lecture #14

Two dimensional Yang-Mills in canonical formalism and many-body systems

Start with two dimensional Yang-Mills in canonical formulation

$G = U(N)$

$$\frac{1}{4g^2}\int \mathrm{Tr}\, F_{\mu\nu}^2 \longrightarrow \int \mathrm{Tr}\, E F_{01} - \frac{g^2}{2}\int \mathrm{Tr}\, E^2 \quad d_x^2$$

& $d>2$

$+ \mathrm{Tr} B_{ij}^2$

$\mathrm{Tr} F_{ij}^2$

$\Sigma = S^1 \times \mathbb{R}^1$

$$// \quad \partial_x A_t - \partial_t A_x + [A_t, A_x]$$

$$\int \mathrm{Tr}\, E\, \partial_t A_x + \mathrm{Tr}\, A_t D_x E - \frac{g^2}{2}\mathrm{Tr}\, E^2 \, dx\, dt$$

$$\int p\, \partial_t q + \langle \mu(p,q), A_t \rangle - H(p,q) \quad dt$$

phase space

Gauss law moment map

hamiltonian

$$\omega = \int dx\, \mathrm{Tr}\, \delta E \wedge \delta A_x$$

$$\mu(E, A_x) = D_x E = \partial_x E + [A_x, E]$$

extension

$$H = \frac{g^2}{2}\oint dx\, \mathrm{Tr}\, E^2$$

generates gauge transformations

$(E, A_x) \longrightarrow (g^{-1}Eg,\ g^{-1}A_x g + g^{-1}\partial_x g)$

$$\mathcal{P} = \mathfrak{g} \times \mathfrak{g}^*$$

Two realizations of two dimensional Yang-Mills

$\mathcal{P} = \mathfrak{g} \times \mathfrak{g}^*$

1) $\mathfrak{g} = \widetilde{L\mathfrak{g}} = \left\{ k\partial + A \;\middle|\; A \in C^\infty(S^1)\otimes \mathfrak{g},\ k \in \mathbb{R} \right\} \ni Q$

finite dim semi simple Lie algebra

$\mathrm{Lie}(LG \rtimes S^1)$ //

$[k_1\partial + A_1, k_2\partial + A_2] = \to 0\cdot\partial + k_1\partial A_2 - k_2\partial A_1 + [A_1, A_2]$

$\mathfrak{g}^* = \left\{ (c, \phi) \;\middle|\; \phi \in C^\infty(S^1)\otimes \mathfrak{g},\ c \in \mathbb{R} \right\} \ni P$

$\langle (c,\phi), k\partial + A \rangle = k\cdot c + \oint \mathrm{Tr}\, \phi A$

$\langle \mathrm{ad}^*_{k\partial+A}(c,\phi), k'\partial + A' \rangle = -\oint \mathrm{Tr}\,\phi\left(k\partial A' - k'\partial A + [A, A']\right) = c'k' + \oint \mathrm{Tr}\,\phi' A'$

$\underset{\substack{\shortparallel \\ (c', \phi')}}{\mathrm{ad}^*_{k\partial+A}(c,\phi)} = \left(\oint \mathrm{Tr}\,\phi\,\partial A,\ k\partial\phi + [A, \phi] \right)$

$H = \frac{1}{2}\langle P, P \rangle := \frac{1}{2}\oint \mathrm{Tr}\,\phi^2\, dz$

Two realizations of two dimensional Yang-Mills

$$\mathcal{P} = \mathfrak{g} \times \mathfrak{g}^*$$

2) $$\mathfrak{g} = \widehat{L\mathfrak{g}} = \{ (c, \phi) \mid \phi \in C^\infty(S^1) \otimes \mathfrak{g}, \ c \in \mathbb{R} \} \ni Q$$

$$\mathrm{Lie}\left(\begin{array}{c} S^1 \to \widehat{LG} \\ \downarrow \\ LG \end{array} \right) \qquad \big[(c_1, \phi_1), (c_2, \phi_2) \big] = \left(\oint \mathrm{Tr}\, \phi_1 d\phi_2 , \ [\phi_1, \phi_2] \right)$$

