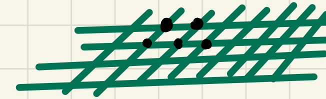
Lecture #13

trigonometry

general QFT

$$\mathcal{L}(\phi, \partial\phi)$$

finite # of terms at high energies (effectively finite in the sense of RG near the UV fixed point)



$$\sum_{\langle ij \rangle} \sigma_i \sigma_j$$

nearest neighbours

$$\mathcal{L}^{eff}_{\Lambda}(\tilde{\phi}, \partial\tilde{\phi}, \partial^2\tilde{\phi}, \dots)$$

∞- many terms ...

Иногда *эффективный Лагранжиан* описывается конечномерной динамической системой

Integrability

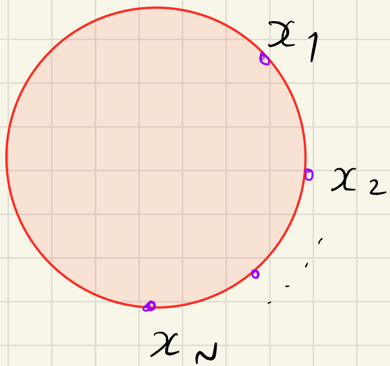
Supersymmetry

Trigonometric Calogero-Moser/Sutherland system

for integer ν
2d Yang-Mills theory

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \underset{\nu(\nu-1)}{\nu^2} \sum_{i<j} U(x_i - x_j)$$

$$\Psi(x_1, \dots, x_N) \sim (x_i - x_j)^{\nu}, \quad x_i \to x_j$$



for all ν
4d super-Yang-Mills
in the weak coupling
limit

$$U(x) = \frac{1}{\sin^2(x)}$$

discrete spectrum

$$U(x) = \frac{1}{\sinh^2(x)}$$

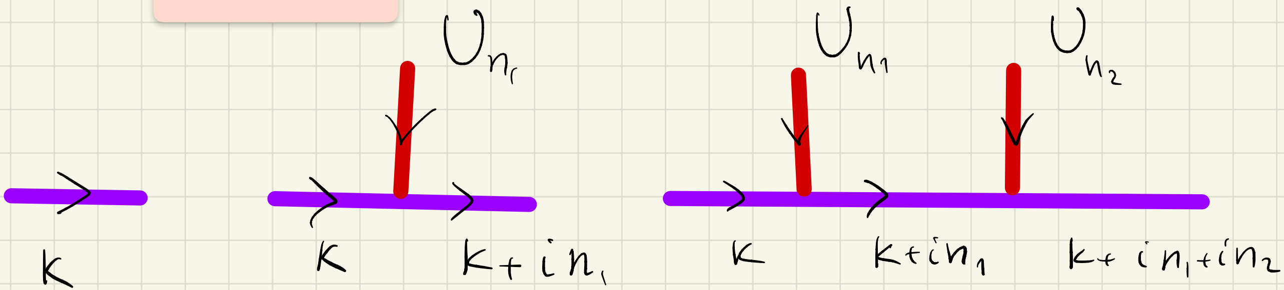
scattering problem

$$U(x_1,\dots,x_N) = \sum_{\vec{n}} U_{\vec{n}}\, e^{-\vec{n}\cdot\vec{x}} \sim$$

$$\sim \sum_{\vec{\alpha}>0} \sum_{m=1}^{\infty} m\, e^{-m\vec{\alpha}\cdot\vec{x}}$$

Jack
Gindikin-Karpelevich
Opdam-Heckmann

$\vec{\alpha}$ -roots of sl(N)



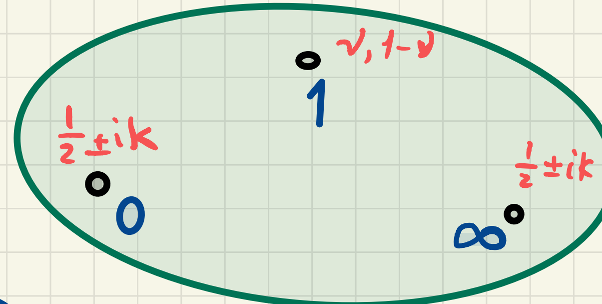
modern viewpoint = weak coupling limit of a surface defect in N=2* theory in 4d

