LECTURE # 8

Feb 11 2021

noncommutative gauge flds 2

$$A : V_1 \longrightarrow V_2$$

$$\operatorname{coker} A = V_2 / \operatorname{im} A$$

$$\operatorname{Index} A = \dim \ker A - \dim \operatorname{coker} A = \dim V_1 - \dim V_2$$

не меняется при деформациях оператора

$$F_A^+ = 0 \qquad \mathbb{R}^4 \cong \mathbb{C}^2 \qquad z_1, z_2$$

$$F^{0,2} \equiv F_{\bar z_1 \bar z_2} = 0 \equiv [\nabla_{\bar z_1}, \nabla_{\bar z_2}] = 0$$

$$\left\{ F^{1,1+} = F_{z_1 \bar z_1} + F_{z_2 \bar z_2} = 0 \right. \qquad \Big/ \left\{ G\text{-gauge transf.} \right\}$$

$$\mathbb{CP}^1$$

выборов комплексной структуры в

$$\mathbb{R}^4$$

$$\approx \quad G_{\mathbb{C}}\text{-gauge}$$

$$\nabla_{z_1}, \nabla_{\bar z_1}, \nabla_{\bar z_1}, \nabla_{z_2}$$

$$u \in \mathbb{CP}^1 = \frac{SU(2)_R}{U(1)}$$

$SU(2)_R$

$$w_1 \cong z_1 + u \bar z_2 \qquad \bar w_1 = \bar z_1 + \bar u z_2$$

$$w_2 \cong z_2 - \bar u z_1 \qquad \bar w_2 = \bar z_2 - u \bar z_1$$

$$[\nabla_{\bar w_1}, \nabla_{\bar w_2}] = 0$$

Белавин Захаров

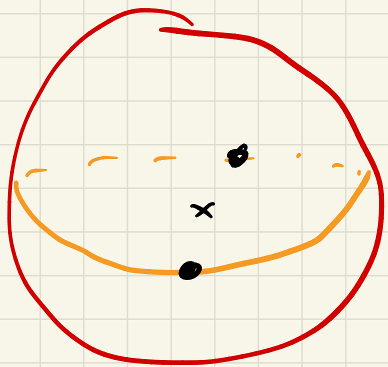
$$|u| = 1 \Rightarrow \bar u = u^{-1}$$

$$\nabla(u)^{+} = \nabla\left(-\frac{1}{\bar{u}}\right)$$

$\mathbb{CP}^3$

$\downarrow$

S^4



$$\nabla_{\bar{w}_\alpha} = \bar{g}^{1}\partial_{\bar{w}_\alpha} g$$

$$g(z_1, z_2, \bar{z}_1, \bar{z}_2; u)$$

$u \neq 0, \infty$

$$\underline{h(w_1, w_2; u)}$$

$$\text{''} \quad g_+^{-1} g_-$$

$$w_1 = z_1 + u\bar{z}_2$$

$$w_2 = z_2 - u^{-1}\bar{z}_1$$

$$F_{\bar z_1 \bar z_2} = 0 \quad \Longrightarrow \quad A_{\bar z_\alpha} = g^{-1} \partial_{\bar z_\alpha} g$$

$\bar z_2$ – время

$$g(z, \bar z)$$

$\Rightarrow$

$$A_{z_\alpha} = \partial_{z_\alpha} g^{+} (g^{+})^{-1}$$

$$F^{1,1+} = 0 \Rightarrow \sum_{\alpha=1}^{2} \partial_{\bar z_\alpha} \left(h^{-1} \partial_{z_\alpha} h \right) = 0$$

калибровка Янга

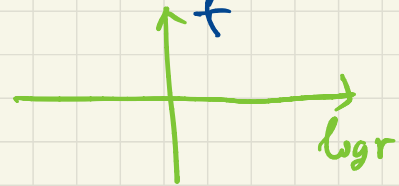
$G = U(1)$

$h = e^{\phi} \Rightarrow \Delta \phi = 0 \quad \Big| \quad h = g g^{+}$ = эрмитова метрика в расслоении

$H_N^{+} = SL(N, \mathbb{C}) / SU(N)$

$\mathbb{R}^4$

$A \to g^{-1} dg$



$$\int \mathrm{Tr}\, F \wedge * F = $$
$$= \int \frac{dr}{r}\left(\dot f^2 + f^2(1-f)^2\right)$$
$$\dot f = r\,\partial_r f$$