$$\mathfrak{g}^* = \{ \underline{k\partial + A} \mid A \in C^\infty(S^1) \otimes \mathfrak{g}, \ k \in \mathbb{R} \} \ni P$$

$$\big\langle (c, \phi), k\partial + A \big\rangle = k \cdot c + \oint \mathrm{Tr}\, \phi A$$

$$\big\langle (c', \phi'), \mathrm{ad}^*_{c,\phi}(k\partial + A) \big\rangle = -k \oint \mathrm{Tr}\, \phi\, d\phi' - \oint \mathrm{Tr}\, A [\phi, \phi'] = k' c' + \oint \mathrm{Tr}\, A' \phi'$$

$$\mathrm{ad}^*_{(c,\phi)}(k\partial + A) = k\partial\phi + [\phi, A]$$

$$H = \frac{1}{2} \langle Q, Q \rangle := \frac{\oint \mathrm{Tr}\, \phi^2 dx}{2}$$

Side remark: dual Hamiltonians, dual systems (later in the course)

1) $\tilde{H} = \mathrm{Casimir}(Q) \sim \mathrm{Tr}_R\, P\exp \oint \frac{A}{k}$

2) $\tilde{H} = \mathrm{Casimir}(P) \sim$

$k \neq 0$
rational
Ruijsenaars
model

Another side remark: dynamics of K Kaluza-Klein gravity

KK ansätze : $ds^2_{d+1} = \lambda(x)\left(d\tau - a_m dx^m\right)^2 + h_{mn}(x)dx^m dx^n$

$\parallel$

$\lambda\, d\tau^2 - 2A_m dx^m d\tau + g_{mn}dx^m dx^n$

h_{mn} ← d dim

Diffs(d) x gauge transformations
of the fiber bundle

$U(1) \to Y^{d+1}$
$\downarrow$
X^d

$$\begin{pmatrix} \tau \\ A_m \\ g_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} \tau + f(x) \\ A_m + \lambda\partial_m f \\ g_{mn} + \dots \end{pmatrix} \rightsquigarrow e^{-f}\boxed{\lambda\partial_m + A_m}\, e^{f}$$

λ -connection

с этого момента k=1, E=Φ

Как разр~~у~~eшать связи

$$D_x E = 0$$

Floquet-Lyapunov theory

1. electric калибровка:

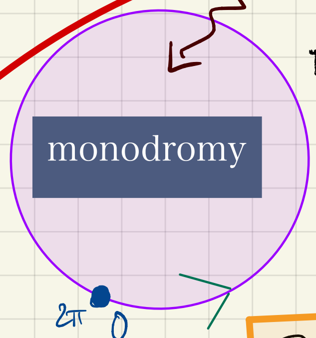
$$\boxed{E(x) = \mathrm{diag}(e_1(x), \dots\dots\dots, e_N(x))}$$

$$e_i' = 0, \quad (e_i - e_j)\, A_{ij}(x) = 0$$

$$E_{ij}(x) = e^{-x(a_i - a_j)}\, E_{ij}(0)$$

2. magnetic калибровка

$$\boxed{A(x) = \mathrm{diag}(a_1, \dots\dots, a_N)}$$



$$\mathrm{Det}\left(\underbrace{P\exp\oint_0^{2\pi} A}_{= g^{-1}} - e^{2\pi i a_\ell}\right) = 0$$

$$\ell = 1, \dots, N$$

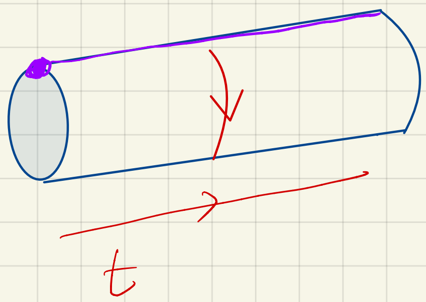
$$\boxed{\partial_x \Psi + A_x \Psi = 0}$$

