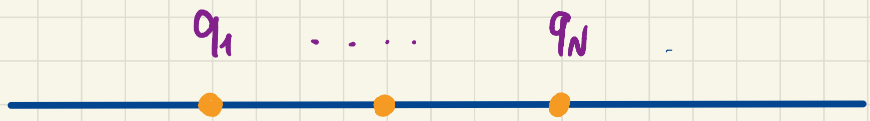
Lecture #11

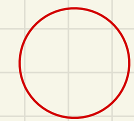
Calogero-Moser spaces and instantons, I

Calogero -Moser   одномерная динамика нескольких идентичных частиц



$$P = T^*(E^N \setminus \Delta)/S(N)$$

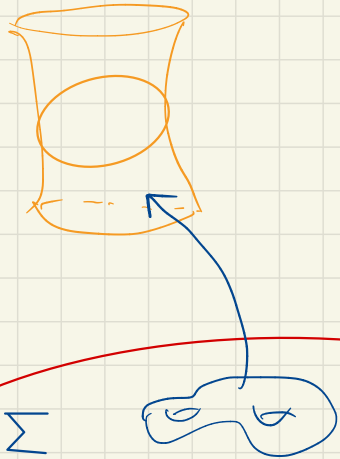
$U = q^{-2} + \omega^2 q^2$	рациональный случай	$\mathbb{R} \subset \mathbb{C}$
$U = \sin^{-2}(q)$	тригонометрический	$S^1 \subset \mathbb{C}^*$
$U = \dfrac{1}{L^2 \sinh^2(\frac{q}{L})}$	гиперболический	$\mathbb{R} \subset \mathbb{C}^*$
$U = \dfrac{1}{L^2}\wp(q/L)$	эллиптический	$T^2 = E$

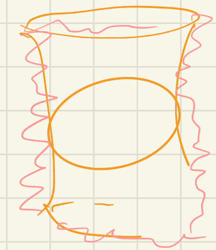


$\tau = iL$

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \nu^2 \sum_{i<j} U(q_i - q_j)$$

Classical geometry





$[X,Y]=Z$
$[Y,Z]=X$
$[Z,X]=Y$

Mirror

Σ

Homological/énumérative geometry

Quantum geometry

Instantons on quotients and defects

Calogero with oscillator = Sutherland

$$\frac{\nu^2}{q^2} + \frac{\omega^2}{2} q^2 \quad \xleftrightarrow{\quad f \quad} \quad \frac{1}{\sin^2(q)}$$

symplectomorphism of phase spaces,
mapping the Hamiltonians to each other

rational

$q = q_1 - q_2$

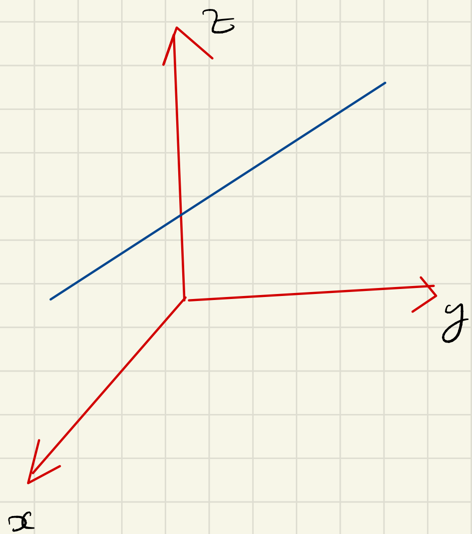
$T^*(\mathbb{R}^N \setminus \Delta) / S(N)$

