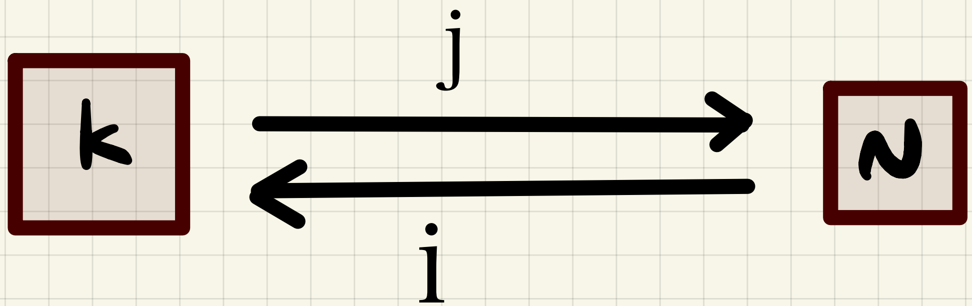
Two sources of hyperkahler manifolds

1) direct, as moduli spaces of solutions to some equations/PDE

2) emergent, as generating functions of instanton/monopole counting

Simplest hyperkähler manifolds

$$\mathbb{R}^{4kN}$$



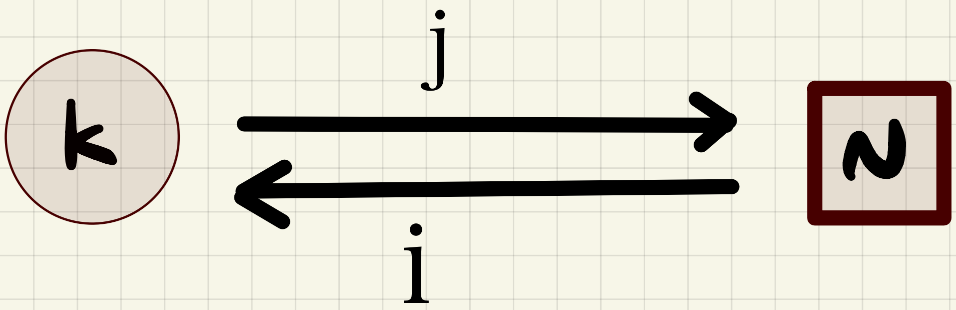
I-holomorphic functions of $\quad i\,,\quad j$

J- holomorphic functions of $\quad i + \sqrt{-1}\, j^{\dagger}, \quad i^{\dagger} + \sqrt{-1}\, j$

K-holomorphic functions of $\quad i + \ j^{\dagger}, \quad i^{\dagger} - \ j$

Simplest hyperkähler quotients

$$\mathbb{R}^{4kN} /\!\!/ U(k)$$



equations: $\vec{\mu} = 0$ modulo U(k)

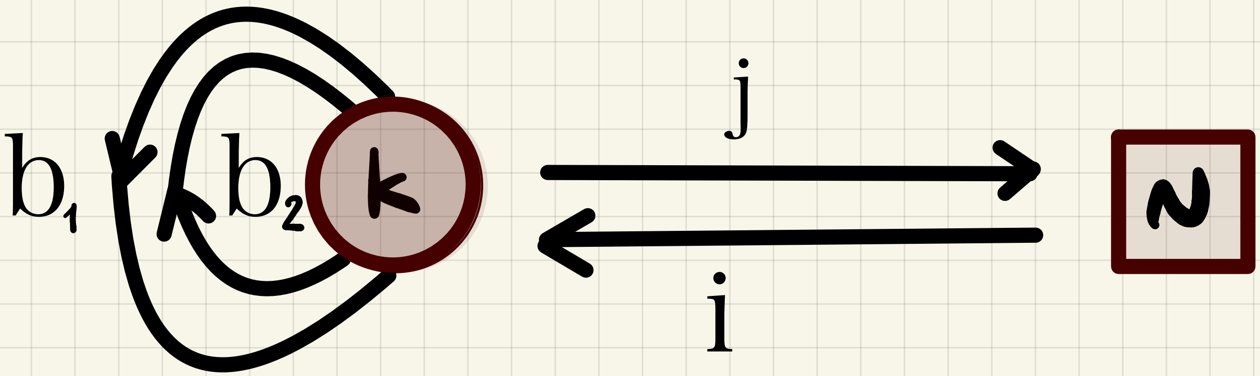
$$\mu_I = i\, i^\dagger - j^\dagger j - \zeta_{\mathbb{R}}\, 1$$