BPST instanton

$G = SU(2)$

$$A^a = f(r)\,\omega^a = \frac{\omega^a}{1+r^2} \qquad A = f(r)\, h^{-1} dh$$

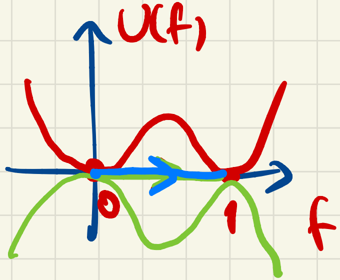
$$F \sim \frac{1}{(1+r^2)^2}$$

$$\mathbb{R}^4 \setminus 0 = \mathbb{R}_+ \times S^3$$

$$h : S^3 \xrightarrow{\sim} SU(2)$$
$$h^{-1} dh = \omega_a \sigma^a$$

$$dr^2 + r^2 d\Omega_3^2 = r^2\left(\left(\frac{dr}{r}\right)^2 + d\Omega_3^2\right)$$

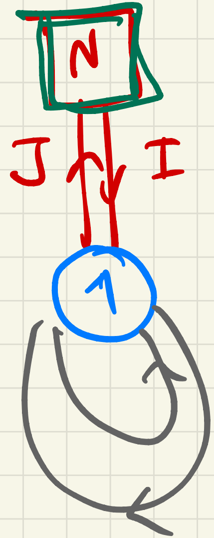
$$\omega_a \in \Omega^1(S^3)$$



$$\left(\dot f \pm f(1-f)\right)^2 \mp \dot f f(1-f)$$

$k=1$

n



модули инстантона

$(b_1,b_2)\in\mathbb{C}^2$

$\rho>0$

$St(2,N)$

$b_1,b_2\in\mathbb{C}$

$I^+\in\mathbb{C}^N$

$J\in\mathbb{C}^N$

$$\begin{cases} IJ=0 \\ II^+-J^+J=0 \end{cases}$$

$$(I,J)\sim(I\underline{e^{i\varphi}},\ J\underline{e^{-i\varphi}})$$

$$II^+=:\rho^2>0,\quad \rho>0$$

$$I=\rho e_2^+$$

$$J=\rho e_1$$

$$\underline{e_2^+e_1=0}$$

$$e_2^+e_2=1$$

$$e_1^+e_1=1$$

$$e_1\in\mathbb{C}^N$$

$$
\mathcal{D}_z^{+} = \begin{array}{cccc} 1 & 1 & \sim & \\ \left(\begin{array}{c} b_1 - z_1 \\ -\bar{b}_2 + \bar{z}_2 \end{array} \right. & \begin{array}{c} b_2 - z_2 \\ \bar{b}_1 - \bar{z}_1 \end{array} & \left. \begin{array}{c} I \\ J^{\dagger} \end{array} \right) & \begin{array}{c} 1 \\ 1 \end{array} \end{array}
$$

$$
\mathcal{D}_z^{\dagger} \psi = 0
$$

$$
(b_1 - z_1)\psi_+ + (b_2 - z_2)\psi_- + I\xi = 0
$$

$$
-(\bar{b}_2 - \bar{z}_2)\psi_+ + (\bar{b}_1 - \bar{z}_1)\psi_- + J^{\dagger}\xi = 0
$$

$$
\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix}
$$

$\psi_\pm \in \mathbb{C}$

$\xi \in \mathbb{C}^n$

$$
(\vec{r} - \vec{r}_0)^2 := |b_1 - z_1|^2 + |b_2 - z_2|^2
$$

$$(b_1 - z_1)\psi_+ + (b_2 - z_2)\psi_- + I\xi = 0$$
$$-(\bar b_2 - \bar z_2)\psi_+ + (\bar b_1 - \bar z_1)\psi_- + J^\dagger \xi = 0$$

$\psi_\pm \in \mathbb{C}$ $\quad (\vec r - \vec r_0)^2 := |b_1 - z_1|^2 + |b_2 - z_2|^2$

$\xi \in \mathbb{C}^n$

$$\Longrightarrow \quad \psi_+ = \frac{-1}{|\vec r - \vec r_0|^2}\left((\bar b_1 - \bar z_1) I - (b_2 - z_2) J^\dagger\right)\xi \quad ✓$$

$$\psi_- = \frac{-1}{(\vec r - \vec r_0)^2}\left((\bar b_2 - \bar z_2) I + (b_1 - z_1) J^\dagger\right)\xi$$

