Lecture 8
Friday, November 19, 2021 8:31 AM

Commuting ODE

Recall KP hierarchy

- Phase space: $\mathcal{L} = \partial + \sum_{i=1}^{\infty} u_i(x)\, \partial^{-i}$ $\quad \partial = \partial_x$
- Flows $\quad \partial_i \mathcal{L} = [\mathcal{L}_+^i, \mathcal{L}]$ $\quad$ Sato form
- Commutativity of flows $\Rightarrow$

$$[\partial_i - B_i, \partial_j - B_j] = 0 \qquad B_j = \mathcal{L}_+^j$$

KP hierarchy in Zakharov-Schabat form

- BA function ψ depends on $\{\Gamma, P, z, D\}$ gives solution of the KP hierarchy

$$\psi \to \mathcal{L} \qquad \psi = e^{z^{-1}x + \cdots}\left(1 + \sum \xi_s z^s\right)$$

$\mathcal{L}$ is the unique p.d.o s.t

$$\mathcal{L}\psi = k\psi \quad \checkmark$$

- Lax (Gelfand-Dickey) reduction $\checkmark$

$$\checkmark \quad \mathcal{L}^n = \mathcal{L}_+^n = L = \partial^n + \sum_{i=1}^{n-2} u_i(x)\, \partial^i \quad \checkmark$$

$$\checkmark \quad \partial_i L = [L_+^{i/n}, L] \quad \checkmark$$

$$\checkmark \quad \partial_n L = 0$$

Q: what is this reduction in terms of the (curves ...) algebraic-geometrical data defining the BA function defining the BA function

Let Γ be a curve on which there is a function with pole of order n at P

$\Rightarrow$ define $\underset{\sim}{z} = E^{-1/n} \Rightarrow \; = z^{-n}$

$$\underline{\psi(t,p) = \tilde{\psi}(\hat{t},p)\, e^{E t_n}} \quad \checkmark$$

$$\hat{t} = (t_1, \dots, t_{n-1}, \underset{\sim}{t_n = 0}, \dots)$$

$$\partial_{t_n}\psi = \underline{E}\,\psi \Rightarrow \underline{(\partial_n - \mathcal{L})\psi = 0} \Rightarrow \underline{\mathcal{L}\tilde{\psi} = E\tilde{\psi}}$$

$\underline{Ex}$ $n=2, n=3$

Two reductions $\quad \underline{\partial_n \mathcal{L} = 0} \quad \underline{\partial_m \mathcal{L} = 0}$

or a linear combination $\quad \underset{\sim}{\partial_m + \sum_{i=1}^{m-1} c_i \partial_i}$ $\qquad \mathcal{L}^n = L_1 L_2^{-1}$

$\Rightarrow$ Commuting ODOs

$$[L_n, L_m] = 0 \qquad L_n = \partial^n + \sum_{i=0}^{n-2} u_i \partial^i \quad \checkmark$$

$$L_m = \partial^m + \sum_{j=0}^{m-1} v_j \partial^j$$

Lemma (Burchnal-Chaundy) $\exists$ polynomial R s.t

$$R(L_n, L_m) = 0 \qquad n = n'r \quad m = m'r$$

$$(n', m') = 1$$

Proof

$$L_n y = E y \qquad y \in \Lambda(E) \quad \dim \Lambda(E) = n$$

$$L_m\big|_{\Lambda(E)} = L_m(E) : \Lambda(E) \to \Lambda(E)$$

$$L_m\Big|_{\Lambda(E)} = L_m(E) : \Lambda(E) \to \Lambda(E)$$

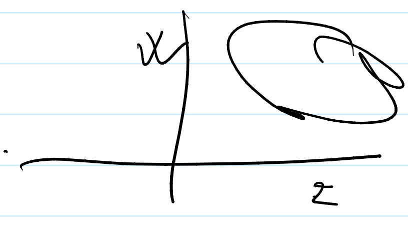
$$R(E,w) = \det\left(w \cdot \mathbb{1} - L_m(E)\right) = 0$$