$$\Psi(2\pi) = g\, \Psi(0)$$

Residual gauge transformations

$$a_\ell \longmapsto a_{\sigma(\ell)} + n_\ell$$

free up to permutations



$$E = \mathrm{diag}(e_1,\dots,e_N), \quad H_2 = \frac{1}{2}\sum e_i^2$$

$$(\sigma,\vec{n}) \in S(N) \ltimes \mathbb{Z}^N$$

$$[P\exp \oint A]$$

The reduced phase space

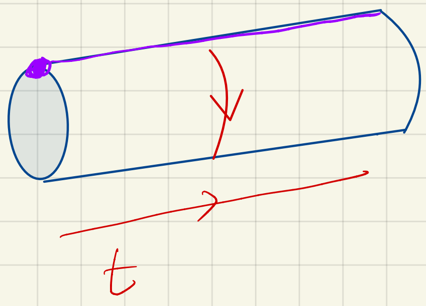
$$\omega = \sum_{i=1}^{N} de_i \wedge da_i$$

$$(T^*T)/W = \frac{\mathbb{R}^N \times U(1)^N}{S(N)} \longrightarrow T/W$$

Residual gauge transformations imply statistics, sometimes

$$a_\ell \longmapsto a_{\sigma(\ell)} + n_\ell$$

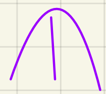
free up to permutations



$$E = \mathrm{diag}(e_1,\dots,e_N), \quad H_2 = \frac{1}{2}\sum e_i^2$$

$$(\sigma, \vec{n}) \in S(N) \ltimes \mathbb{Z}^N$$

$$\left[P\exp \oint A \right]$$



The reduced phase space

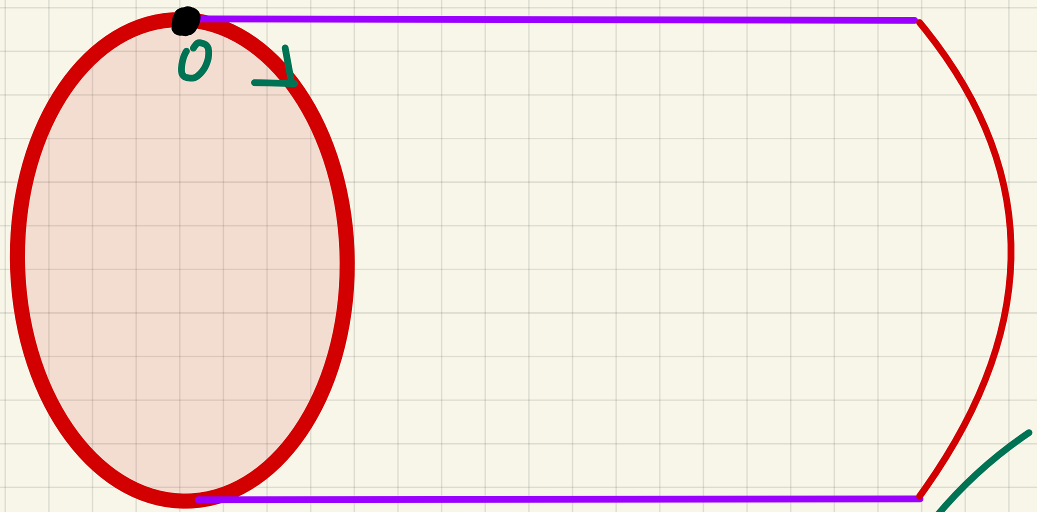
$$\omega = \sum_{i=1}^{N} de_i \wedge da_i$$

$$(T^*T)/W = \frac{\mathbb{R}^N \times U(1)^N}{S(N)} \longrightarrow \mathbf{T}/W$$

Free (?) system of indistinguishable particles:

Question of statistics

Reduction in two steps



$$\mathcal{G} = \mathcal{G}^{\text{framed}} \times G$$

$$\mathcal{G}^{\text{framed}} = \{\, g(x) \mid g(0) = 1 \,\}$$

$$Q = P\exp \oint_0^{2\pi} A$$

$$P = E(0)$$

invariants

T^*G

moment

$$Q^{-1} P Q - P$$

$$\int_G dQ\, |\Psi(Q)|^2 = \int_T \frac{dt}{|W|} |\Delta(t)|^2 |\Psi(e^t)|^2$$

$$Ad_Q(P) - P = 0 \Rightarrow \Psi(Q) = \Psi(g^{-1} Q g)$$

no nontrivial cocycles

$$\int_G dQ = 1$$

N free fermions on a circle

$$\rho(\alpha_1, \ldots\ldots, \alpha_N) = (-)^{\sigma} \rho(\alpha_{\sigma(1)} + n_1, \ldots\ldots, \alpha_{\sigma(N)} + n_N)$$