The two-body case

$$-\frac{\partial^2}{\partial x^2}\psi + \frac{\nu(\nu-1)}{4\sinh^2(x/2)}\psi = k^2\psi$$

$$S(k) = e^{i\pi\nu}\,\frac{\Gamma(1-2ik)\,\Gamma(1-\nu+2ik)}{\Gamma(1+2ik)\,\Gamma(1-\nu-2ik)}$$

$$z = e^{x}$$



$$\psi(x) = \chi(z)\, z^{-1/2}$$

$$\left(-\frac{\partial^2}{\partial z^2} + \frac{\nu(\nu-1)}{(z-1)^2} - \frac{k^2+\frac{1}{4}}{z^2} + \frac{\nu(\nu-1)}{z(1-z)}\right)\chi = 0$$

$$g_\infty g_1 g_0 = 1$$

$$\mathrm{Tr}\, g_0 = \mathrm{Tr}\, g_\infty = -2\cosh 2\pi k$$

$$\mathrm{Tr}\, g_1 = 2\cos 2\pi\nu$$

Physically, need $\chi(z) \sim (z-1)^{\nu}$

$$\begin{cases} 2\,\mathrm{Re}\,\nu > -1 \\ 2\,\mathrm{Re}(1-\nu) < -1 \end{cases}$$

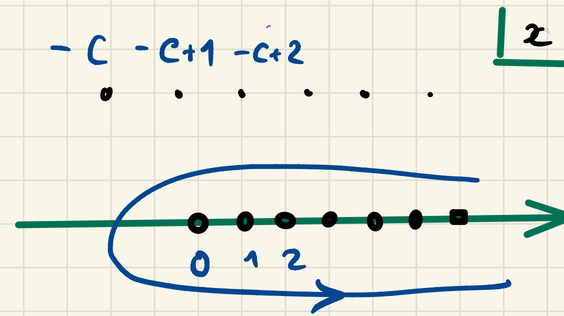
$$\mathrm{Re}\,\nu > \frac{3}{2}$$

χ -eigenfunction

g_1

$$\chi(z) \simeq z^{\frac{1}{2}+ik} + S(k)\, z^{\frac{1}{2}-ik}, \quad z \to 0$$

$$\chi^{\pm} \sim (z-1)^{\nu} z^{\frac{1}{2} \mp ik} \, {}_2F_1(a,b,c;z)$$



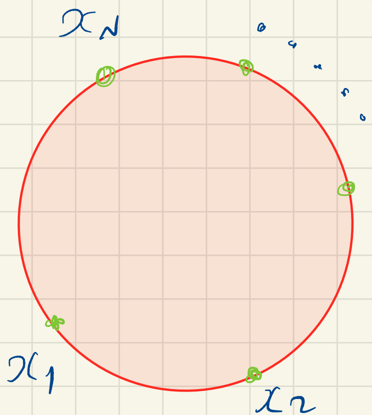
$$a = \nu$$
$$b = \nu \mp 2ik$$
$$c = 1 \mp 2ik$$

Barnes trick

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \sim \int_{\gamma} \frac{\Gamma(-x)\Gamma(-x-c)}{\Gamma(a+x)\Gamma(b+x)} z^{-x} \, dx$$

T-duality/mirror symmetry in 2d σ-model on $T^*\mathbb{P}^1$ (more generally, sum of line bundles)

trigonometric case



$\mathrm{diag}(e^{ix_1}, e^{ix_2}, \ldots\ldots, e^{ix_N}) = g \in U(N) = G$

Phase space $\mathscr{P}$ = **symplectic quotient**

$(T^*G \times \mathcal{O})//G$

$$\mu = g^{-1}Pg - P + \nu(zz^{+} - 1) = 0$$

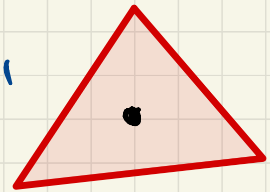
modulo

$$(P, g; z) \longmapsto h\cdot(P, g; z) = (h^{-1}Ph, h^{-1}gh; h^{-1}z)$$

$$\boxed{z \in \mathbb{C}^N \qquad z^+ z = 1}$$

$$H_2 = \frac{1}{2} \operatorname{Tr} P^2$$

$\mathbb{CP}^{N-1}$



$$\frac{\nu}{2i}\, dz^+ \wedge dz = p^*(\nu \cdot \omega_{FS}) \qquad p : S^{2N-1} \to \mathbb{CP}^{N-1}$$