$$\mu_J + i\, \mu_K = ij - \zeta_{\mathbb{C}}\, 1$$

Simplest hyperkähler quotients

$$\mathcal{M}_k(N) = \mathbb{R}^{4k(N+k)} /\!\!/ U(k)$$

ADHM



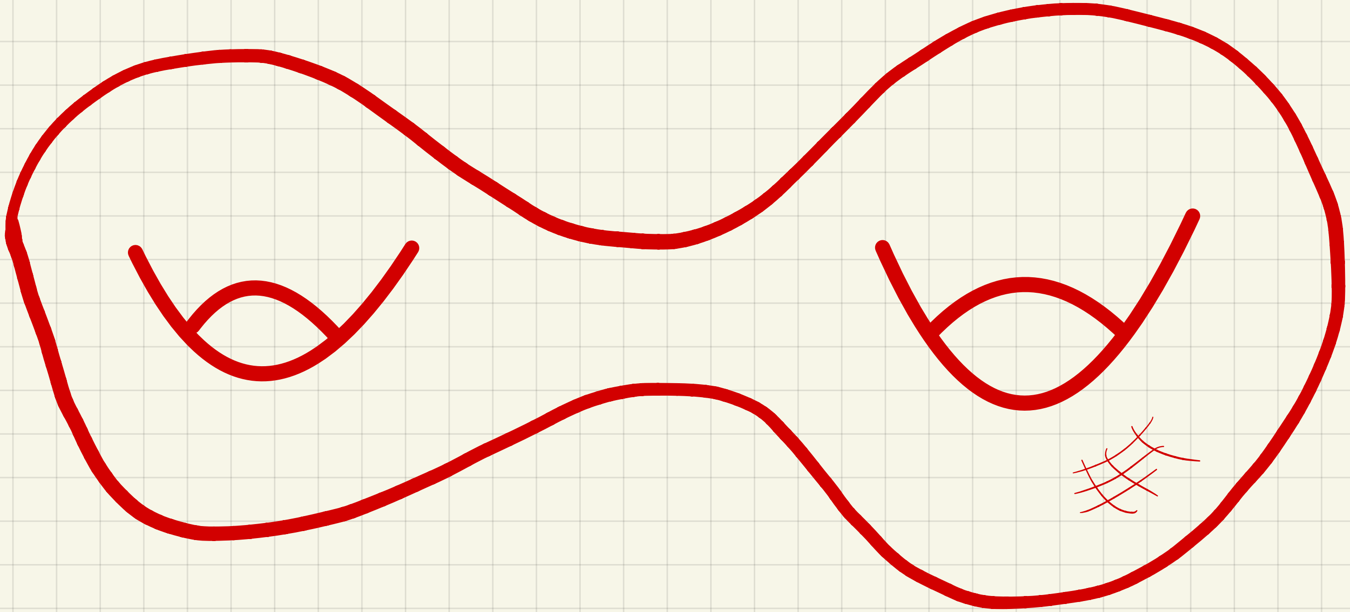
equations: $\vec{\mu} = 0$ modulo U(k)

$$\mu_I = i\, i^\dagger - j^\dagger j + [b_1, b_1^\dagger] + [b_2, b_2^\dagger] - \zeta_{\mathbb{R}}\, 1$$

$$\mu_J + i\, \mu_K = ij + [b_1, b_2] - \zeta_{\mathbb{C}}\, 1$$

Infinite dimensional hyperkahler quotients: solutions to PDE

We fix a Riemann surface Σ and gauge group G



Space of stable Higgs pairs

$(A\ ,\ \phi)$

A-connection on a principal G-bundle P

$\phi \in \Omega^1(\Sigma) \otimes \mathrm{ad}\, P$

$$\Omega_I = \omega_J + i\,\omega_K = \int_\Sigma \mathrm{Tr}\left(\delta\,\phi_z \wedge \delta A_{\bar z}\right) d^2 z$$

(2,0)-form in complex structure I

$$\omega_I = \int_\Sigma \mathrm{Tr}\left(\delta A_z \wedge \delta A_{\bar z} - \delta\,\phi_z \wedge \delta\,\phi_{\bar z}\right) d^2 z$$

symplectic structure I

$\mathcal{G}$ сохраняет $\omega_I, \omega_J, \omega_K$

$$\mu_I = F_{\bar{z}\bar{z}} + [\phi_{\bar{z}}, \phi_{\bar{z}}]$$

$$\mu_J + i\mu_K = D_{\bar{z}} \phi_{\bar{z}} \qquad \vec{\mu}^{-1}(0)/\mathcal{G} = \mathcal{M}$$

$$\dim \mathcal{M} = 4 \dim(G)\,(g-1)$$

$$\varepsilon \in \mathrm{LieStab}(A, \phi)$$

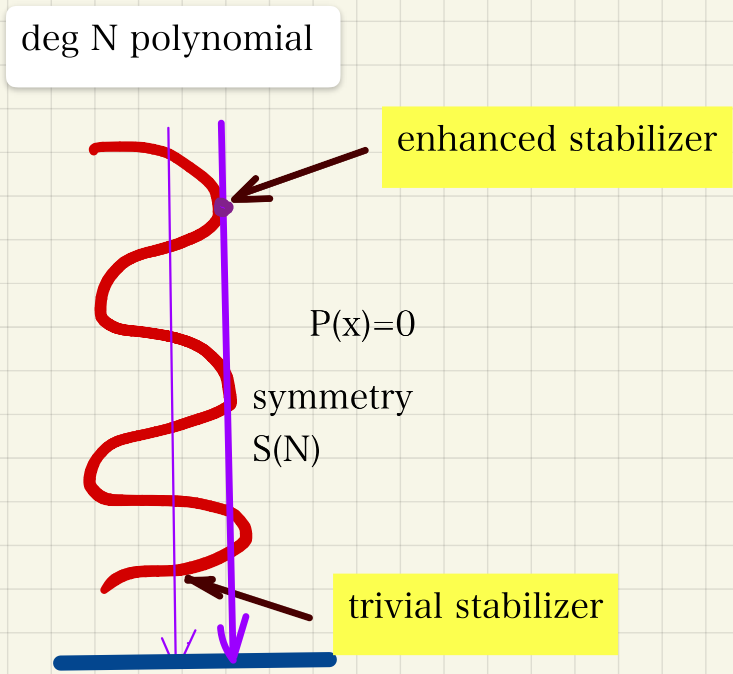
$$\partial_m \varepsilon + [A_m, \varepsilon] = 0$$