$$\rho(\alpha) = \Delta(e^{\alpha}) \Psi(e^{\alpha})$$

$$/\!/$$

$$\prod_{i>j} \sin\left(\pi(\alpha_i - \alpha_j)\right)$$

$$\forall \, (\vec{n}, \sigma) \in \mathbb{Z}^N \rtimes S(N)$$

$$\rho\Big|_{\Delta} = 0$$

$$\Delta = \{ \alpha_i = \alpha_j + n \mid j \neq i, n \in \mathbb{Z} \}$$

Spectrum

N free fermions on a circle

$$\rho(\alpha_1, \ldots\ldots, \alpha_N) = (-)^{\sigma} \rho(\alpha_{\sigma(1)} + n_1, \ldots\ldots, \alpha_{\sigma(N)} + n_N)$$

$$\rho(\alpha) = \Delta(e^{\alpha}) \psi(e^{\alpha})$$

$$// \quad \prod_{i>j} \sin\left(\pi(\alpha_i - \alpha_j)\right)$$

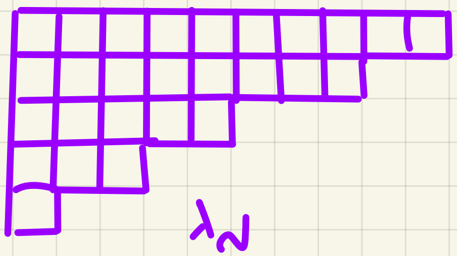
Basis: Schur functions

$$\rho_\lambda(\alpha) = \det \left\| e^{i(\lambda_k + N - k)\alpha_m} \right\|_{k,m=1}^{N}$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

$$E_\lambda = \sum_{k=1}^{N} (\lambda_k + N - k)^2 + \text{const}$$

$$= c_2(V_\lambda) + \text{const}'$$



λ_1

λ_N

$$\psi_\lambda(g) = \chi_{V_\lambda}(g)$$

$$\Big\langle \underbrace{D_x E + \sum_{i=1}^{L} \mu_{\mathcal{O}_i}\, \delta(x - x_i)}_{\in\, \Omega^1(S^1)\otimes \mathfrak{g}}\,,\ \xi \Big\rangle \in \mathrm{Lie}\, LG$$

$$\bigcap$$

$$\hat{\mathfrak{g}}^{*}$$

$$= \left(\int_{S^1} \operatorname{Tr} \xi D_x E \right) + \sum_{i=1}^{L} \langle \mu_{\mathcal{O}_i}, \xi(x_i) \rangle$$

evaluation

$$\mathcal{O}_i \qquad \xi$$

конечномерные орбиты калибровочной группы

evaluations at the points. x_i

$$\delta_\xi E = [\xi, E]$$

$$\delta_\xi A = \partial_x \xi + [A, \xi]$$

$$D_x E + \sum_{i=1}^{L} \mu_{0_i} \delta(x-x_i) = 0$$

periodic, but jumps at $x=x_i$

$$\mu_{0_i} = -\left(E_{x_i+0} - E_{x_i-0}\right)$$

тяжелые заряды

$x \in \mathbb{R} \Rightarrow \exists$ gauge $A_x = 0$

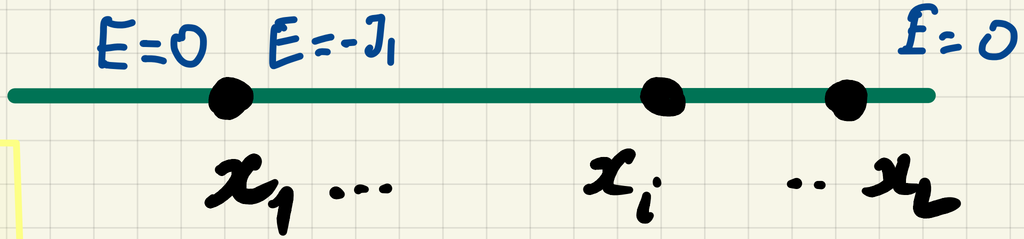
U(1) electric field (string)