$N = 2$

$H_{\text{eff}} = p^2 + \dfrac{\nu^2}{q^2}$

$(p, q) \sim (-p, -q)$

$q = |\vec{r}| > 0$



$H = \vec{p}^{\,2} = p_q^2 + \frac{1}{q^2} L^2$

$|\vec{p} \times \vec{r}| = \nu$

$\vec{L}$ / $SO(3)$

$T^* \mathbb{R}^3$

$ds^2_{\mathbb{R}^3} = dq^2 + q^2 d\Omega^2_{S^2}$

$iP, iQ \in \mathrm{Lie}\, U(N) = \mathfrak{g}$

P, Q $N \times N$ эрмитовы матрицы

$\omega = \mathrm{Tr}\, dP \wedge dQ$

$(P, Q) \mapsto (g^{-1} P g, g^{-1} Q g)$ (*)

порождается

свободная частица на $T^*\mathfrak{g}$

сохраняется эволюцией по времени

$H_2 = \frac{1}{2} \mathrm{Tr}\, P^2$ G-inv

$$\begin{cases} Q \mapsto Q + tP \\ P \mapsto P \end{cases}$$

$\mu = [P, Q]$

фиксируем орбиту μ по отношению к (*)

$\mu \mapsto g^{-1} \mu g$

Проекция на $(T^*\mathfrak{g})/G \xrightarrow{\mu} \mathfrak{g}^*/G$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$\underline{\underline{P}} = \mu^{-1}(\mathcal{O})$ $\qquad \mathcal{O}$ – орбита коприсоединенного действия

$X: \mathcal{O} \to \mathfrak{g}^*$

<u>CM</u>

$\mathcal{O} \simeq \mathbb{CP}^{N-1}$ = пространство эрмитовых матриц N xN

Tr=0

$\|$

$\{X \mid g^{-1}Xg =$ diag($\nu,\ldots,\nu,(1-N)\nu$) собственные значения кратности $(N-1,1)$

$z^+z = N \quad / \quad \mathbb{C} z \subset \mathbb{C}^N$

$z \sim z e^{i\theta} \qquad z \in \mathbb{C}^N$

$X = \nu(1-\Pi) \qquad \Pi = z \otimes z^+$

$$[P, Q] = \nu\left(1 - z \otimes z^+\right) \qquad (**)$$

modulo

$$(P, Q, z) \mapsto \left(g^{-1} P g,\ g^{-1} Q g,\ g^{-1} z e^{i\theta}\right) \qquad g \in U(N) \qquad e^{i\theta} \in U(1)$$

$z_i = 1$ $\Uparrow$ $\forall i$ $|z_i|^2 = 1$

$$H_2 = \frac{1}{2} \mathrm{Tr}\, P^2 = \frac{1}{2} \sum p_i^2 + \frac{1}{2} \nu^2 \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}$$

$$\sum_{i=1}^{N} |z_i|^2 = N$$

$\mathcal{P}$ — symplectic manifold

$$\text{“} \ T^*(\mathbb{R}^N \setminus \Delta) / S(N)$$

$p_i = P_{ii}$ $\qquad q_i \neq q_j$

явно $\quad Q = \mathrm{diag}(q_1, \ldots, q_N)$

$$(***) \quad P_{ij}(q_i - q_j) = -\nu z_i \bar{z}_j \qquad i \neq j$$

$$0 = \nu(1 - |z_i|^2)$$

$\mathbb{C}$ – version

P, Q = complex N x N matrices

$z \otimes z^{+} \longrightarrow z \otimes \tilde{z}$

$z \in \mathbb{C}^N, \quad \tilde{z} \in (\mathbb{C}^N)^*$

$\tilde{z}(z) = N$

$$[P, Q] = \nu \left(1 - z \otimes \tilde{z}\right)$$

$(P, Q, z, \tilde{z}) \sim (g^{-1} P g, g^{-1} Q g, g^{-1} z, \tilde{z} g)$

$g \in GL(N, \mathbb{C})$

теперь Q не всегда диагонализуется

но если, то $Q = \mathrm{diag}\,(q_1, \dots q_N)$ $\quad q_i \in \mathbb{C}$

$g = \mathrm{diag}\,(g_1, \dots g_N)$

$g_i \in \mathbb{C}^{\times}$

комплексные CM частицы могут сталкиваться!