modulo $z \mapsto z e^{i\varphi}$ $U(1)$

$$(P(t), g(t); z(t)) = (P(0), e^{tP(0)} g(0); z(0))$$

гамильтонова редукция

$$\mu = g^{-1} P g - P + \nu (z z^+ - 1) = 0$$

$$\mu(t) = \mu(0)$$

modulo $(P, g; z) \longmapsto h \cdot (P, g; z) = (h^{-1} P h, h^{-1} g h; h^{-1} z)$

Quantization

$$T^*G \times \mathcal{O} \longrightarrow L^2(G) \times \mathcal{H}_\nu$$

$$\Psi(g, z) = 0$$

$$(*) \quad \boxed{\Psi(h^{-1} g h, h^{-1} z) = (\det h)^{\nu}\, \Psi(g)}$$

If $\nu \notin Z$ then impose (*) for h=exp(ε ξ)

$\varepsilon^K = 0$, $\xi \in$ LieG

$$\tilde{z}^{+} \tilde{z} = \nu N$$

$$\tilde{z}^{+} = \frac{\partial}{\partial \tilde{z}}$$

non abelian Fourier

$$\Psi(g, z) = \sum_{\substack{\alpha \in \mathrm{Rep}(G) \\ m_\alpha, n_\alpha \in V_\alpha}} \tilde{\psi}^{(\alpha)}_{m_\alpha, n_\alpha}(z) \langle m_\alpha | T_\alpha(g) | n_\alpha \rangle$$

Thm: space of intertwiners

$$\Phi : \alpha \to \alpha \otimes \mathcal{H}_\nu$$

intertwiner, i.e. for any $\xi \in \mathfrak{g}$

$$\left[T_{V_\alpha}(\xi) \otimes 1 + 1 \otimes T_{\mathcal{H}_\nu}(\xi)\right]\Phi = $$

$$= \Phi \, T_{V_\alpha}(\xi)$$

Span of $J^i_i - J^{i+1}_{i+1}$

$\mathfrak{h} \subset \mathfrak{sl}(n)$

Cartan subalgebra

$\mathfrak{h}$ -invariants

$$\{\Phi\} \simeq \mathcal{H}_\nu[0] \simeq \mathbb{C}$$

реализация $\mathfrak{gl}_N$
дифференциальными операторами 1го порядка определенными на клетке

$\mathcal{H}_\nu$

HW-module

«квантование» $\mathbb{CP}^{N-1}$
при нецелом $N\gamma$

$$J_a^{\,b} = \omega_\nu^{-1}\left(z_a \frac{\partial}{\partial z_b}\right)\omega_\nu$$

при целом мы имеем геометрическое квантование — представление группы SU(N)

$$\omega_\nu = (z_1 \dots z_N)^{\nu}$$

$$\boxed{(\mathbb{C}^\times)^{N-1} \subset \mathbb{CP}^{N-1}}$$

$$\mathrm{Sym}^{N\nu}\mathbb{C}^N \simeq \mathcal{H}_\nu / \mathcal{H}_0$$

noncompactness leads to infinite dimensional space

$$h \cdot (P, g, z) = (h^{-1} P h, h^{-1} g h, h^{-1} z)$$

$$\mu = g^{-1} P g - P + \tilde{z} \tilde{z}^{+} - \nu \cdot \mathbb{1} = 0$$

remaining gauge transformations

go the gauge $g = \operatorname{diag}(e^{ix_1}, \dots, e^{ix_N}) \Rightarrow$ $h = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$

$$\mu_{ij} = (e^{ix_{ji}} - 1) P_{ij} + \tilde{z}_i \tilde{z}_j^* - \nu \delta_{ij} = 0$$

$i = j \Rightarrow |\tilde{z}_i|^2 = \nu$

use

$$\tilde{z}_i = \sqrt{\nu}$$

вместо T^*G берём $T^*\hat{\mathfrak{g}}$

придём к тому же результату методами топологической теории поля