$$q^2 \Delta l = \int_{\mathbb{R}} \mathrm{Tr}\, E^2 \, dx$$

$$\mathcal{D}_x E + \sum_{i=1}^{L} \mu_{\mathcal{O}_i}\, \delta(x-x_i) = 0$$

periodic, but jumps at $x=x_i$

$$\mu_{\mathcal{O}_i} = -\left(E_{x_i+0} - E_{x_i-0}\right)$$



Warm-up

infinite volume = tropical Gaudin system

finite energy H implies E(x) vanishes outside (x_1, x_L)

$$E(x) = -\sum_{j=1}^{i-1} J_j \qquad x_{i-1} < x < x_i$$

$$J_i = \mu_{\mathcal{O}_i}$$

fixed orbits $\mathcal{O}_i$ = fixed eigenvalues

$$\sum_i J_i = 0$$

$$(J_1, \dots, J_L) \to (g^{-1} J_1 g, \dots, g^{-1} J_L g)$$

thus, reduced phase space

$$\mathcal{P} = (\mathcal{O}_1 \times \dots \times \mathcal{O}_L) /\!/ G$$

$$\int_{\mathbb{R}} \operatorname{Tr} E^2 \, dx =$$

Yangian

$$H = \sum_{1 \le j < i \le L-1} (x_L - x_i) \operatorname{Tr}(J_j J_i)$$

up to additive constants

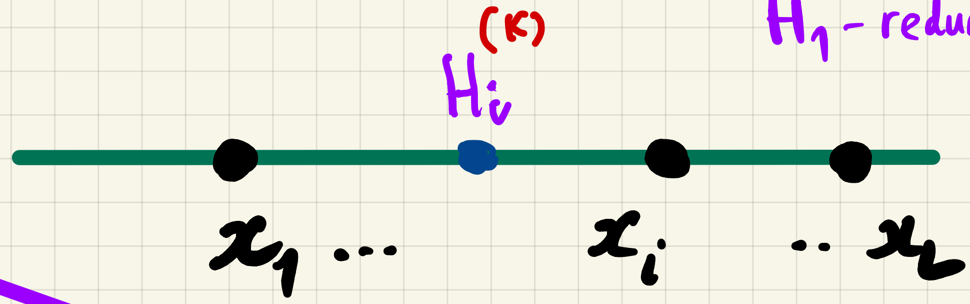
$$\boxed{H_i^{(k)} = \mathrm{Tr}\, E(x)^{k+1}}, \quad x_i < x < x_{i+1}, \quad i=1,\dots,L-1$$

$(L-2)(N-1)$

$i=2,\dots,L-1$, $k=1,\dots,N-1$ $\qquad \{H_i^{(k)}, H_j^{(l)}\} = 0$

откуда взять все интегралы?