$$[\Phi_m, \varepsilon] = 0$$

если $g=1$, то ε = абелево

$A_{\bar z}$, Φ_z

в общем положении



если $g \geqslant 2$, то $\varepsilon = 0$

в общем положении

Stability vs real moment map

$$\mathbb{C}^N /\!/ U(1)$$

$$(z_1, z_2, \dots, z_N) \longmapsto (e^{i\vartheta} z_1, e^{i\vartheta} z_2, \dots, e^{i\vartheta} z_N)$$

$$\mu = \sum_{i=1}^{N} |Z_i|^2 - \zeta$$

$\zeta > 0$

$\vec{z} \neq 0 \Longleftrightarrow \exists t,\ t^* \mu = 0$

complexified group action $\vec{z} \mapsto t\vec{z},\ t \in \mathbb{C}^\times$

$$(\mathbb{C}^N \setminus 0)/\mathbb{C}^\times$$

$$\parallel$$

stable

$$(\mathbb{C}^N)/\mathbb{C}^\times = \mu^{-1}(0)/U(1) = \mathbb{CP}^{N-1}$$

Решаем комплексное уравнение моментов.
($G_{\mathbb{C}}$- invariant)
выбрасываем нестабильные решения
делим по комплексной группе

$$D_{\bar{z}} \phi_z = 0 \Rightarrow \partial_{\bar{z}} P_i(\phi_z) = 0$$

$$P_i \in \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}} = \mathbb{C}[\mathfrak{t}_{\mathbb{C}}]^W$$

P_i - homogeneous G-invariant polynomial on $\mathfrak{g}$ =LieG

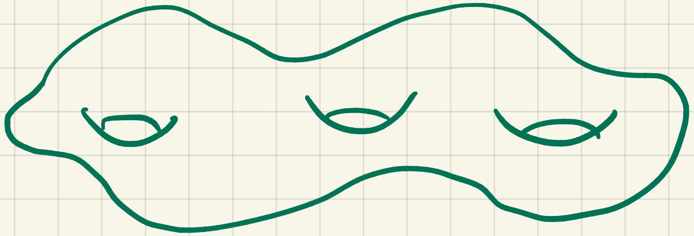
$\sum_{i=1}^{r} (2d_i+1) = \dim(G)$

$i = 1, \ldots, r$

$$P_i(\phi_z) = u_i \in H^0(\Sigma, K_\Sigma^{\otimes d_i})$$

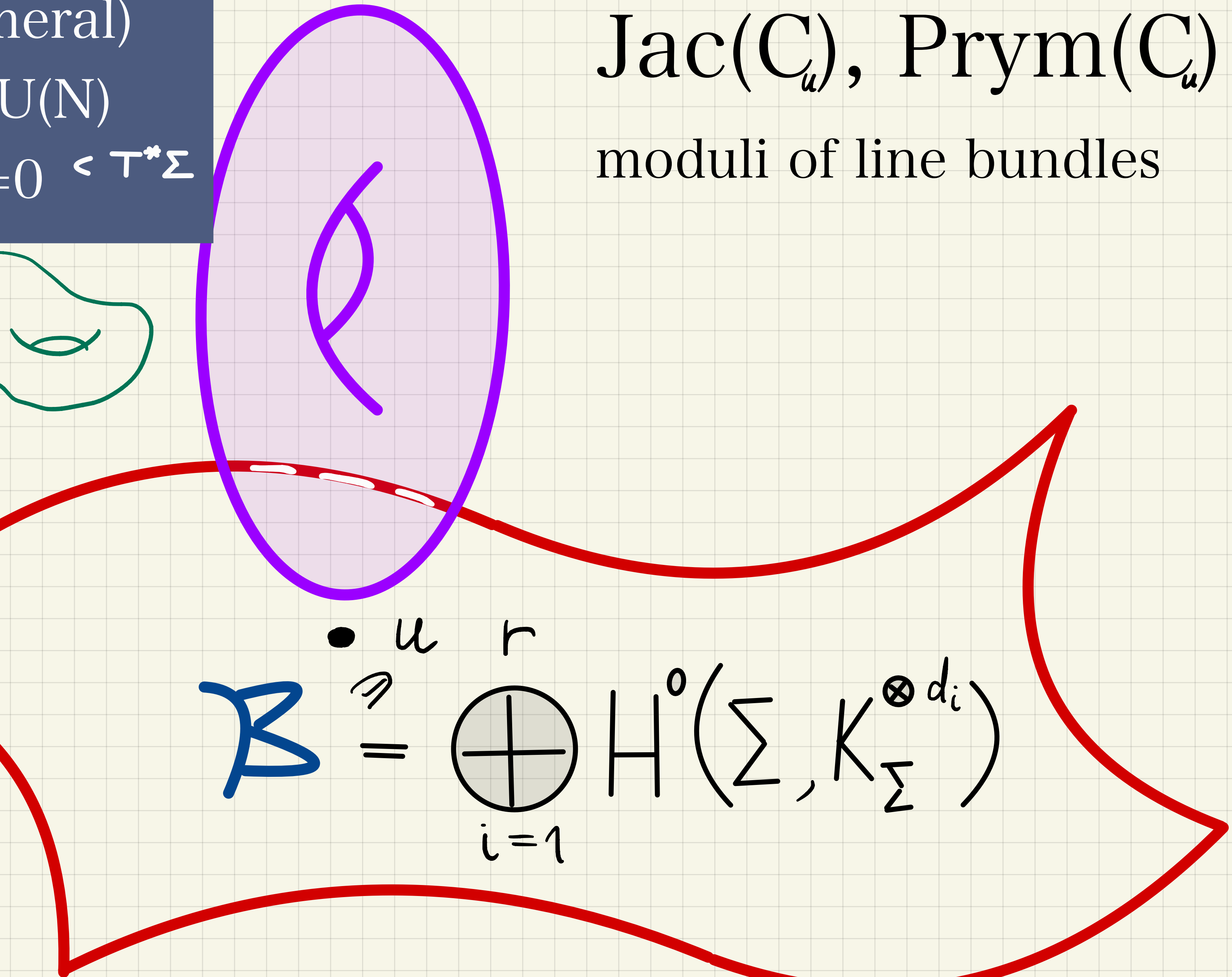
C_u spectral (cameral) curve. For G=SU(N)

