Lecture #7

Non-commutative gauge theories

Зачем?

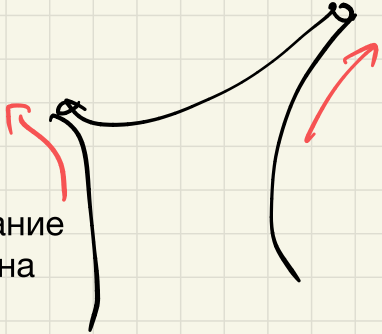
интерпретация пространства $\overline{\mathcal{M}}_k(n)$ как пространства модулей

$\mathcal{M}_K(n)$ framed (оснащённые) инстантоны

$G = SU(n)$ на $\mathbb{R}^4$,

$F_A \to 0$ на ∞

$/g_\infty \quad g \to 1$ на ∞

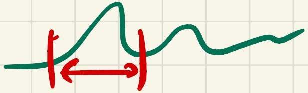


схлопывание инстантона в точку

убегание на ∞

$F_A^+ = 0$

инв. $\mathrm{Conf}(\mathbb{R}^4 \cup \infty)$

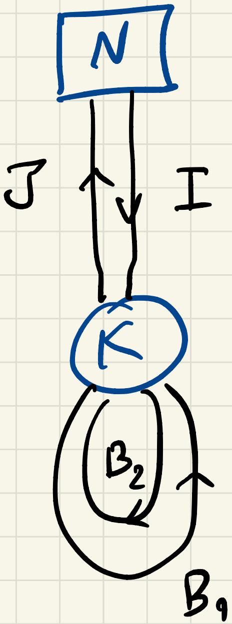


$\rho = \mathrm{Tr}\, F_{\mu\nu}^2$

ADHM

$$\mathcal{M}_k(n) = \left\{ (B_1, B_2, I, J)^s , \ \mu_{\mathbb{C}} = 0, \ \mu_{\mathbb{R}} = 0 \right\} / U(k)$$

$$U(n) \curvearrowright \mathcal{M}_k(n)$$



$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ$$

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J$$

$$g \cdot (B_1, B_2, I, J) =$$

$$= (g^{-1} B_1 g, \ g^{-1} B_2 g, \ g^{-1} I, \ Jg)$$

$(\cdot \ldots)$ стабильна $\Longleftrightarrow \{ g \cdot (\ldots) = (\ldots)$

$\Longleftrightarrow g = 1 \}$

$$\widetilde{\mathcal{M}_k(n)} = \{(B_1, B_2, I, J),\ \mu_{\mathbb{C}}=0,\ \mu_{\mathbb{R}}=0\} / \underline{U(k)} \swarrow$$

Uhlenbeck

$$\widetilde{\mathcal{M}_k(n)} = \overbrace{\mathcal{M}_k(n) \cup \mathcal{M}_{k-1}(n) \times \mathbb{R}^4 \cup \dots \cup \boxed{\mathcal{M}_{k-\ell}(n)} \times S^\ell \underline{\mathbb{R}^4} \cup}^{} $$

$$\cup\ S^k\mathbb{R}^4$$

$k-\ell$

$$\mathrm{Stab} = U(1)^\ell$$

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{diag} \end{array}\right) = g$$

$$B_\alpha \left(\begin{array}{c|c} b_\alpha & 0 \\ \hline 0 & \mathrm{diag}_\alpha \end{array}\right) \begin{array}{l} k-\ell \\ \ell \end{array}$$

$$I = \left(\begin{array}{c} i \\ \hline 0 \end{array}\right) \begin{array}{l} k-\ell \\ \ell \end{array}$$

$$J = \begin{array}{cc} k-\ell & \ell \\ \left(j\right. & \left.\mid 0\right) \end{array}$$

$\alpha = 1, 2$

$$\underset{\uparrow}{S^\ell}\mathbb{C}^2 \cong \underset{\uparrow}{S^\ell}\mathbb{R}^4$$

