Lecture 5

Friday, October 22, 2021 8:21 AM

Inverse problem

Given $y^2 = R_{2g+2}$ and a generic divisor

$$\mathcal{D} = \gamma_1 + \cdots + \gamma_g$$

$\exists!$ $\psi_n(p)$ which is meromorphic on Γ

with poles $\gamma_1 \dots \gamma_g$ (simple if $\gamma_i \neq \gamma_j$)

with pole of order n at P_+

zero of order n at P_-

$$\psi_n = e^{\pm x_n} E^{\pm n}\left(1 + \sum \xi_s^{\pm} E^{-s}\right)$$

RR $\quad \dim \mathcal{L}(\mathcal{D} + n(P_+ - P_-)) = \deg(\quad) - g + 1 =$

$$\deg \mathcal{D} - g + 1 = 1$$

Th $\quad c_n = e^{x_n - x_{n+1}}$, $v_n = \xi_1^+(n) - \xi_1^+(n+1)$

$$c_n \psi_{n+1} + v_n \psi_n + c_{n-1}\psi_{n-1} = E\psi_n$$

Toda flow

$\Gamma, \mathcal{D}(t)$

$$(L\psi)_n = c_n \psi_{n+1} - v_n \psi_n + c_{n-1}\psi_{n-1} = E\psi_n$$

$$\psi_{n+N} = z\,\psi_n$$

Th $\quad z^2 \quad \dots \quad \Omega = \Omega(\dots)$

$\psi_{n+N} = z\,\psi_n$

Toda $\qquad z^2 - 2Qz + 1 \qquad Q = Q_N(E)$

$$(c_n, v_n) \approx Q_N, \ \Theta_N(E) = \prod (E - \gamma_s)$$

Toda flow $\quad \underline{c_n(t), v_n(t)}$

$$\dot{\Gamma} = 0$$

$$(\partial_t - A)\hat{\Psi} = 0 \qquad \hat{\Psi}(t{=}0) = 1 \qquad A = \begin{pmatrix} c_n & 0 & -c_{n-1} \end{pmatrix}$$

$$L(t)\hat{\Psi}(t) = \hat{\Psi}(t) \cdot G \ \text{- const}$$

$$t = 0 \Longrightarrow G = L(0)$$

$$L(t) = \hat{\Psi}(t)\, L(0)\, \hat{\Psi}^{-1}(t)$$

$$\mathcal{D}(t) = \prod (E - \gamma_s(t))$$

$$\psi_n(t, p) \qquad\qquad p \in \Gamma$$

$$\begin{cases} L\psi_n = E\psi_n \\ \psi_{n+N} = z\psi_n \end{cases} \qquad \psi_0 = 1$$

$$(\partial_t - A)\psi_n = \psi_n \cdot f(p, t)$$

$$\left(\partial_t \psi_n - c_n \psi_{n+1} + c_{n-1}\psi_{n-1}\right) = \psi_n f(t, p)$$

- $f(t)$ has simple at $\gamma_s(t)$

$$\psi_n = \frac{\alpha_s}{E - \gamma_s(t)} + \cdots$$

$\therefore$

$$E - \gamma_s(t)$$

$$f(t,p) = \frac{\dot{\gamma}_s}{E - \gamma_s(t)} + \cdots$$

$$\tilde{\psi}_n = \psi_n \left(e^{-\int_0^t f(t',p)} \right)$$

$$\begin{cases} L\tilde{\psi} = E\tilde{\psi} \\ (\partial_t - A)\tilde{\psi} = 0 \end{cases}$$

- $\tilde{\psi}_n$ is holomorphic at $\gamma_s(t)$

$$\int_0^t f(t',p)\,dt' = \int \left(\frac{\dot{\gamma}}{E - \gamma} + \cdots \right) dt$$

$$= -\ln(E - \gamma) + \cdots$$

$$\tilde{\psi} = \psi \cdot (E - \gamma(t)) (\cdot)$$

$f(t)$ near $P_\pm$

$$\left(\partial_t \psi_n - c_{n+1}\psi_n + c_{n-1}\psi_{n-1} \right) = \psi_n f$$