$\mathrm{Det}(\lambda \cdot 1 - \Phi_z)=0 \subset T^*\Sigma$

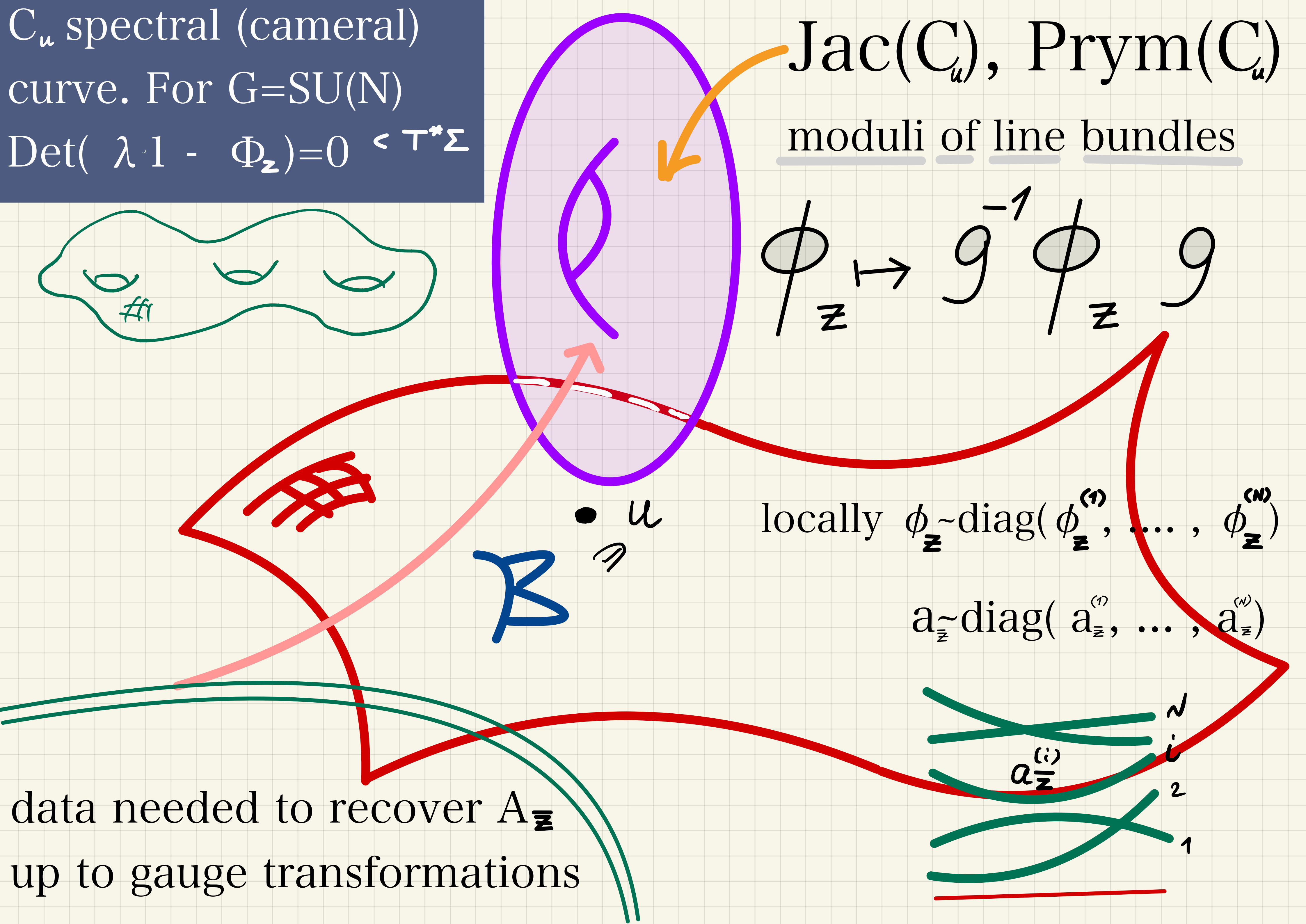


Jac(C_u), Prym(C_u)

moduli of line bundles



$$\mathcal{B} \cong \bigoplus_{i=1}^{r} H^0(\Sigma, K_\Sigma^{\otimes d_i})$$



C_u spectral (cameral) curve. For G=SU(N)
$\mathrm{Det}(\lambda 1 - \Phi_z)=0 \subset T^*\Sigma$
$\mathrm{Jac}(C_u)$, $\mathrm{Prym}(C_u)$
moduli of line bundles
$\phi_z \mapsto g^{-1} \phi_z g$
u
B
locally $\phi_z \sim \mathrm{diag}(\phi_z^{(1)}, \ldots, \phi_z^{(N)})$
$a_{\bar z} \sim \mathrm{diag}(a_{\bar z}^{(1)}, \ldots, a_{\bar z}^{(N)})$
$a_{\bar z}^{(i)}$
N
i
2
1
data needed to recover $A_{\bar z}$
up to gauge transformations