$H_1^{(k)}$ – redundant



$\sum_i J_i = 0$

topological field theory => integrability

thus, reduced phase space

$$P = (O_1 \times \dots \times O_L)//G$$

$$\dim P = L \dim(G/T) - 2\dim G = (L-2)N(N-1) - 2(N-1)$$

$G = SU(N)$

$D_x E + \sum \mu_{O_i} \delta(x - x_i) = 0$

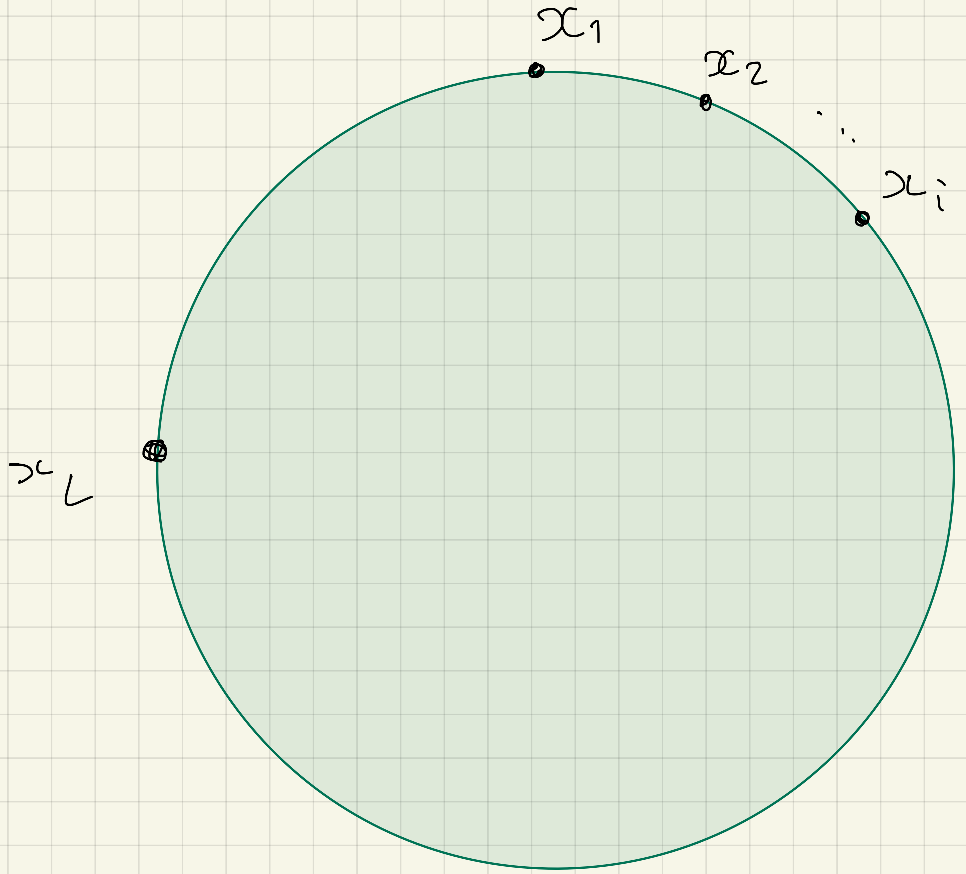
$D_{\bar z} \phi + \sum \mu_{O_i} \delta^{(2)}(z - z_i) = 0$

$2 \times \sum_{p=0}^{N-2} (L-2)(N-p) - 1$

$$\phi(z) = \sum_{i=1}^{L} \frac{J_i}{z - z_i}, \quad \mathrm{Det}(\phi(z) - \lambda) = \sum_{p=0}^{N} \frac{Q_{(L-2)(N-p)}(z)\, \lambda^p}{\prod_{i=1}^{L} (z - z_i)^{N-p}}$$

