Lecture 7

Friday, November 5, 2021 8:54 AM

KP, 2D Toda hierarchies

Recall 1-pt BA function depends on $t = (t_1, t_2, \dots)$ one "set" of times

$$\psi_D(t,p) = e^{\sum t_i \Omega_i(p)} \times$$
$$\times \frac{\Theta(A(p) + \sum U_i t_i + Z)}{\Theta(A(p) + Z)} \cdot \frac{\Theta(A(p_1) + Z)}{\Theta(A(p_1) + \sum U_i t_i + Z)} \qquad Z = -\sum A(\gamma_s) + K$$

Uniqueness of BA function

Let $\tilde{\psi}$ be a BA function

$\Rightarrow \tilde{\psi}/\psi$ is a meromorphic function with poles at zeros of ψ

RR $\quad h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) =$

$$= d - g + 1$$

$$\dim \mathcal{L}(D) \geq d - g + 1$$

$$\deg D \geq g$$

def D nonspecial if inequality is the equality

Ex $\quad \Gamma = CP^1 \qquad \psi = e^{\sum t_i z^{-i}}$

$$\left(\partial_{t_n} - \partial_{t_1}^{n}\right)\psi = 0 \qquad t_1 = x$$

$\left(\partial_{t_n} - \partial_{t_1}\right) \psi = 0 \qquad t_1 = x$

$\underset{\approx}{\mathcal{E}x}$ $\quad \forall n \quad \exists! \; L_n = \partial_{t_1}^n + \sum_{i=0}^{n-2} \underline{\underline{u_{n,i}}}(t)\, \partial_{t_1}^i$

s.t $\quad \left(\partial_{t_n} - L_n\right) \psi = 0$

Step 1 $\quad \exists! \; L_n \quad \left(\partial_{t_n} - L_n\right) \psi = O(z^{\cdot}) \psi$

Step 2 $\quad$ congruence $\Rightarrow$ equality $\quad \left(\partial_x^a + \dots\right) \psi = \left(\underline{\underline{z}}^{-a} \dots\right)$

$\underline{\underline{\text{Cor}}}$ $\quad \underbrace{\left[\partial_n - L_n, \partial_m - L_m\right]} = 0$

$\underline{\underline{\mathcal{E}x}}$

$n = 2, \; m = 3$

$L_2 = \partial_x^2 - u \qquad L_3 = \partial_x^3 - \frac{3}{2} u \partial_x + w$

system of two equations for two unknown function u, w

After exclusion of w we get KP for $u(x, t_2, \dots)$

$u = 2 \partial_x \xi_1$

$$\psi = \frac{\theta\left(A(p) + \sum U_i t_i + z\right) \theta\left(A(p_1) + z\right)}{\theta\left(A(p) + z\right) \theta\left(A(p_1) + \sum U_i t_i + z\right)} e^{\dots}$$

$\mathcal{E}x$ $\quad$ Bilinear R relation $\qquad$ prove

$\underbrace{A(p) = A(p_1) - \sum \frac{1}{i} U_i z^i}$

$\xi_1 = -\partial_x \ln \theta\left(A(p_1) + \sum U_i t_i + z\right) + c$

$$u = -2\partial_x^2 \ln \Theta\left(A(p_1) + \sum V_i t_i + Z\right)$$

$$\left[\partial_{t_n} - L_n, \partial_{t_m} - L_m\right] = 0$$

For each n, m # equations = # coeff

That is called KP hierarchy in Zakharov - Schabat form

Sato - form

Phase space =

The space of pseudo-differential operators of the form

$$\mathcal{L} = \partial_x + \sum u_i \partial_x^{-i} \qquad u_i(x)$$

$$\partial_x^{-1} \cdot w = w\partial_x^{-1} - w_x \partial_x^{-2} + w_{xx}\partial_x^{-3}.$$

Flows

$$\partial_{t_n} \mathcal{L} = \left[\mathcal{L}_+^n, \mathcal{L}\right] \qquad u_i(x, t)$$