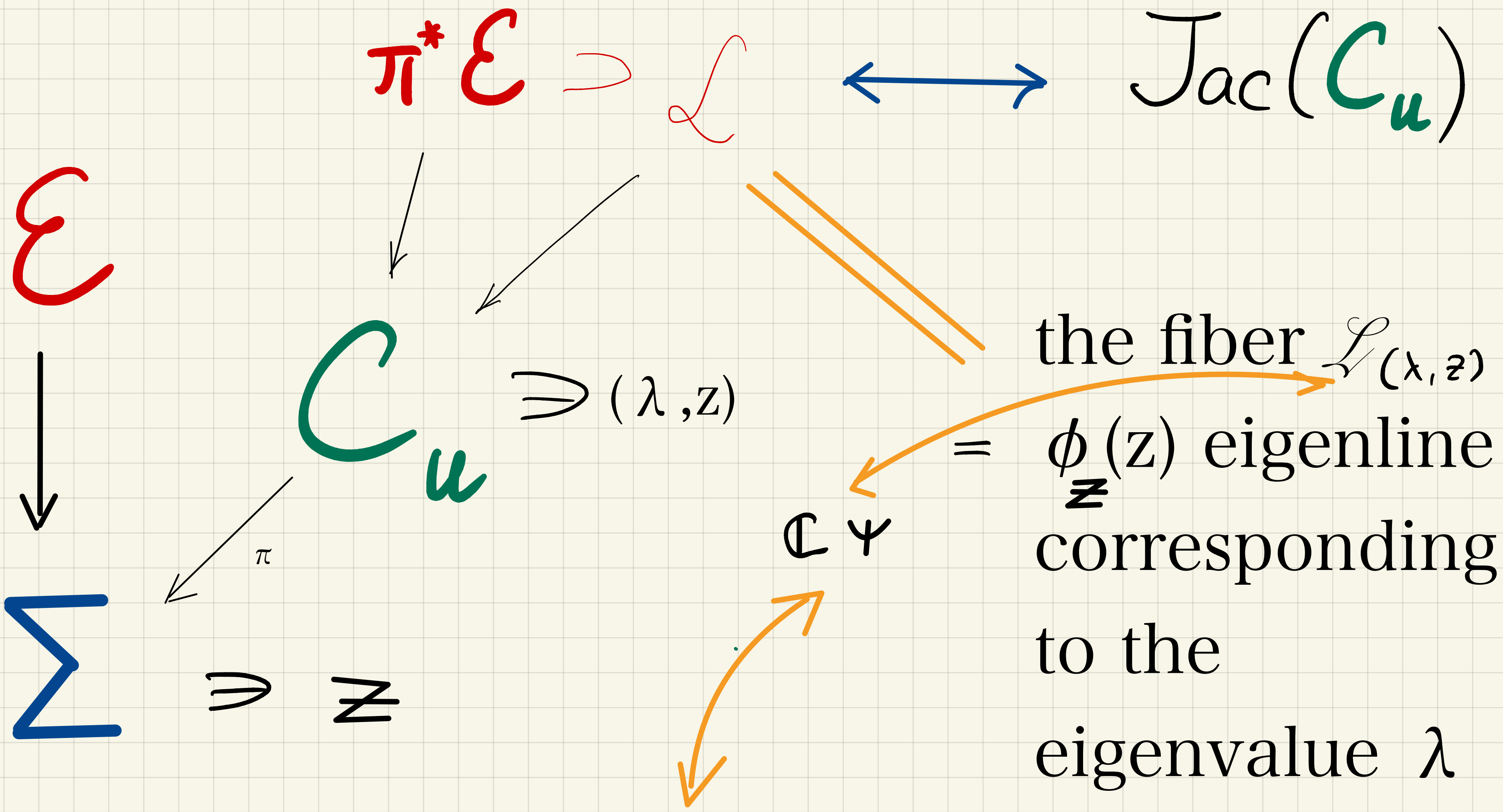
$$(A_z , A_{\bar z}) \longmapsto \left(g^\dagger A_z (g^\dagger)^{-1} + g^\dagger \partial_z (g^\dagger)^{-1} ,\ g^{-1} A_{\bar z} g + g^{-1} \partial_{\bar z} g \right)$$

$$F_{z\bar z} \longrightarrow g^{-1} \left(\partial_{\bar z} (h^{-1} \partial_z h) + \dots \right) g$$