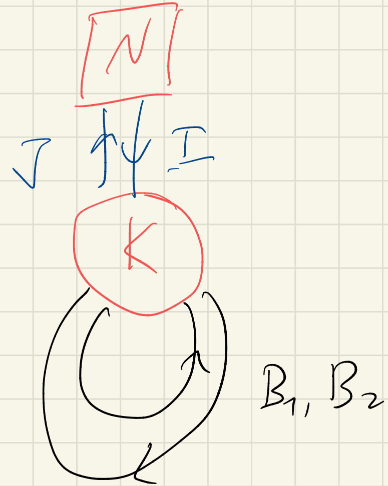
$H = p^2 + \frac{\nu^2}{q^2}$ $\quad q \to 0, p \to \infty$, H - finite

$$[P, Q] = \nu\,(1 - z \otimes \tilde{z})$$

Одно из уравнений ADHM в случае некоммутативной деформации с

$$\zeta_{\mathbb{C}} \approx \nu \neq 0$$

$$(B_1, B_2, I, J)$$



$$N = \mathbb{C}^n$$

$$K = \mathbb{C}^k$$

$$[B_1, B_2] + IJ = 0 = \mu_{\mathbb{C}}$$

$$\zeta \cdot 1 = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \mu_{\mathbb{R}}$$

$$\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\zeta) / U(k) = \widetilde{\mathcal{M}}_k^{\text{framed}}(n)$$

$$\| $$

$$\mu_{\mathbb{C}}^{-1}(0)^{\text{stable}} / GL(k)$$

(B, I, J) stable $\Leftrightarrow$ $\mathbb{C}[B_1, B_2]\, I(N) = K$

$\zeta > 0$

$$\left(\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}} \cdot 1) \cap \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}} \cdot 1) \right) / U(k) = \tilde{\mathcal{M}}_k(n)$$

framed

$\vec{\zeta} \in \mathbb{R}^3$

$$= \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}} \cdot 1) \,/\, GL(k)$$

stable

$SU(2)_L \times SU(2)_R$

stable $\zeta_{\mathbb{R}} > 0$ $= \{ \mathbb{C}[B_1, B_2] I(N) = K \}$

$$g_L \begin{pmatrix} B_1 & -B_2^+ \\ B_2 & B_1^+ \end{pmatrix} g_R$$

$\vec{\zeta}$ — триплет $SU(2)_R$

можем повернуть так $\vec{\zeta} = (\zeta_{\mathbb{R}}, 0, 0)$

а можем повернуть так $\vec{\zeta} = (0, \zeta_{\mathbb{C}}, \bar{\zeta}_{\mathbb{C}})$

in the second case

$P = B_1$

$Q = B_2$

$\nu = \zeta_{\mathbb{C}}$

$I = \sqrt{\gamma}\, Z$

$J = \sqrt{\gamma}\, \tilde{Z}$

$N = k$

$n = 1 \Rightarrow$

$\downarrow$

$n > 1$

framed

$\tilde{\mathcal{M}}_N(1)_{\vec{\zeta}=(0,\nu,\bar{\nu})}$

$\parallel$

CM_N

$B_1 \to B_1 + t B_2$

$$H_2 = \frac{1}{2} \operatorname{Tr} B_2^2$$

$\tilde{Z} Z : \mathbb{C}^n \to \mathbb{C}^n$

$(Z, \tilde{Z}) \mapsto (Zh, h^{-1}\tilde{Z})$

$Z : \mathbb{C}^n \to \mathbb{C}^N$

$\tilde{Z} : \mathbb{C}^N \to \mathbb{C}^n$

$$[P, Q] = \zeta_{\mathbb{C}}(1 - Z\tilde{Z})$$

$(**)$

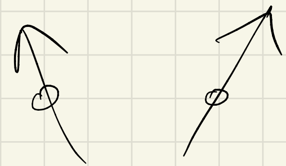
$\tilde{Z} Z = 1_n$

generically $Q = \operatorname{diag}(q_1, \dots q_N)$, $(Z\tilde{Z})_{ii} = 1,\ i = 1 \dots N$

modulo

$g = \operatorname{diag}(g_1, \dots g_N)$ $(Z, \tilde{Z}) \to (g^{-1} Z, \tilde{Z} g)$