$\mathcal{L}_+^n$ - diff operator such

$$\mathcal{L}^n - \mathcal{L}_+^n = \mathcal{L}_-^n = O(\partial_x^{-1})$$

$$\mathcal{L}^n = \partial^n + \cdots \quad w_1\partial + w_0 + \cdots$$

$$\mathcal{L}^n = \underbrace{\partial_x^n + \cdots \; w_1 \partial_x + w_0}_{\mathcal{L}^n_+} + \cdots$$

$$\partial_{t_n} \mathcal{L} = \sum \partial_{t_n} u_i \, \partial_x^{-i} \qquad O(\partial_x^{-1})$$

$$[\mathcal{L}^n, \mathcal{L}] = 0 \Rightarrow [\mathcal{L}^n_+, \mathcal{L}] = -\underbrace{[\mathcal{L}^n_-, \mathcal{L}]}$$

$$\partial_{t_n} u_i = F_i(u, \ldots u')$$

<u>Lemma</u> Flows commute with each other

$$\partial_{t_n} \mathcal{L} = [\mathcal{L}^n_+, \mathcal{L}] \Rightarrow [\underline{\partial_{t_n} - L_n}, \underline{\mathcal{L}}] = 0$$

$$L_n := \mathcal{L}^n_+$$

$$0 = [\partial_{t_n} - \mathcal{L}^n_+, \partial_{t_m} - \mathcal{L}^m_+] \quad \text{- diff operator}$$

$$[\partial_{t_n} - \mathcal{L}^n_+, \partial_{t_m} - \mathcal{L}^m_+] \hat{=} O(\partial^{-1})$$

$$-\partial_{t_n} \mathcal{L}^m_+ + \partial_{t_m} \mathcal{L}^n_+ + [\mathcal{L}^n - \mathcal{L}^n_-, \mathcal{L}^m - \mathcal{L}^m_-] \hat{=}$$

$$-\cancel{\partial_{t_n} \mathcal{L}^m} + \cancel{\partial_{t_m} \mathcal{L}^n} - \cancel{[\mathcal{L}^n, \mathcal{L}^m_-]} - \cancel{[\mathcal{L}^n_-, \mathcal{L}^m]} \hat{=} O(\partial^{-1})$$

$$\partial_{t_n} \mathcal{L}^m = [\mathcal{L}^n_+, \mathcal{L}^m]$$

$$\partial_{t_1} \mathcal{L} = [\mathcal{L}_+, \mathcal{L}] \qquad \Rightarrow x = t_1$$

$$\left[\partial_x, w\,\partial_x^k\right] = w_x\,\partial_x^k$$

$n=2$ $\quad \mathcal{L}_+^2 = \partial^2 + u \qquad u = 2u_1$

$$\mathcal{L}_+^3 = \partial^3 + 3u_1\partial + w$$

$$L = \partial_{t_2} - \partial_x^2 + \underline{u(x,t_2)}$$

Reductions

Consider the subspace of $\mathcal{L}$

s.t. $\quad \mathcal{L}_+^n = \mathcal{L}^n \qquad \underbrace{\mathcal{L}_-^n = 0}$

This space is identified with

$$\mathcal{L}^n = \underbrace{L_n = \partial_x^n + \sum v_i \partial_x^i} \qquad v_i = v_i(x)$$

$$\mathcal{L} = L_n^{1/n}$$

$$\partial_{t_n}\mathcal{L} = 0$$

$$\underline{\underline{\partial_m L_n}} = \left[L_{n+}^{m/n}, L_n\right]$$

Consider locus of stationary points of two flows

$$\left[L_{\cdot}^{m/n}\ L_n\right] = 0$$

$$\left[L_{n+}^{m/n}, L_n \right] = 0$$

Classify commutative ring of ODO

<u>Lemma</u> (Burchnall-Chaundy) Let

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i \partial_x^i$$

$$L_m = \partial_x^m + \sum_{j=0}^{m-2} v_i \partial_x^i$$

be diff operators s.t

$$[L_n, L_m] = 0$$

$$\Longrightarrow \exists \; R(L_n, L_m) = 0$$