$$h = g g^\dagger$$

$$\mu_{\mathbb{R}} \sim F_{z \bar z} - [\phi_z , \phi_{\bar z}]$$

$= 0$ gauge condition for $G_{\mathbb{C}} / G$



$$\Phi_z(z)\ \psi = \lambda\ \psi$$

Baker-Akhiezer function

$$\mathcal{E}_z = \bigoplus_{i=1}^{N} \mathcal{L}_{(\lambda_i, z)}$$

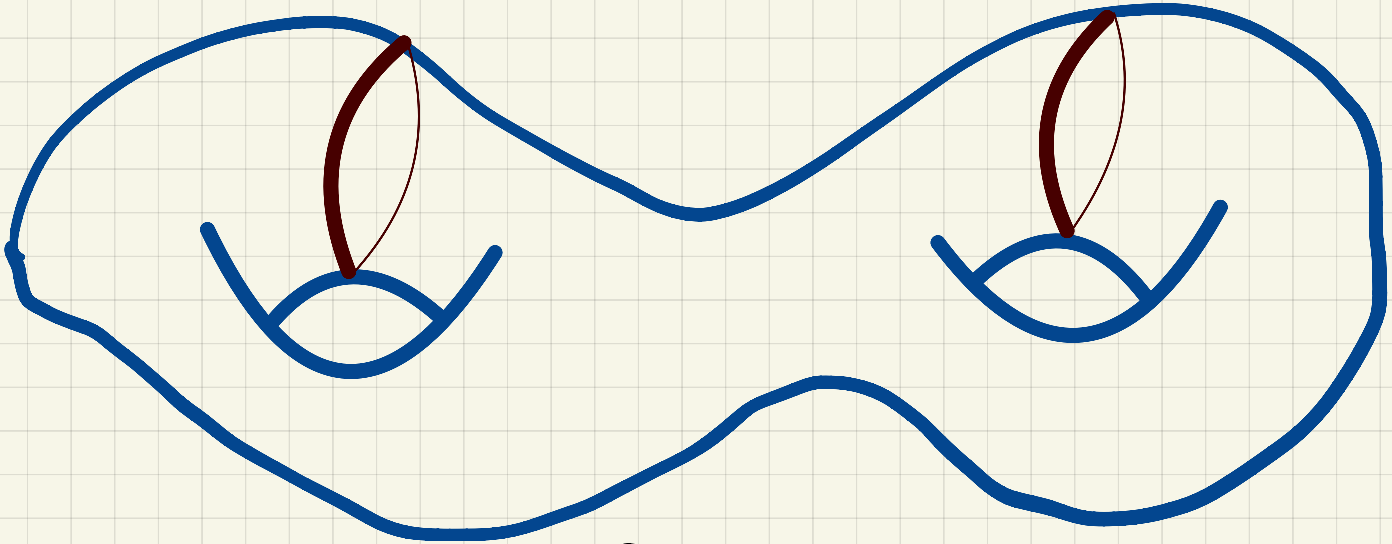
complex structure J picture

$$\mathcal{F}_{\mathcal{A}} = 0 = \mu_J$$

$$\mathcal{A} = A + i\,\Phi$$

$$\Omega_J = \int_\Sigma \delta\mathcal{A} \wedge \delta\mathcal{A}$$

$$W_R(C) = \mathrm{Tr}_R\, P\exp \oint_C \mathcal{A}$$



$$C_1 \cap C_2 = \emptyset$$

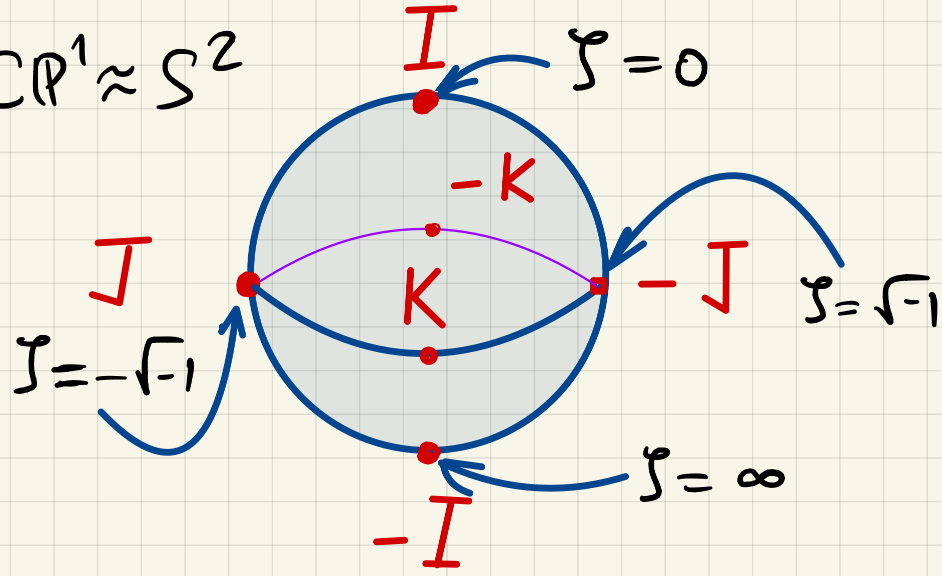
