Lecture 4
Friday, October 15, 2021 8:47 AM

Periodic Toda lattice III

Recall

$$(L\psi)_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} = E\psi$$

$$c_n = c_{n+N} \qquad v_n = v_{n+N} \qquad c_n \neq 0$$

$$(c_n, v_n) = (\mathbb{R}^*)^n \times \mathbb{R}^n \longrightarrow$$

$$Q(E), \quad \theta_N(E)$$

$$T(E) = \begin{pmatrix} \varphi_N(E) & \theta_N(E) \\ \varphi_{N+1}(E) & \theta_{N+1}(E) \end{pmatrix}$$

$$\det(z\cdot\mathbb{1} - T(E)) = 0 \qquad\qquad y = z - Q$$

$$z^2 - 2Q(E)z + 1 = 0 \qquad p = (z, E) \in \Gamma$$

$$\begin{cases} (L\psi)_n = E\psi_n \\ \psi_{n+N} = z\psi_n \end{cases} \quad \text{Bloch solution}$$

$$\psi_n = \varphi_n + \frac{z - \varphi_N}{\theta_N}\theta_n$$

$$Q^2(E) = 1$$

forbidden

$E_0 \quad E_1$

$$\theta_N(\gamma_s) = 0$$

$$E_0 < E_1 \le \gamma_1 \le E_2 < E_3 \ldots$$

Finite-gap (g-gap) potentials

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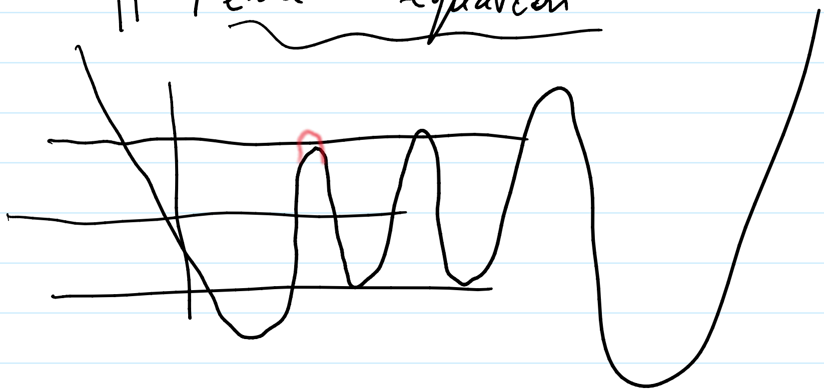
$(0+1)$ $\qquad$ $(1+1)$

$$Q^2 - 1 = R_{2g+2}\, r^2_{N-g-1} \qquad r(E) \text{ polynomial}$$

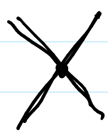
Pelle equation



$$Y = z - Q$$

$$Y^2 = Q^2 - 1 = R r^2$$

$$r(e_j) = 0 \qquad j = N - g - 1$$



$Y \sim \pm r$

$$y = \frac{Y}{r} \qquad \Big|\; y^2 = R_{2g+2}(E) \qquad \Gamma$$

$$(y, E) \longrightarrow (Y = yr,\ E)$$

The map is one-to-one everywhere exept at e_j

$$\Gamma \longrightarrow \tilde{\Gamma}$$

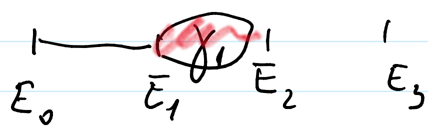
$$y^2 = R \qquad Y^2 = Q^2 - 1$$

$$y^2 = R \xleftarrow{f} Y^2 = Q^2 - 1 \qquad \psi_n$$

$$f^*(\psi_n)$$

Th ψ_n is a meromorphic function on Γ with g poles at γ_s, with pole of order n at $P_+ = E=\infty, \varepsilon=\infty$

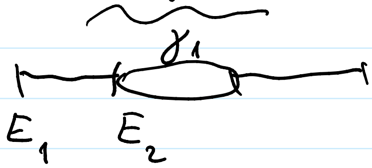
zero — " — n at P_- $E=\infty, \varepsilon=0$



$$E_1 = E_2 = e_1 = \gamma_1$$

$$\left(R_{2g+2}, \tilde{\gamma}_1, \dots, \tilde{\gamma}_g\right) \longleftarrow (c_n, v_n)$$

g-gap



At e_j $\qquad T(e_j) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$

$$\Theta_N = \tilde{\Theta}_N r$$

$$Q - \varphi_N(E) = \tilde{Q} r$$

$$\psi_n = \varphi_n + \frac{Q + \sqrt{Q^2-1} - \varphi_N}{\Theta_N} \Theta_n =$$

$$= \varphi_n + \frac{\tilde{Q} \cancel{r} + \sqrt{R}\, \cancel{r}}{\tilde{\Theta}_N \cancel{r}} \Theta_n$$