$U(k-\ell) \times U(\ell)$

$$\underline{\underline{S(\ell)}} \subset U(\ell)$$

$$\Rightarrow \text{Stability}$$

$$\overline{\mathcal{M}}_k(n) = \left\{ (B_1, B_2, I, J) \;\middle|\; \mu_{\mathbb{C}} = \zeta_{\mathbb{C}} \cdot 1,\ \mu_{\mathbb{R}} = \underline{\zeta_{\mathbb{R}}} \cdot 1 \right\} \Big/ U(k)$$

$$\vec{\zeta} = (\zeta_{\mathbb{R}}, \mathrm{Re}\,\zeta_{\mathbb{C}}, \mathrm{Im}\,\zeta_{\mathbb{C}}) \in \mathbb{R}^3$$

$$U_R \xrightarrow{\text{с помощью}} \vec{\zeta} = (\zeta_{\mathbb{R}}, 0, 0)$$

Симметрии ADHM данных

трансляции $\mathbb{R}^4$ $\quad B_\alpha \longmapsto B_\alpha + b_\alpha \cdot 1_k \quad (b_1, b_2) \in \mathbb{C}^2$

вращения

$$\mathrm{Spin}(4) = SU(2)_L \times SU(2)_R$$

$$(U_L\ ,\ U_R)$$

$$U_L\,(B_1, B_2, I, J) = \left(\alpha B_1 + \beta B_2,\ -\bar{\beta} B_1 + \bar{\alpha} B_2,\ I,\ J\right)$$

Если $I = J = 0$ $\quad \mu_{\mathbb{C}} = \mu_{\mathbb{R}} = 0$

$$\Rightarrow \quad [X_i, X_j] = 0 \qquad 1 \le i, j \le 4$$

$$B_1 = X_1 + i X_2 \qquad X_\mu = X_\mu^\dagger$$

$$B_2 = X_3 + i X_4$$

$$g(X_\mu) = (g^{-1} X_\mu g)$$

$$g \in S(k)$$

$$(x_\mu^i) \mapsto (x_\mu^{(\sigma(i))})_{\alpha=1}^{k} \qquad \sigma \in S(k)$$

$Spin(4) = SU(2)_L \times SU(2)_R$

$(U_L \ , \ U_R)$

$U_L\,(B_1, B_2, I, J) =$

$= (\alpha B_1 + \beta B_2,\ -\bar{\beta} B_1 + \bar{\alpha} B_2,\ I,\ J)$

$$U_L = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$\mu_{\mathbb{C}}\,,\ \mu_{\mathbb{R}}$ — синглеты $SU(2)_L$

$$U_R(B_1, B_2, I, J) = (\alpha B_1 + \beta B_2^{\dagger},\ -\beta B_1^{\dagger} + \alpha B_2,\ \alpha I + \beta J^{\dagger},\ -\beta I^{\dagger} + \alpha J)$$

$(\mu_{\mathbb{R}}, \mu_{\mathbb{C}}, \bar{\mu}_{\mathbb{C}})$ — триплет

$SU(2)_R$

$\begin{pmatrix} I \\ J^{\dagger} \end{pmatrix}$ — дублет

$SU(2)_R$

Напоминалка

$$Spin(4) = SU(2)_L \times SU(2)_R$$

"γ- матрицы"

$S_L \otimes V \to S_R$

$S_R \otimes V \to S_L$

$(0,0)$ скаляр

$S_L = (\frac{1}{2}, 0)$ левый спинор

$S_R = (0, \frac{1}{2})$ правый спинор

$V = (\frac{1}{2}, \frac{1}{2})$ вектор

$\Lambda^{2-} V = (1, 0)$ антисамодуальные 2 формы

$\Lambda^{2+} V = (0, 1)$ самодуальные 2 формы

$\omega^{+} = 0$

$\omega^{-} = 0 \longleftrightarrow \vec{\xi}$

$\overline{\mathcal{M}_K(n)}$ зависит от выбора $\omega \in \Omega^{2,+}(\mathbb{R}^4)$

$(B_1, B_2, I, J) \in \mathcal{M}_K(n) \Longrightarrow A_\mu \quad , F_A^+ = 0$

в $\mathbb{R}^4$

$$D_z^+ = \begin{pmatrix} B_1 - z_1 \cdot 1 & B_2 - z_2 \cdot 1 & I \\ -B_2^+ + \bar{z}_2 \cdot 1 & B_1^+ - \bar{z}_1 \cdot 1 & -J^+ \end{pmatrix} : K \otimes S_L \oplus N \to K \otimes S_R$$

$$z = (z_1, z_2) \in \underset{\displaystyle \mathbb{R}^4}{\overset{\displaystyle \mathbb{C}^2}{\wr\wr}}$$