$$\Downarrow$$

$$\mathcal{O} = \{ W_{R_1}(C_1), W_{R_2}(C_2) \}^J$$

предел WKB

$$A_z + \zeta^{-1}\phi_z$$
$$A_{\bar z} - \zeta\,\phi_{\bar z}$$

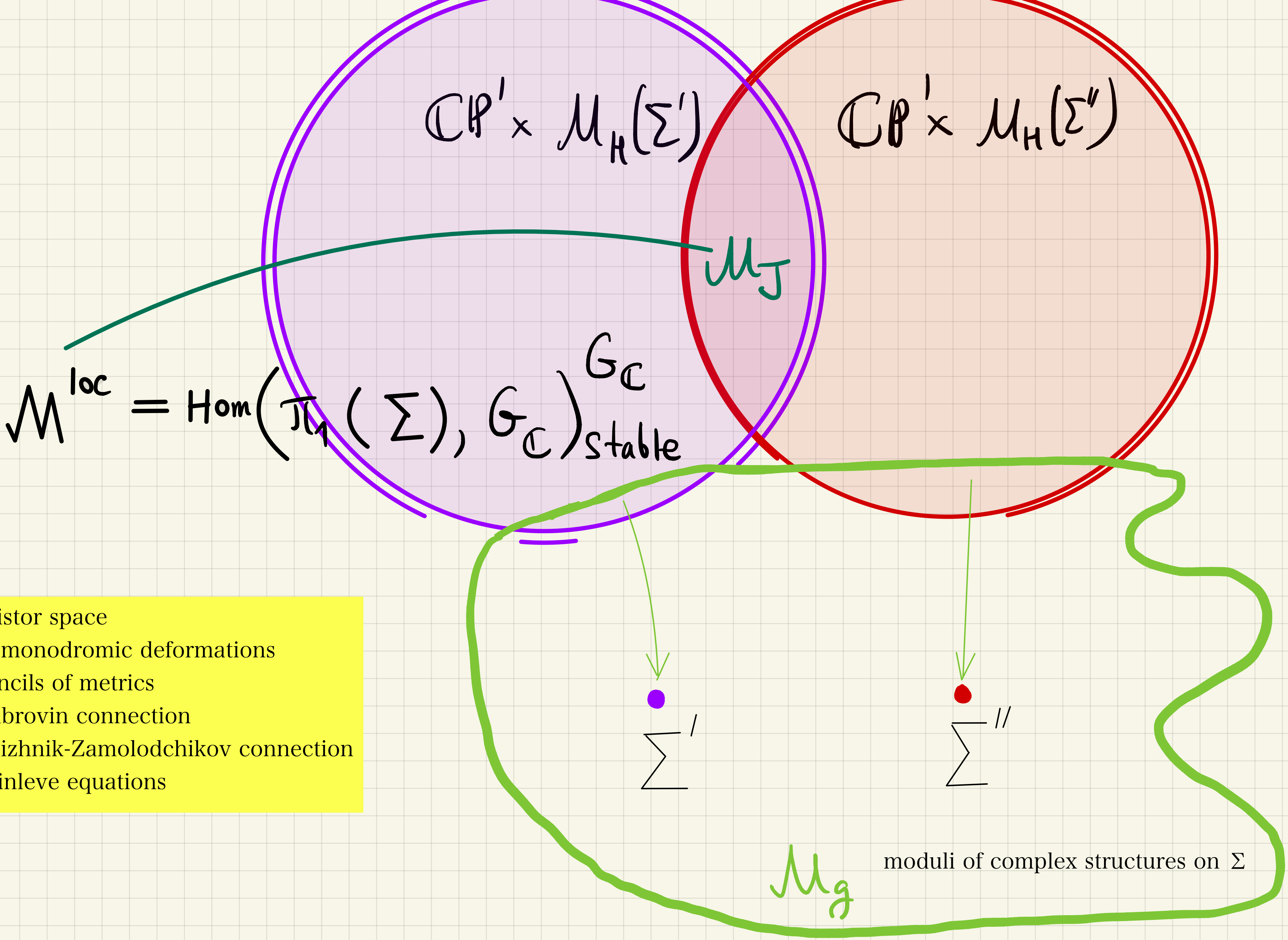
$$\zeta \in \mathbb{CP}^1 \approx S^2$$



$$\zeta \approx \hbar \to 0$$

invariant J_ζ- holomorphic functions

$$\mathrm{Tr}_R\, \mathrm{Pexp} \oint_C \left(A_z + \zeta^{-1}\phi_z\right)dz + \left(A_{\bar z} - \zeta\,\phi_{\bar z}\right)d\bar z$$



twistor space
isomonodromic deformations
pencils of metrics
Dubrovin connection
Knizhnik-Zamolodchikov connection
Painleve equations