$$1_N = \Psi^\dagger \Psi = \xi^\dagger\xi + \frac{\xi^\dagger}{|r-r_0|^4}\left((b_1 - z_1) I^\dagger - (\bar b_2 - \bar z_2) J\right)\left(\left((\bar b_1-\bar z_1)I - (b_2-z_2)J^\dagger\right)\xi\right)\xi +$$
$$+ \xi^\dagger \frac{\left((\bar b_2 - \bar z_2)I + (b_1-z_1)J^\dagger\right)^\dagger \left((\bar b_2-\bar z_2)I + (b_1 - z_1)J^\dagger\right)}{|r-r_0|^4}\xi =$$

$$I^\dagger I = \rho^2$$

$$= \xi^\dagger\left(1 + \frac{\rho^2}{|r-r_0|^2}\right)\xi \quad \Rightarrow \quad \xi = \frac{|r-r_0|}{\sqrt{|r-r_0|^2+\rho^2}}\, u$$

$$u^\dagger u = 1$$

$$A = \psi^\dagger d\psi$$

$$A_+ = U^{-1}dU + U^{-1}A_-U$$

U хорошо определено в окрестности S_0^3

вне $\vec{r} = \vec{r}_0$ несингулярно

A_+ внутри $D^4_{r_0}$, A_- снаружи

$$U : S^3 \xrightarrow{\sim} SU(2) \hookrightarrow SU(N)$$

$$(I, J^\dagger) \to (e_1, e_2)$$

$SU(2) \hookrightarrow SU(N)$ подгруппа преобразований, (e_1, e_2) не трогая $(e_3, e_4, \dots, e_N)$

$n = 1$

$b_1, b_2 \in \mathbb{C} \longrightarrow$ сдвигом сделаем, для простоты $b_1, b_2 = 0$

$$\begin{cases} IJ = 0 \\ II^+ - J^+J = \zeta > 0 \end{cases} \qquad I, J \in \mathbb{C} \qquad \Longrightarrow J = 0 \quad I = \sqrt{\zeta}$$

$$D_z^+ = \begin{pmatrix} -z_1 & -z_2 & \sqrt{\zeta} \\ \bar{z}_2 & -\bar{z}_1 & 0 \end{pmatrix}$$

$$[a_i, a_j^+] = \delta_{ij}$$

$$\begin{cases} [z_1, z_2] = 0 \\ [z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -\zeta \end{cases}$$

первая реализация, симметричная

$$z_1 = \sqrt{\zeta/2}\, a_1^+$$
$$z_2 = \sqrt{\zeta/2}\, a_2^+$$
$$\bar{z}_1 = \sqrt{\frac{\zeta}{2}}\, a_1$$
$$\bar{z}_2 = \sqrt{\frac{\zeta}{2}}\, a_2$$

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix}$$

$\psi_\pm \in \mathcal{A}(a_i, a_i^\dagger)$

$\xi \in \mathcal{A}$

$$v = \frac{\sqrt{2}}{\hat{N}} \xi$$

$$\Psi^\dagger \Psi = 1 = \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \xi^\dagger \xi = \xi^\dagger \left(1 + \frac{2}{\hat{N}}\right) \xi$$

$$D^\dagger \Psi = 0$$

$$\begin{cases} \underline{a_1^\dagger \psi_+ + a_2^\dagger \psi_-} = \sqrt{2}\, \xi \\ a_2 \psi_+ - a_1 \psi_- = 0 \end{cases} \not\Rightarrow \mathcal{H}/_{\mathrm{im}\, \xi} \cong \mathbb{C}^k$$

$\psi_+ = a_1 v$

$\psi_- = a_2 v$

$$(a_1^\dagger a_1 + a_2^\dagger a_2) v = \sqrt{2}\, \xi$$

$$\text{"}\hat{N} v = \sqrt{2}\, \xi$$

$\underline{\langle 0,0 | \xi = 0}$

$$\sim 1 + \frac{\rho^2}{r^2}$$

$$1 = \xi^{+} \frac{\hat{N}+2}{\hat{N}} \xi \implies$$

$$\boxed{\psi_{+} = a_1 \frac{\sqrt{2}}{\hat{N}} \xi = a_1 \frac{\sqrt{2}}{\sqrt{\hat{N}(\hat{N}+2)}} S}$$

$$\xi = \frac{\sqrt{\hat{N}}}{\sqrt{\hat{N}+2}} S$$

$$\boxed{S^{+} S = 1}$$

$$\not\Downarrow$$

$$S S^{+} = 1$$

$\underline{\mathcal{A}}$ = алгебра операторов
в гильбртовом пространстве

$\mathcal{H} = \mathbb{C}[a_1^{+}, a_2^{+}]\,|0,0\rangle$

$\simeq L^2(\mathbb{C}^2)$

$*$ - алгебра

$f \in \mathcal{A} \to f^{+} \in \mathcal{A}$

нам нужно, чтобы имели смысл формулы

$$S \xrightarrow{?} SU$$

$$U \in U(\infty)$$
$$U^+U = 1$$
$$UU^+ = 1$$

достаточно

$$\psi_+ = a_1 \frac{\sqrt{2}}{\sqrt{\hat{N}(\hat{N}+2)}} S \quad \Leftarrow \quad \begin{cases} S^+S = 1 \\ SS^+ = 1 - |0,0\rangle\langle 0,0| \end{cases}$$