$\mathrm{Tr}\, J_i^k$ fixed

вспомогательные переменные



$$A = \mathrm{diag}(\alpha_1, \dots, \alpha_N)$$

const$_x$

$$E_{ab}(x) = e^{-(x-x_i)(\alpha_a - \alpha_b)} E_{ab}(x_i + 0)$$

$$x_i < x < x_{i+1}$$

$$E(x_{i+1} - 0) - E(x_{i+1} + 0) = J_i$$

$$\left(\sum_{i=1}^{L} J_i\right)_{aa} = 0 \quad , \quad a = 1, \dots, N$$

$$E_{ab}(x_1 + 0) = -J_1 + E_{ab}(x_1 + 2\pi - 0)$$

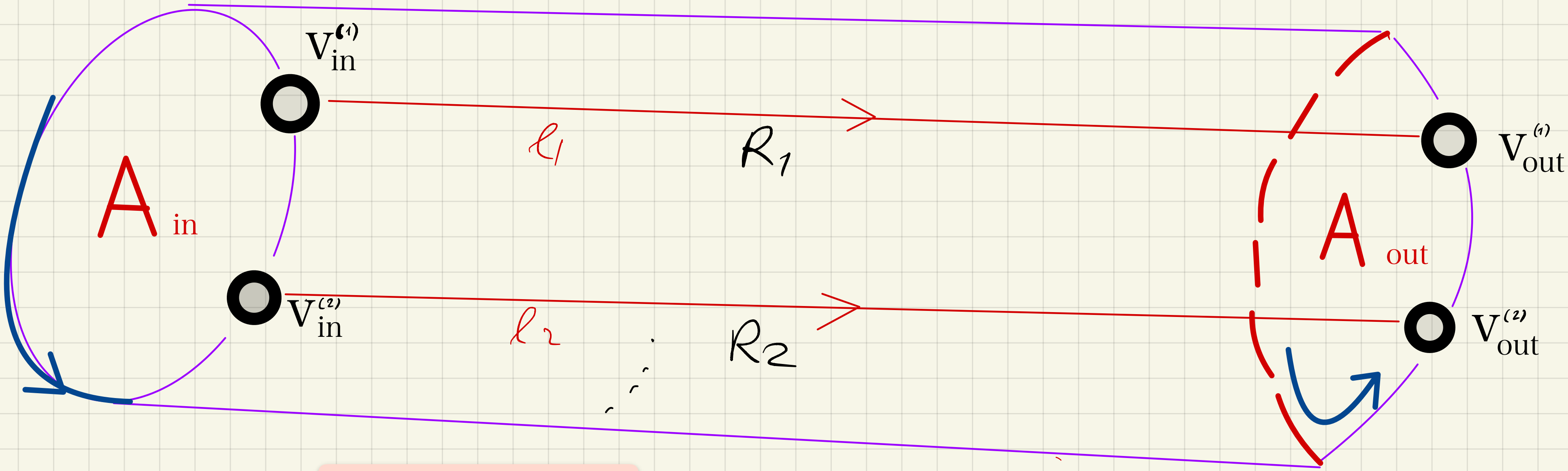
$$= -J_1 + e^{-(2\pi + x_1 - x_L)\alpha_{ab}} E_{ab}(x_L + 0)$$

$$\left(e^{-2\pi i(\alpha_i-\alpha_j)} - 1\right) E_{ij}(+0) = \nu\, z_i \bar{z}_j \qquad , \qquad i \neq j$$

$$E_{ii}(x) = p_i \qquad \text{const}$$

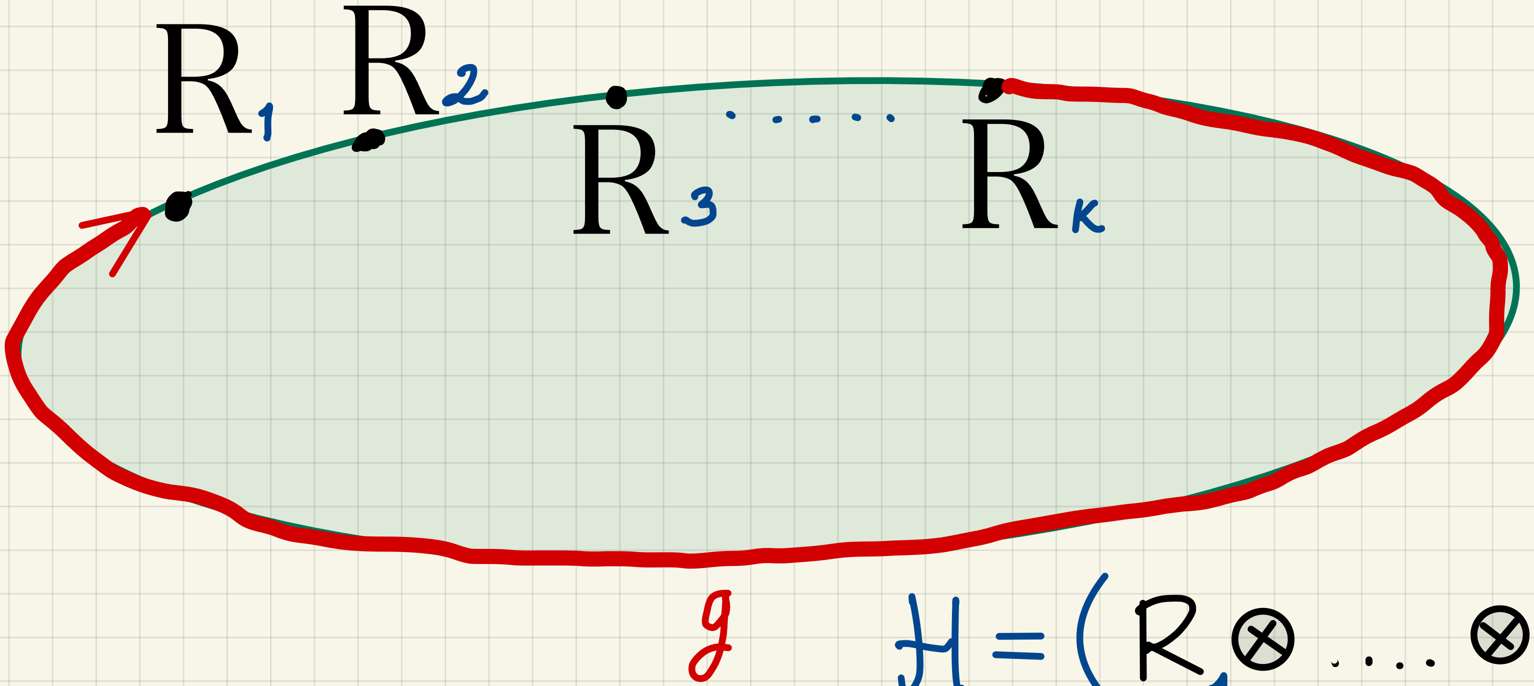
$$\frac{1}{2}\int \operatorname{Tr} E^2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \nu^2 \sum_{i<j} \frac{1}{4\sin^2(\pi\langle\alpha_i-\alpha_j\rangle)}$$