$$\underset{\sim}{R(L_n, L_m)}\, \underset{\sim}{y} = 0$$

$y \in \Lambda(E)$ linear independent for $E \neq E'$

$$\Downarrow$$

$$R(L_n, L_m) = 0$$



$$R(E,w) = w^n + \sum_{i=0}^{n-1} r_i(E)\, w^i$$

$$= \underline{w^n + E^m + \sum_{ni+mj \le mn-1} r_{ij}\, E^i w^j}$$

$$L_n\, c_i(x, E, x_0) = E\, c_i(x, E, x_0) \qquad c_i \in \Lambda(E)$$

$$i = 0, \dots, n-1$$

$$\partial^j c_i\Big|_{x_0} = \delta_{ij}$$

· Claim $L_m^{i,j}(E)$ are polynomials in E

$$L_m^{i,j} = \partial^j \left(L_m c_i\right)\Big|_{x=x_0}$$

$$\partial^j c_i \qquad j \le n-1$$

$$\partial^n c_i\Big|_{x=x_0} = \left(-\sum \underset{\sim}{u_k}\, \partial^k\right) c_i + E c_i$$

$\Rightarrow$ $R(E, w)$ is polynomial in E

Let $h_j(p, x_0)$ be an eigenvector $p \in (E, w) \in \Gamma$ of $L_m(E)$

$$L_m^{ij}(E)\, h_j = w\, h_i$$

$$\psi = \sum h_i(p, x_0)\, \underbrace{c_i(x, E, x_0)}$$

$$L_n \psi = E\psi \qquad \underline{L_m \psi = \underline{w}\, \psi}$$

$$L_m\left(\sum h_i c_i\right) = \sum h_i \left(L_m c_i\right) = \sum h_i L_m^{ij} c_j$$

$$= w \sum_j h_j c_j = w \psi$$

ψ is a meromorphic function on Γ with poles independent on x (but depending on x_0)

$$h_i = \frac{\Delta_i\left(L_m^{ij} E - w \cdot 1\right)}{\Delta_1\left(\qquad\qquad\right)}$$

<u>Compactification</u>

$$R(E, w) = w^n + E^m + \sum r_{ij}\, w^i E^j = 0$$

$$\exists!\ \psi(x, z, x_0) = \left(1 + \sum_{s=1}^{\infty} \xi_s(x) z^s\right) e^{z^{-1}(x - x_0)}$$

$$L_n \psi = E\psi \qquad E = z^{-n}$$

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$$\psi(x_0, z, x_0) = 1 \iff \xi_s(x = x_0, x_0) = \underline{\underline{0}}$$

$$n\,\xi'_{s+n-1} = F(\xi_0, \ldots \xi_{s+n-2})$$

$$n = 2 \qquad 2\xi'_{s+1} = \xi''_s + u\xi_s$$

$\underline{\underline{\xi_x}}$

$$\Rightarrow \quad \tilde{\psi} = e^{z^{-1}(x-x_0)}\left(\underbrace{\sum_{s=-N}^{\infty} \tilde{\xi}_s\, z^s}\right)\Big)$$

$$L_n \tilde{\psi} = E\tilde{\psi} \Rightarrow \tilde{\psi} = \psi \cdot A(z)$$

$$A(z) = \sum_{s=-N}^{\infty} a_s\, z^s$$

$$L_m \psi(x, z, x_0) = a(z)\,\psi(x, z, x_0)$$

$$a(z) = z^{-m} + \sum_{s=-m+1}^{\infty} a_s z^s$$

$$\boxed{R(E, w) = \prod_{k=0}^{n-1} \big(w - a(\varepsilon_k z)\big)}$$

$$= w^n + (z^{-m})^n + \ldots = w^n + \underline{E}^m +$$

$\tilde{\Lambda}(E)$ linear space over the field of Laurent series of z spaned

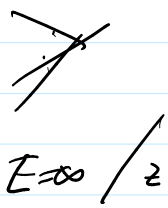
$$\underline{\psi(x, \varepsilon_k z, x_0)} \qquad \varepsilon_k^n = 1$$

$$L\psi(x, \varepsilon_k z, x_0) = E\,\psi(x, \varepsilon_k z, x_0) \qquad E = z^{-n}$$

ε^m

ε_k^m

$$w_k(E) = \varepsilon_K^m \, E^{m/n} + \dots$$



$E=\infty$ / $z=0$

If $(n,m)=r$ then there might be the case when eqs

$$L_n \psi_i = E\psi_i \qquad L_m \psi_i = w\psi_i$$

have r linear independent sol.

$$R(w,E) = \tilde{R}^r(w,E)$$

$$\tilde{R} = 0$$

Ex Show that ψ has g poles on $\Gamma \setminus P_0$ (if Γ is smooth)