- $f(p,t) = \pm E + O(1)$ near $P_\pm$

$$\psi_n = e^{\pm x_n} E^{\pm n} \left(1 + \sum_{s=1}^{\infty} \xi_s^{\pm}(n) E^{-s} \right)$$

$$c_n = e^{x_n - x_{n+1}}$$

$$\int^t \; \cdots$$

$$\int_0^t f \, dt = \pm E t + O(1)$$

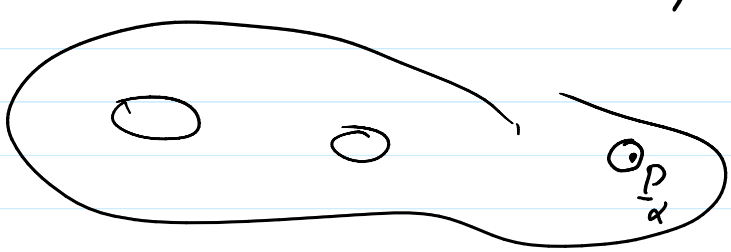
$$\tilde{\psi}_n = e^{\pm x_n(t)} E^{\pm n} e^{\pm E t} \left(1 + \sum \xi_s^{\pm}(t) E^{-s}\right)$$

$$\tilde{\psi}_0(t) \neq 1$$

$\tilde{\psi}(t,p)$ on $(\Gamma \setminus P_\pm)$ has $\gamma_1(0), \ldots, \gamma_R(0)$

$$e^{-\int f(t,p)} = \left(\frac{E - \gamma(t)}{E - \gamma(0)}\right)$$

Γ smooth genus g algebraic curve with fixed local coordinates $z_\alpha(p)$ near marked point P_α, $\alpha = 1, \ldots N$



$\mathcal{D} = \gamma_1, \ldots, \gamma_{g+r-1}$

Define BA $\psi(\underset{\sim}{t}, p)$ $t = (t_{\alpha,i})$

[illegible] of $\psi(\underset{\sim}{t}, p)$ $\quad \underset{\sim}{t} = \left(t_{\alpha,i}\right)_{i \geq 1}$

as a function on $\Gamma \ni p$

ψ on $\left(\Gamma \setminus P_\alpha\right)$ is meromorphic

$\mathcal{D} + (\psi) \geq 0$

near P_α $\quad \psi$ has the form

$$\boxed{\psi(t,p) = e^{\left(\sum_{i=1}^{\infty} t_{\alpha,i}\, z_\alpha^{-i}\right)} \left(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t)\, z_\alpha^{s}\right)} \qquad z_\alpha = z_\alpha(p)$$

Lemma $\quad$ Let $\mathcal{L}(\mathcal{D})$ be a linear space of BA function

For generic $\mathcal{D}$ $\quad \dim \mathcal{L}(\mathcal{D}) = \deg \mathcal{D} - g + 1$

Proof $\quad$ by explicite construction

$$u(x,y,t) = -2\partial_x^2 \ln \theta(Ux + Vy + Wt + Z)$$

$$L\psi = E\psi$$

$\varphi_n(E; n_0)$ $\theta_n(E; n_0)$

$\varphi_{n_0}(\ , n_0) = 1$ θ_n

$\varphi_{n_0+1}(\ , n_0) = 0$

L $\psi_n(p, n_0)$

$\psi_{n+N}(p, n_0) = z\ \psi_{n+N}$

$\psi_{n_0} = 1$

$\psi_n(\ , n_0) = \psi_n(\ , n_1) \cdot \psi_{n_1}(n_0)$

$\psi_n(p) = \boxed{\ldots\ \psi_2(1)\,\psi_1(0)}$