отталкивающий потенциал



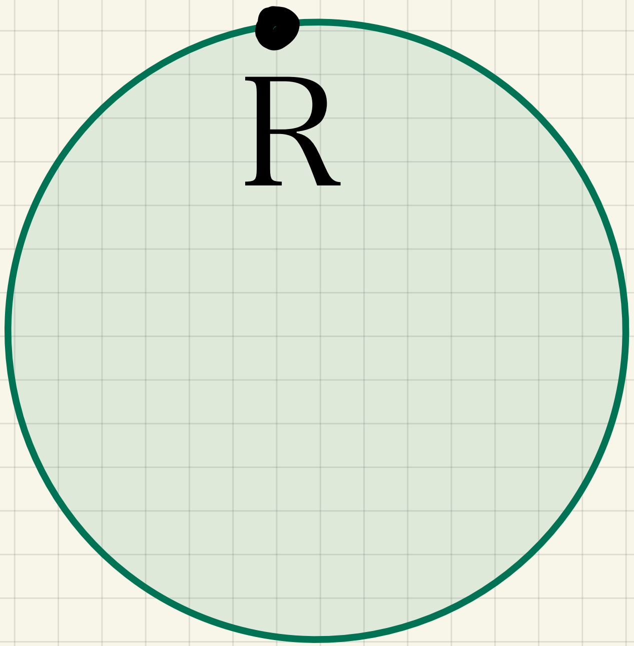
Time-like Wilson lines

$$K\left(A_{out}\, A_{in} \middle| \begin{matrix} V_{out}^{(1)}\otimes\dots\otimes V_{out}^{(L)} \\ V_{in}^{(1)}\otimes\dots\otimes V_{in}^{(L)}\end{matrix}\right) = \int e^{-\frac{1}{4g^2}\int \mathrm{Tr} F_{\mu\nu}^2} \prod_{i=1}^{L} \left\langle V_{out}^{(i)} \middle| \mathrm{Tr}_{R_i}\left(P\exp\int_{l_i} A\right) \middle| V_{in}^{(i)} \right\rangle$$



$$\Psi = \mathrm{Tr}_{V_\lambda}\left(\Phi\, T_\lambda(g)\right)$$

$$\mathcal{H} = \left(R_1 \otimes \cdots \otimes R_k \otimes \mathrm{Fun}(G)\right)^G$$



$$\mathcal{H} = \left(R \otimes \mathrm{Fun}(G)\right)^G =$$

$$\bigoplus_\lambda \{\Phi : V_\lambda \to R \otimes V_\lambda\}$$

$$\wr\wr$$

$$R[0]$$

space of intertwiners