Lecture 6

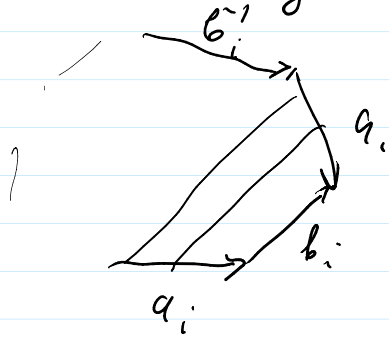
Friday, October 29, 2021 9:00 AM

Baker-Akhiezer functions

Let Γ be a complex curve

Let a_i, b_i be cycles on Γ $i=1,\dots,g$

$$a_i \circ a_j = 0 \quad b_i \circ b_j \quad a_i \circ b_j = \delta_{ij}$$



$$\partial \Pi = a_i b_i a_i^{-1} b_i^{-1} \dots = 1$$

- ω_i basis of holom. diff

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad B_{ij} = \oint_{b_i} \omega_j \text{ matrix of}$$

b-periods $\qquad \underline{B_{ij} = B_{ji} \quad \operatorname{Im} B > 0}$

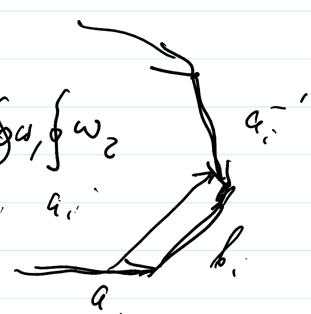
examples of Bilinear Reaman identities

Let ω_1 ω_2 be two holomorphic diff

$$\omega_1 = d\tau$$

Consider

$$0 = \oint_{\partial \Pi} \tau \omega_2 = \oint_{b_i} \omega_1 \oint_{a_i} \omega_2 - \oint_{a_i} \omega_1 \oint_{b_i} \omega_2$$



Ex Check that $\Longrightarrow B_{ij} = B_{ji}$

[illegible]

$$0 < \int \omega \wedge \bar{\omega} = \oint_{\partial \Pi} v \bar{\omega}$$

$$\theta(z \mid B) = \sum \ \ldots$$ theta-function

<u>Jacobi inversion theorem</u>

$$A : \Gamma \longrightarrow C_p^g \to \mathbb{C}^g / e_i, B_i = \mathcal{J}(\Gamma)$$ Abel-Jacobi map

$$A_k(p) = \int_{p_0}^{p} \omega_k$$

$\theta(A(p) + Z) \to$ multivalued holomorphic function

If $\underline{\theta(A(p) + Z)} \not\equiv 0 \implies$ it has g ~~zeros~~

$$\theta(A(\gamma_s) + Z) = 0 \qquad s = 1, \ldots, g$$

$$\underline{Z = -\sum_{s=1}^{g} A(\gamma_s) + K}$$

$\underline{\underline{Ex}}$ Show $\theta(A(p) \underbrace{- A(q) + \tilde{Z}})$

$$\tilde{Z} = Z + A(\gamma_1)$$

has zeros at $q, \gamma_2, \ldots, \gamma_g$

$\underline{Ex}$ $\gamma_1, \ldots, \gamma_g, \gamma_{g+1}$ q

f_q meromorphic function with poles at γ_s and zero at q

$$\frac{\theta(A(p) - A(q) + \tilde{Z})\, \theta(A(p) + Y)}{}$$

$$f_p = \frac{\theta(A(p) - A(q) + \tilde{Z})\,\theta(A(p)' + Y)}{\theta(A(p) + Z)\quad \theta(A(p) - A(\gamma_{g+1}) + \tilde{Z})}$$

$\gamma_1 \ldots \gamma_g$ $\qquad\qquad$ $\gamma_2, \ldots \gamma_g, \gamma_{g+1}$

$$\cancel{\tilde{Z}} + Y = Z + \overset{\gamma_2, \ldots \gamma_g}{\cancel{\tilde{Z}}} - A(\gamma_{g+1})$$

Ex $\quad$ Let $\quad \gamma_1 \ldots \gamma_{g+r-1} \quad \dim \mathcal{L}(\gamma.) = g+r-1$

Find θ-funct. formula $\quad -g+1 = r$

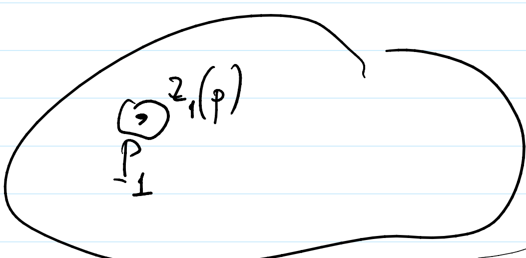
for $r_i(p)$ that has poles at γ_s

~~an~~ $r_i(q_j) = \delta_{ij} \quad q_j \quad j = 1, \ldots, r$

Basic $\quad$ B A function $\quad \psi_{\mathcal{D}}(t, p) = \psi(t, p)$

1° ψ has poles on

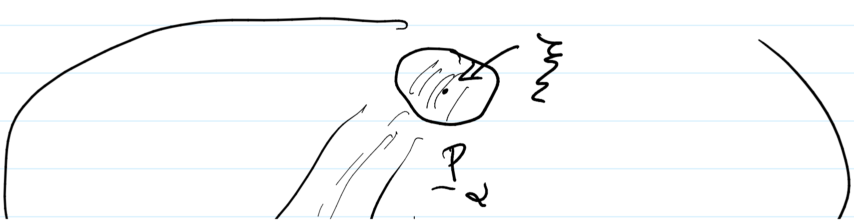
$\Gamma \setminus P_\alpha$ at $\gamma_1 \ldots, \gamma_g$