Система Калоджеро со спинами



$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i\neq j} \frac{\mathrm{Tr}_n S_i S_j}{(q_i - q_j)^2}$$

Poisson brackets of $\mathfrak{gl}_n$

S_i is an $n\times n$ matrix

$$(S_i)_a{}^b = Z_a{}^i \, \tilde{Z}_i{}^b$$

$a, b = 1, \dots, n$

$i = 1, \dots, N$

ADHM n, k — $\mathrm{Spin}_n\ CM_k$

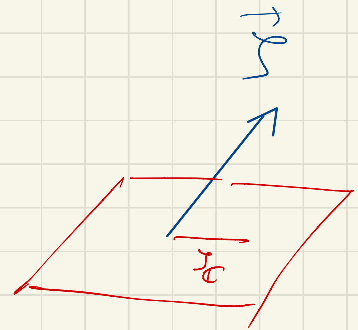
$$[B_1, B_2] + IJ = \zeta_{\mathbb{C}}$$

$\mathrm{Tr}\, B_2^2$

новые

$$B_1 = \alpha B_1 + \beta B_2^\dagger$$

$$B_2 = +\bar{\alpha} B_2 - \bar{\beta} B_1^\dagger$$



твисторная сфера

$$(\alpha : \beta) \in \mathbb{P}^1$$

тригонометрический/эллиптический случай

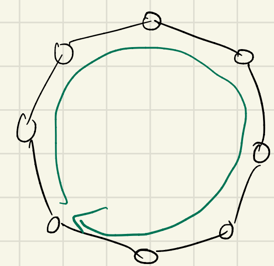
$\mathbb{R}^4 \longrightarrow \mathbb{R}^4/\Gamma$

$\Gamma = \mathbb{Z}$ $\qquad$ $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$

нам нужны и другие случаи

$\Gamma \subset SU(2)$

конечная подгруппа

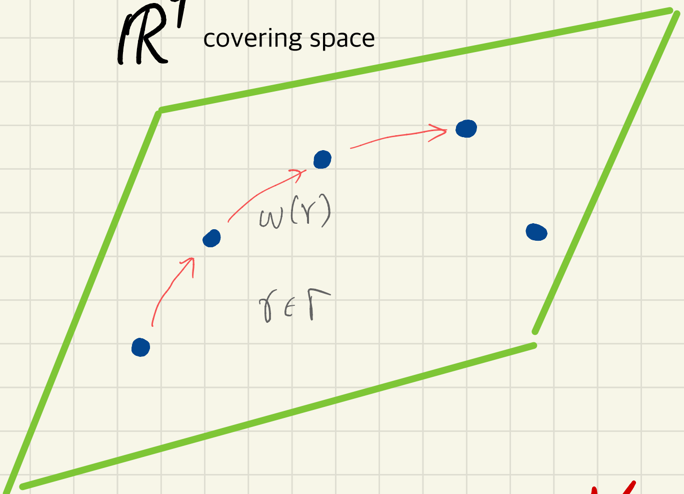


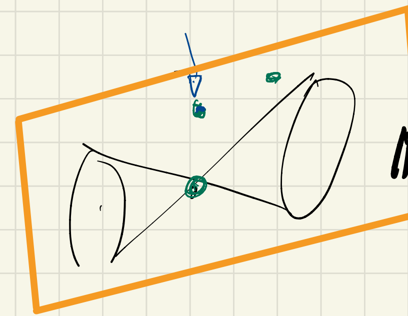
инстантоны на (некоммутативном) $\mathbb{R}^4/\Gamma$

cross-product construction
Morris equivalence

$\Gamma \subset$ isometries $\mathbb{R}^4$

$\mathbb{R}^4$ covering space =





$\mathbb{R}^4/\Gamma$

Chan-Paton space K complex valued functions on the collection of k points in $\mathbb{C}^2$