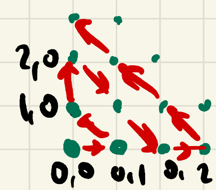
$$\psi_- = a_2 \frac{\sqrt{2}}{\sqrt{\hat{N}(\hat{N}+2)}} S$$

$$S$$

$$S|0,0\rangle = |0,1\rangle$$

$$\xi = \sqrt{\frac{\hat{N}}{\hat{N}+2}} S$$

$$\varkappa$$



$$\xi\xi^+$$

Пример: произвольное k, n=1

$\zeta > 0$ ADHM $\Rightarrow$ $J=0$, $[B_1, B_2]=0$

$\mathcal{I}_k \subset \mathbb{C}[z_1, z_2]$

$f \in \mathcal{I}_k \Leftrightarrow$

$f(B_1, B_2)I=0$

$$\begin{cases} (B_1 - a_1^+)\psi_+ + (B_2 - a_2^+)\psi_- + I\zeta = 0 \\ -(B_2^+ - a_2)\psi_+ + (B_1^+ - a_1)\psi_- = 0 \end{cases}$$

идеал $\mathcal{I}_k$

$$\zeta^+ I^+ = \psi_+^+(B_1^+ - a_1) + \psi_-^+(B_2^+ - a_2)$$

подпространство $\mathcal{H}_{\mathcal{I}_k} \subset \mathcal{H}$

$$\zeta^+ I^+ e^{B_1^+ a_1^+ + B_2^+ a_2^+} = \\ = \tilde{\psi}_+^+ a_1 + \tilde{\psi}_-^+ a_2$$

подпространство $\mathcal{H}_K \subset \mathcal{H}$

$\mathcal{H}_K \simeq \mathbb{C}^k$

$i: K \to \mathcal{H}$

подпространство гильбертова пространства, соответствующее идеалу

$\mathcal{H}_{\mathcal{I}_K}$

$$i = P \frac{1}{(P^+ P)^{1/2}}$$

$\mathcal{I}_K$

$$v = \sum_{n_1, n_2 \geq 0} v_{n_1, n_2} |n_1, n_2\rangle$$

$$v \overset{\in}{} \iff v \perp i(\lambda) = 0 \quad \forall \lambda \in K$$

$$P = I^+ e^{B_1^+ a_1^+ + B_2^+ a_2^+} |0,0\rangle = \sum_{n_1, n_2 \geq 0} \frac{I^+ (B_1^+)^{n_1} (B_2^+)^{n_2}}{\sqrt{n_1! \, n_2!}} |n_1, n_2\rangle$$

если полином $f \in \mathcal{I}_K$

$$f = \sum_{n_1, n_2 \geq 0} \overset{\in \mathbb{C}}{f_{n_1, n_2}} z_1^{n_1} z_2^{n_2} \Longrightarrow \sum_{n_1, n_2 \geq 0} \sqrt{n_1! \, n_2!} \, f_{n_1, n_2} |n_1, n_2\rangle \in \mathcal{H}_{\mathcal{I}_K}$$

$$\Updownarrow \qquad\qquad \| $$

$$\sum_{n_1, n_2 \geq 0} f_{n_1, n_2} B_1^{n_1} B_2^{n_2} I = 0 \qquad f(a_1^+, a_2^+)|0,0\rangle$$

$\mathcal{H} \supset K$

$S^+ S = 1$

$\underline{SS^+} = 1 - \Pi_K \sim \hat{\mathbb{C}}^2$

Пусть $\mathcal{H} = \mathbb{C}[a^+]|0\rangle$

$a^+ \sim z$

$SS^+ = 1 - |0\rangle\langle 0| = \sum_{n=1}^{\infty} |n\rangle\langle n|$

$S^+ S = 1$

$S^+ = \frac{1}{\sqrt{\hat{N}+1}} a$

$\mathbb{C} = \mathbb{R}^2$

$S = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$

$S = \frac{1}{\sqrt{\hat{N}}} a^+ = a^+ \frac{1}{\sqrt{\hat{N}+1}}$

квазиклассически $S \sim \exp(i\varphi)$