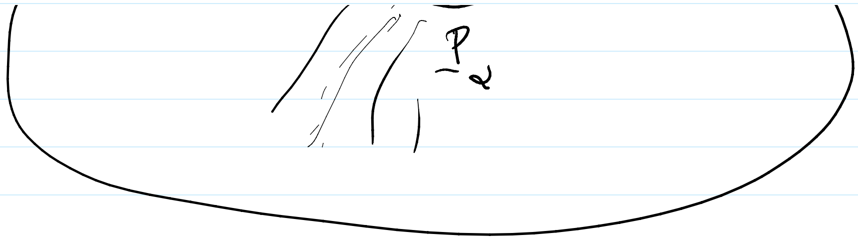


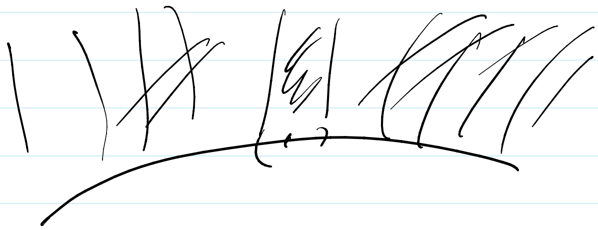
2° = near P_α

$$\psi = e^{\sum t_{\alpha,i} z_\alpha^{-i}} \left(\sum \xi_s(t) z_\alpha^s \right)$$

ξ







$$\psi = e^{\sum_\alpha \sum_i t_{\alpha i} \Omega_{\alpha i}(p)} \frac{\theta(A(p) + X + Z)}{\theta(A(p) + Z)}$$

$\gamma_1 \dots \gamma_g$

$d\Omega_{\alpha, i}(p)$ – normalized meromorphic dif with pole at P_α of the form $dz_\alpha^{-i}(p)$ + holom

$$\Omega_{\alpha, i}(p) = z_\alpha^{-i}(p) + O(1)$$

$$\oint_{a_i} d\Omega_{\alpha, i} = 0$$

$$\sum t_{\alpha i} \oint_{b_k} d\Omega_{\alpha i} - 2\pi i X_k = 0$$

$$X = \sum \sum t_{\alpha i} U_{\alpha i}$$

$$U^k_{\alpha i} = \frac{1}{2\pi\sqrt{-1}} \oint_{b_k} d\Omega_{\alpha, i}$$

ℓ_x $\quad x_1, \dots, x_{g+r-1}$

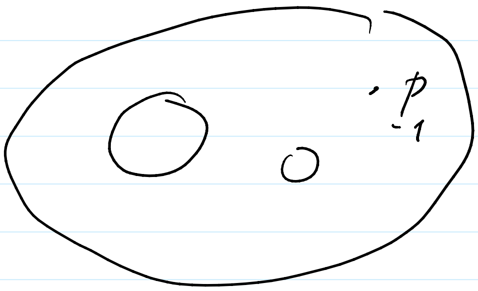
$- k$

$\mathcal{E}x$ $\quad \gamma_1 .. , \gamma_{g+r-1}$

Find r linear independent BA functions

$\underline{\mathcal{E}x}$ $\quad N = 1$

$$\psi(t,p) = e^{\sum t_i z^{-i}} \left(\underline{\underline{1}} + \sum \xi_s(t) z^{-s} \right)$$



$$\psi(t,p) = e^{\sum t_i \Omega_i} \frac{\theta(A(p) + t_i U_i + Z)\, \theta(A(p_1) + Z)}{\theta(A(p) + Z)\, \theta(A(p_1) + t_i U_i + Z)}$$

$\underline{\underline{\text{Th}}}$ $\quad \forall n \quad \exists!\ L_n = \partial_{t_1}^n + \sum_{i=0}^{n-2} u_i(t)\, \partial_{t_1}^i$

$$\partial_{t_n} \psi = L_n \psi$$

$\mathcal{E}x$ $\quad n = 2$

$$\partial_y \psi = \left(\partial_x^2 + u\right) \psi \qquad y = t_2 \quad x = t_1$$

$$u(x, y, t_3 ..)$$

$$\tilde{\psi} = \left(\partial_y - \partial_x^2 + u\right) \psi = \left(\partial_y - \partial_x^2 + u\right) e^{xz^{-1} + yz^{-2} + \dots} (1 ..)$$

$$= e^{xz^{-1}+yz^{-2}+\dots}\left(\sum_{s=0}^{\infty}\chi(t)z^{s}\right)$$

$$\chi_0 = -2\xi_1' + u = 0$$

$$\tilde{\psi} \equiv 0$$

Ex Find explicite formula for $\xi_1(t)$

$u(t)$

$n=3$

$$\left(\partial_{t_3} - \partial_x^3 - \frac{3}{2}u\partial_x + w\right)\psi = 0$$

$$\Rightarrow \left[\partial_{t_n} - L_n, \partial_{t_m} - L_m\right] = 0$$

$n=2 \quad m=3 \quad \Rightarrow u$

$$\frac{3}{4}u_{yy} = \left(u_{t_3} - \frac{3}{2}u_x u + u_{xxx}\right)_x$$

$$\left[\partial_{t_n} - L_n, \partial_{t_m} - L_m\right]\psi = 0$$

~~Op~~ linear operator in ∂_x

$$\underline{-\partial_{t_n}L_m} + \partial_{t_m}L_n + [L_n, L_m]$$