$B_1, B_2 : K \circlearrowleft$ operation of multiplication by coordinates z_1, z_2

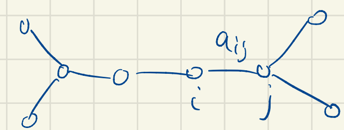
K covering space = representation of Γ

$$\hat{N} = \bigoplus_{\Gamma^\vee} N_i \otimes R_i$$

$$\hat{K} = \bigoplus_{\Gamma^\vee} K_i \otimes R_i$$

irreducible representations of Γ

$$\mathbb{C}^2 \otimes R_i = \bigoplus_j \mathbb{C}^{a_{ij}} \otimes R_j$$



$$\varphi : \Gamma \longrightarrow U(\hat{K})$$
$$\omega : \Gamma \longrightarrow \mathrm{Iso}(\mathbb{R}^4)$$

$$\omega(\gamma) \circ \vec{B} = \varphi(\gamma)^{-1} \vec{B} \varphi(\gamma)$$

represent geometrically the action of Γ on the covering space

$$I : \hat{N} \longrightarrow \hat{K}$$
$$J : \hat{K} \longrightarrow \hat{N}$$

asymptotics of an instanton at infinity

$\mathbb{R}^4/\Gamma$ at infinity

$\partial(\mathbb{R}^4/\Gamma) = Y$

$\Gamma \subset SU(2)$

$Y = S^3/\Gamma$

$r \to \infty$

$A \to A_0$

$F_{A_0} = 0$

$\pi_1(S^3/\Gamma) = \Gamma$

$A_0 \leftrightarrow \rho : \Gamma \to U(\hat{N})$

$\updownarrow$

$\hat{N} = \bigoplus_{i \in \Gamma^\vee} N_i \otimes R_i$

$\Gamma = \mathbb{Z}$

$Y = S^2 \times S^1$

$\Gamma = \mathbb{Z} \times \mathbb{Z}$

$Y = S^1 \times S^1 \times S^1$

$\pi_1(S^2 \times S^1) = \mathbb{Z}$

$A_0 \leftrightarrow \rho : \mathbb{Z} \to U(\hat{N}) \leftrightarrow [g] \in U(N)/\mathrm{Ad}\,U(N)$

$$\mathbb{R}^4/\Gamma \qquad (z_1, z_2) \longrightarrow (z_1 + n, z_2)$$

$$\begin{cases} B_1 + 1 = \phi^{-1} B_1 \phi \\ B_2 = \phi^{-1} B_2 \phi \end{cases}$$

$$IJ \longrightarrow IJ\,\delta(\theta)$$

$$A, B \in \mathrm{Fun}(S^1) \otimes \mathrm{End}(\mathbb{C}^k)$$

$$K = \mathrm{Fun}(S^1) \otimes \mathbb{C}^k$$

$$\mathrm{Fun}(S^1) = \mathbb{Z}^\vee$$

$$B_1 = \partial_\theta + A(\theta)$$

$$B_2 = B(\theta)$$

$\theta \in S^1 \mapsto n \in \mathbb{Z}$ acts by multiplication by

$$R_\theta \cong \mathbb{C} \qquad T_{R_\theta}(n) = \exp(2\pi i n \theta)$$

$\mathbb{R}^4/\mathbb{Z}$

$\mu_{\mathbb{C}}$

$$\longrightarrow \quad [\partial_\theta + A(\theta), B(\theta)] = \gamma(1 - IJ)\,\delta(\theta)$$

trigonometric Calogero-Moser=Sutherland