$$S_{L,R} \cong \mathbb{C}^2$$

$$\mathcal{D}_z^{\dagger}\mathcal{D}_z : K\otimes S_R \to K\otimes S_R$$

$$\parallel$$

$$\Delta\otimes 1_2$$

$$= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$B=C=0$

$A=D=\Delta\cdot 1$

$$\nearrow$$

$\mu_{\mathbb{C}}=0,\ \mu_{\mathbb{R}}=0$

$$\Delta : K \to K$$

$$\Delta = (B_1-z_1)(B_1^{\dagger}-\bar{z}_1) + (B_2-z_2)(B_2^{\dagger}-\bar{z}_2) + II^{\dagger}$$

$$= (B_1^{\dagger}-\bar{z}_1)(B_1-z_1) + (B_2^{\dagger}-\bar{z}_2)(B_2-z_2) + J^{\dagger}J$$

ADHM

stabiliy $\Longleftrightarrow$ Δ^{-1} существует

$$\Delta = (B_1 - z_1)(B_1^\dagger - \bar{z}_1) + (B_2 - z_2)(B_2^\dagger - \bar{z}_2) + I I^\dagger =$$

$$= (B_1^\dagger - \bar{z}_1)(B_1 - z_1) + (B_2^\dagger - \bar{z}_2)(B_2 - z_2) + J^\dagger J$$

$$\Delta \xi = 0$$

$$\Rightarrow \quad J\xi = 0$$

$$I^\dagger \xi = 0$$

$\exists\, \xi \in K.$

$$B_i \xi = z_i \xi$$

$$B_i^\dagger \xi = \bar{z}_i \xi$$

can only happen at some point

stabilizer at least U(1)

$$g = 1 + (e^{i\phi} - 1)\xi\xi^\dagger \in U(K)$$

$\mathcal{E}_z = \ker \mathcal{D}_z^+$ —

постоянный ранг

$rk = n = \dim_{\mathbb{C}} N$

$$\begin{tikzcd} \mathcal{D}_z^+ : K\otimes S_L \oplus N \arrow[r] & K\otimes S_R \arrow[l, bend right] \end{tikzcd}$$