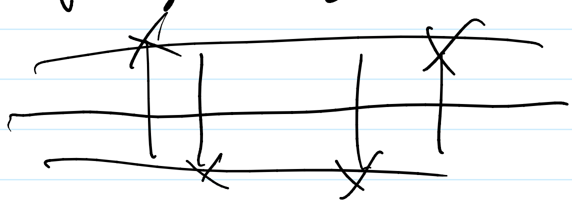
Ex Prove that ψ_1 has simple pole at P_+ and simple zero at P_-

Ex ψ_n has pole of order n, zero of order n

Inverse Problem

$$y^2 = R_{2g+2}(E), \quad \gamma_1 \dots \gamma_g \to (c_n, v_n)$$

Let Γ be a hyperelliptic curve of genus g and let $\gamma_1 \dots \gamma_D$



$$E(\gamma_i) = E(\gamma_j)$$

Then the dim of a linear space of functions with simple poles at γ is $D - g + 1$ if $D \geq g$ and 0 if $D < g$

- $f = y\, g_1(E) + g_2(E)$ $\quad g_1, g_2$ meromorphic function of E

I $\quad \sigma : \Gamma \to \Gamma \quad \sqrt{R} \to -\sqrt{R}$

$\underline{I}$ $\quad \sigma: \Gamma \to \Gamma \quad \sqrt{R}' \to -\sqrt{R}^{\gamma}$

$$\boxed{\frac{f(E,\sqrt{R}) - f(E,-\sqrt{R})}{y}} = \frac{1}{2} g_1$$

$$f(E,\sqrt{R}) + f(E,-\sqrt{R}) = \frac{1}{2} g_2$$

$$f = \frac{P_1(E)y + P_2(E)}{\prod_{s=1}^{\mathcal{D}} (E - E(\gamma_s))}$$

$$\deg P_1 = \mathcal{D} - g - 1$$
$$\deg P_2 = \mathcal{D}$$

$$P_1(E(\gamma_s))\sqrt{R(\gamma_s)} - P_2(E(\gamma_s)) = 0$$

f has a pole $\quad E(\gamma_s), \sqrt{R(\gamma_s)}$

$\underline{Ex}$ $\quad g = 1 \quad$ There is no function with one simple pole with non-zero residue

$$y^2 = R_4(E)$$

ψ_n has g poles at γ_s
pole of order n at P_+
zero of order n at P_-

RR $\Rightarrow$ ψ_n exists and unique
up to multiplication by a const.

$$\psi_n = \frac{P_1 y + P_2}{\prod (E - E(\gamma_s))}$$

$y \sim \pm E^{g+1}$

$$\overline{E^g}$$

$\deg P_1 = n-1$

$\deg P_2 = n-1+g$

g equation to ensure ψ_n has
pole at γ_s

$$\psi_n \sim e^{x_n} E^n \left(1 + \sum_{s=1}^{\infty} \xi_s^+(n) E^{-s}\right) \quad \text{at } P_+$$

$$e^{-x_n} E^{-n} \left(1 + \sum_{s=1}^{\infty} \xi_s^-(n) E^{-s}\right) \quad P_-$$

$\underline{\underline{\mathrm{Th}}}$ $\exists !\ c_n, v_n$ s.t.

$$c_n \psi_{n+1} + v_n \psi_n + c_{n-1}\psi_{n-1} = \underline{E}\psi_n$$

$$c_n \psi_{n+1} + v_n \psi_n + c_{n-1} \psi_{n-1} = E\psi_n$$

$$c_n = e^{x_n - x_{n+1}} \qquad v_n = \xi_1^+(n) - \xi_1^+(n+1)$$

Consider $\tilde{\psi}_n = L\psi_n - E\psi_n$

Claim $\tilde{\psi}_n$ = has pole of order $n-1$

$\tilde{\psi}_n$ has zero of order n at P_-

$$\Rightarrow \quad \tilde{\psi}_n \equiv 0$$

Congruence $\Longrightarrow$ equality