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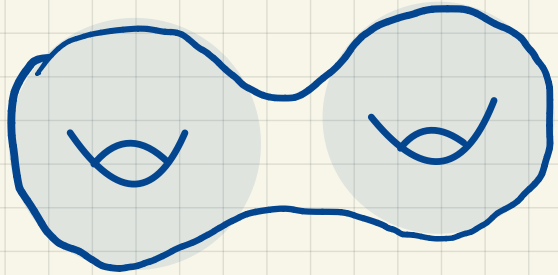
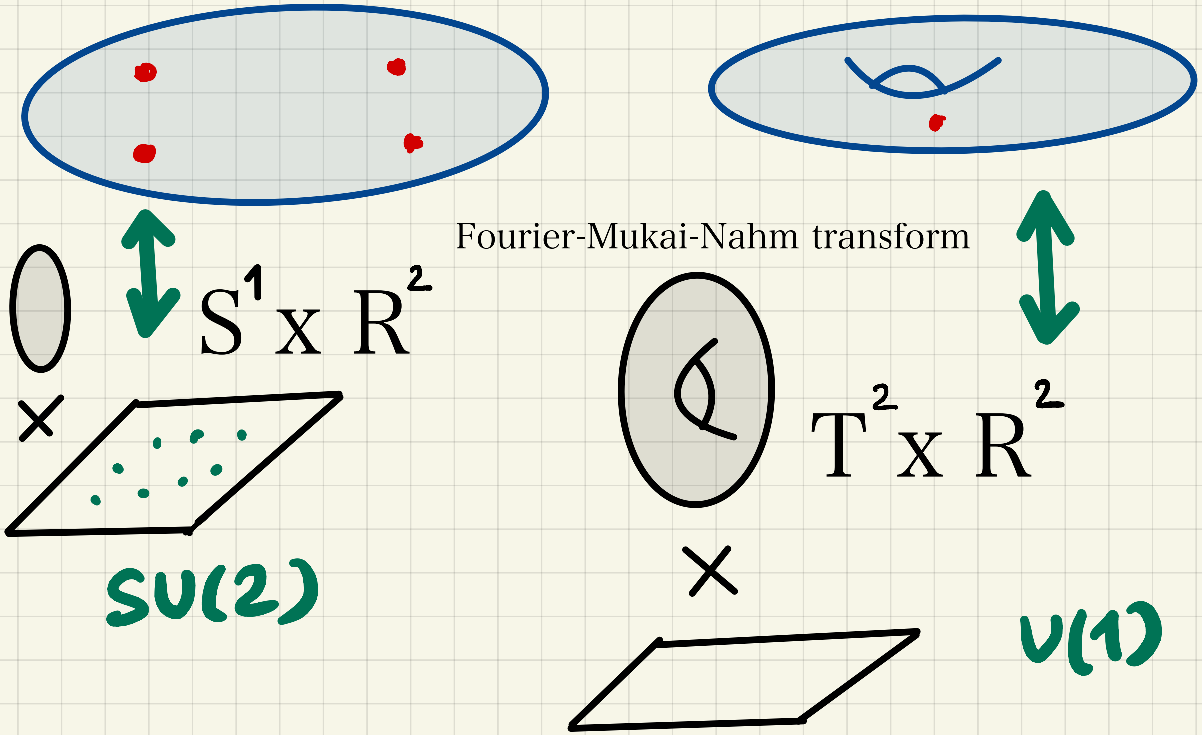
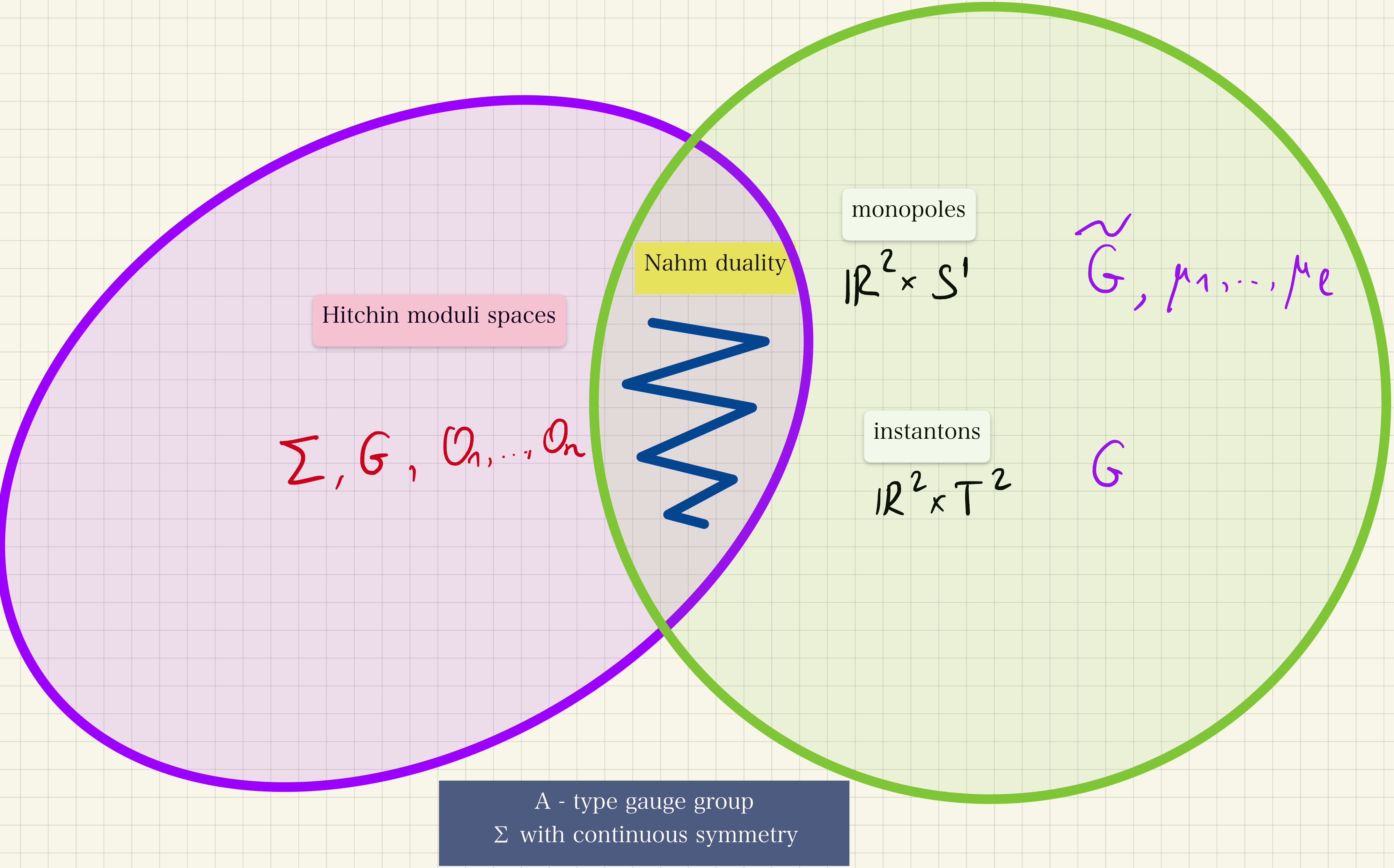
SU(N) g=0, n=4; g=1, n=1

G=SU(2), g=2






Hitchin moduli spaces
Σ, G, 𝒪1, …, 𝒪n
Nahm duality
monopoles
ℝ2 × S1
G̃, μ1, …, μl
instantons
ℝ2 × T2
G
A - type gauge group
Σ with continuous symmetry