$\mathcal{E}_z = \dfrac{\ker \alpha_z}{\operatorname{im}\beta_z}$

$\ker\Delta = 0 \Rightarrow \mathcal{D}_z^+$ имеет максимальный ранг $\forall z$

$\mu_{\mathbb{C}} = \underline{\alpha_z\beta_z} = 0$

$\ker$

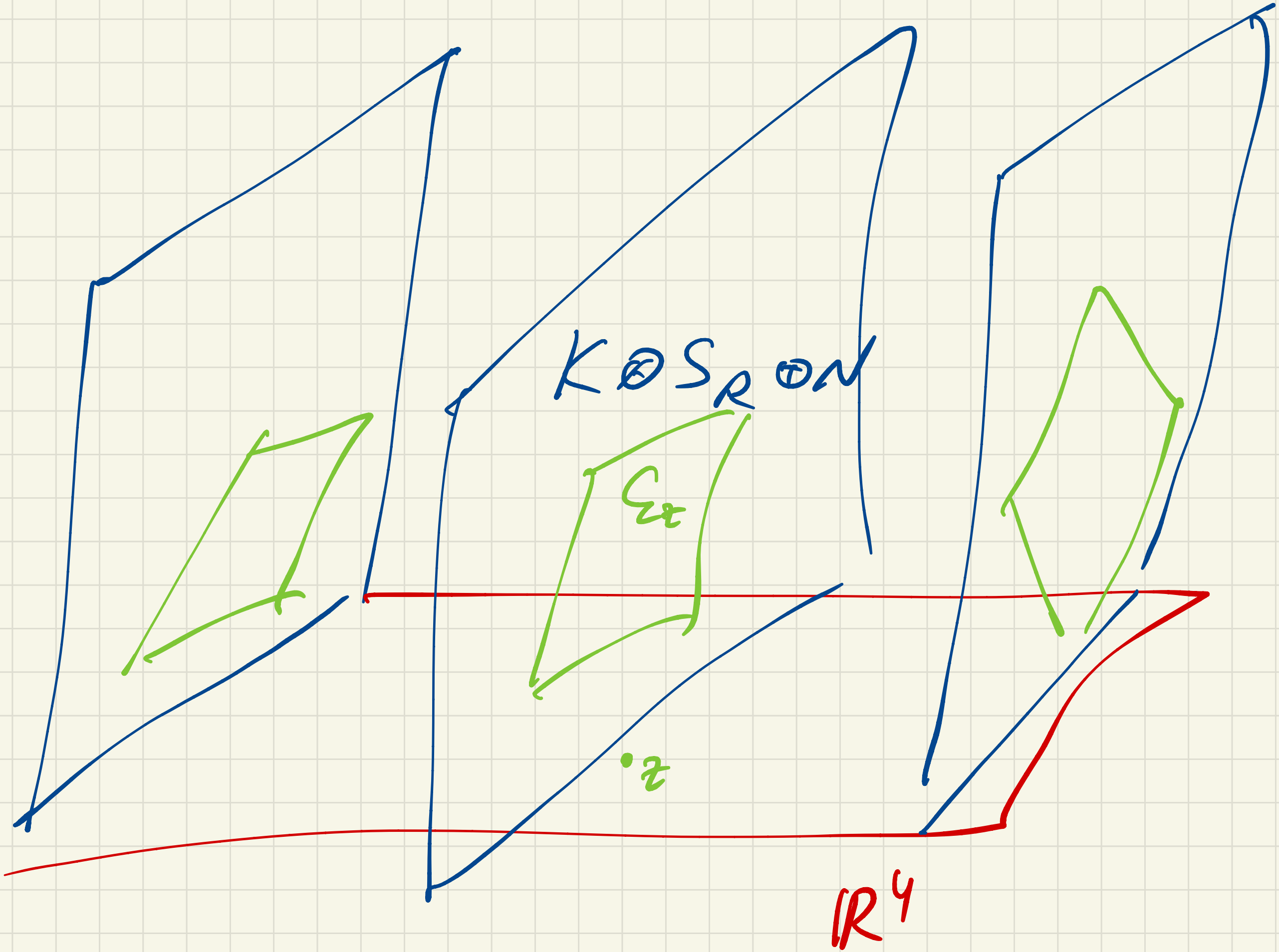
$$\mathcal{D}_z^+ = \begin{pmatrix} \alpha_z \\ \beta_z^+ \end{pmatrix}$$

$\alpha_z = \begin{pmatrix} B_1 - \underline{z_1} & B_2 - \underline{z_2} & I \end{pmatrix} :$

$K\otimes S_R \oplus N \to K$

$$\beta_z = \begin{pmatrix} -B_2 + \underline{z_2} \\ B_1 - \underline{z_1} \\ -J \end{pmatrix}$$

$\beta_z : K \to K\otimes S_L \oplus N$


$K \otimes S_R \oplus N$
ε_z
z
$\mathbb{R}^4$

инстантонная связность на $\mathcal{E}$

$=$ проекция тривиальной связности на $K \otimes S_R \oplus N$

$$\Psi_z : N \to \underline{K \otimes S_R \oplus N}$$

$$\operatorname{im} \Psi_z = \mathcal{E}_z = \ker \mathcal{D}_z^\dagger$$

$$\Pi_z = \Psi_z \Psi_z^\dagger \quad . \quad \Pi_z^2 = \Pi_z$$

$$\Pi_z^\dagger = \Pi_z$$

$$\begin{cases} \mathcal{D}_z^\dagger \Psi_z = 0 \\ \Psi_z^\dagger \Psi_z = 1_N \end{cases}$$

$$\Psi_z \sim \Psi_z \, g^{-1}_{(z)}$$

$$g(z, \bar{z}) \in U(n)$$

$$\to A^\dagger = -A$$

$$\operatorname{Tr} A = 0$$

$$A_\mu = \Psi_z^\dagger \partial_\mu \Psi_z \quad : \quad N \to N$$

$$A_\mu \in \operatorname{Lie} SU(n)$$

$$\Pi_z = 1 - \mathcal{D}\frac{1}{\mathcal{D}^\dagger\mathcal{D}}\mathcal{D}^\dagger = \Psi\Psi^\dagger$$

ebaaa
упор

$$\mathcal{D}_z^\dagger d\Psi = -(d\mathcal{D}_z^\dagger)\Psi$$

$$\mathcal{D}^\dagger\Psi = 0$$

$$\xi \in K \otimes \underline{S_L} \oplus N$$

$$\xi \in \mathcal{E}_z \qquad \mathcal{D}_z^\dagger\xi = 0$$

$$\Pi_z\xi = \xi$$

$$\boxed{F_A = d\mathcal{D}\left(\Psi^\dagger\frac{1}{\Delta}\Psi\right)d\mathcal{D}^\dagger} \Rightarrow \boxed{F_A^+ = 0}$$

$$\Psi^\dagger\Psi = 1$$

$$A = \Psi^\dagger d\Psi \Rightarrow F_A = dA + A^2 = d\Psi^\dagger d\Psi + \underline{\Psi^\dagger d\Psi\,\Psi^\dagger d\Psi}$$

$$= d\Psi^\dagger(1 - \Psi\Psi^\dagger)d\Psi = d\Psi^\dagger\mathcal{D}\frac{1}{\Delta}\mathcal{D}^\dagger d\Psi \qquad = -d\Psi^\dagger\cdot\Psi$$

$\zeta \neq 0$

$\widehat{\mathbb{C}}^2$

X

$\zeta \neq 0$ конструкция ADHM работает, если $[z_1, z_2] = -\zeta_{\mathbb{C}}$

$$[z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -\zeta_{\mathbb{R}}$$

то есть мы теперь работаем над некоммутативным пространством $\mathbb{R}^4_\theta$

ассоциативная алгебра $\mathcal{A}$ $=$ "алгебра функций"

представление $\mathcal{A} \sim \mathcal{H}$

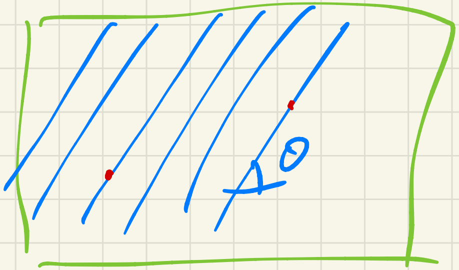
$xy = qyx$

некоммутативный тор

$\simeq \quad T^2 /$ иррациональная намотка

θ

$q = e^{2\pi i \theta}$



X — топологическое пространство

$C(X) = \mathcal{A}$ коммутативная ассоциативная алгебра

π_x неприводимое представление $\mathcal{A} \longleftrightarrow x \in X$

$$f \in \mathcal{A} \longmapsto f(x) \in \mathbb{C}$$

$$I_x = \{ f \mid f(x) = 0 \}$$

$$\mathbb{C} \simeq \pi_x = A / I_x$$

$$\begin{tikzcd}
\mathcal{A} = C(S^2) = \mathbb{C}[X, Y, Z] \big/ X^2 + Y^2 + Z^2 = 1 \arrow[d] \\
\mathcal{A}_\hbar = U(sl_2) \big/ X^2 + Y^2 + Z^2 = 1
\end{tikzcd}$$

$$[X, Y] = \hbar Z$$

$$[Y, Z] = \hbar X$$

$$[Z, X] = \hbar Y$$

$$\mathbb{C}^d$$

$$d = 2j + 1$$

$$C = j(j+1)$$

$\mathbb{R}^4_\theta$ = алгебра Гейзенберга, порожд. $x^\mu = (x^\mu)^+$

$$[x^\mu, x^\nu] = \theta^{\mu\nu} \cdot 1$$

$\theta = -\theta^t$ $\qquad \theta \in \Lambda^2 \mathbb{R}^4 \qquad \theta \to \begin{pmatrix} 0 & \theta_1 & 0 & \\ -\theta_1 & 0 & & \\ & 0 & 0 & \theta_2 \\ & & -\theta_2 & 0 \end{pmatrix}$

Точки ?

неприводим. представлен.

$a_1, a_2, a_1^+, a_2^+ \qquad [a_i, a_j] = 0$

$[a_i, a_j^+] = \theta_i \delta_{ij} \qquad \underline{\theta_i > 0}$

комм.

$\downarrow$

$z = (z_1, z_2) \in \mathbb{C}^2$

$\mathcal{H}_z = \mathbb{C}[a_1^+, a_2^+] \quad \psi_z$

$a_i \psi_z = z_i \psi_z$

$$\boxed{\psi_z = e^{\sum_i \frac{z_i a_i^+}{\theta_i}} |0,0\rangle}$$

У y $\mathcal{A} = \mathbb{R}^4_\theta$ имеется

коммутативная алгебра

симметрий $\mathbb{R}^4$

$$x^\mu \mapsto x^\mu + c^\mu \cdot \mathbb{1}$$

$$c \in \mathbb{R}^4$$