Lecture 3
Friday, October 8, 2021 8:54 AM

Periodic Toda lattice part 2

$$(L\psi)_n = c_n \psi_{n+1} + v_n \psi_n + c_{n-1} \psi_{n-1} \qquad L: \psi_n \to (L\psi)_n$$

$$L(z) = L\Big|_{(\psi_n = z\psi_{n+N})} \qquad c_n = c_{n+N} \quad v_n = v_{n+N}$$

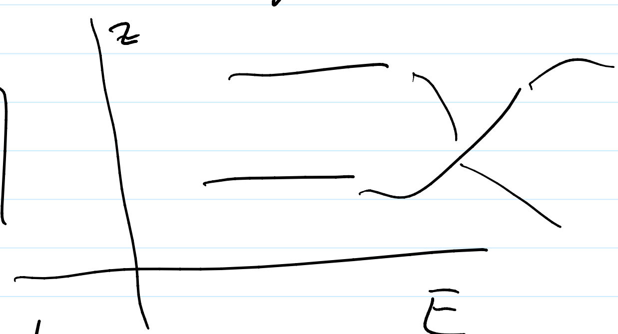
$$L(z) = \begin{pmatrix} \ddots & & zc \\ c_n & v_n & c_{n-1} \\ z^{-1}c_0 & & \ddots \end{pmatrix} \qquad \psi_0, \dots \psi_{N-1}$$

$$\det\left(L(z) - E\cdot\mathbb{1}\right) = \qquad L\psi = E\psi$$

$$Cz + Cz^{-1} = 2Q(E) \qquad \deg Q = N$$

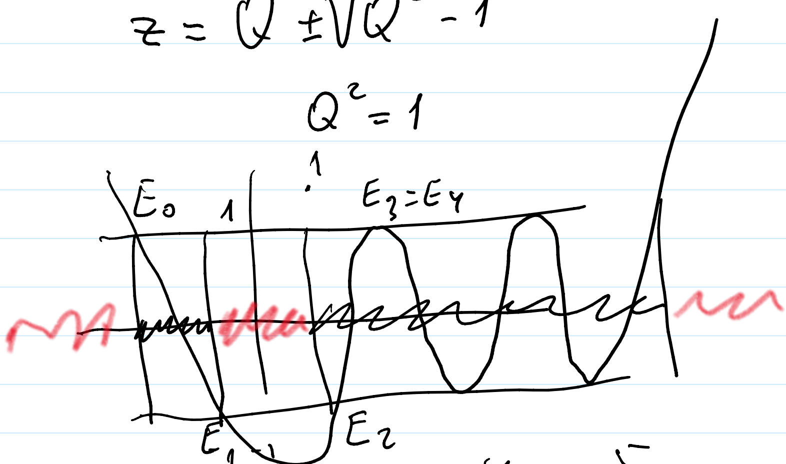
$$C = c_0 \cdots c_{N-1} = 1$$

$$\boxed{z^2 - 2Qz + 1 = 0}$$



$$z = Q \pm \sqrt{Q^2 - 1}$$

$$Q^2 = 1$$



$N = 2n$

$Q = 1 \qquad z = 1$

$Q = -1 \qquad z = -1$

$Q(E) = \pm 1$ has N solutions counting with multip.

$$Q'(e) = 0 \Rightarrow |Q| \geq 1$$

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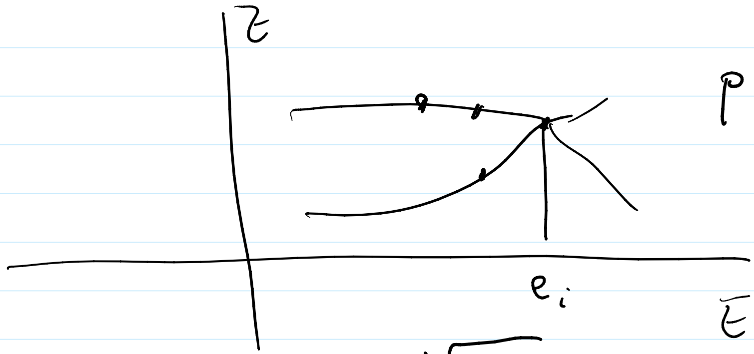


$p \in (z, E)$: $\det(L(z) - E) = 0$

$z(p)$ - functions

$E(p) \longrightarrow$

$w = \sqrt{E - e_i}$

Compactification $E = \infty$

$$z^2 - 2Q(E)z + 1 = 0$$

$$z = Q(E) \pm \sqrt{Q^2 - 1} \qquad Q^2 \neq 1$$

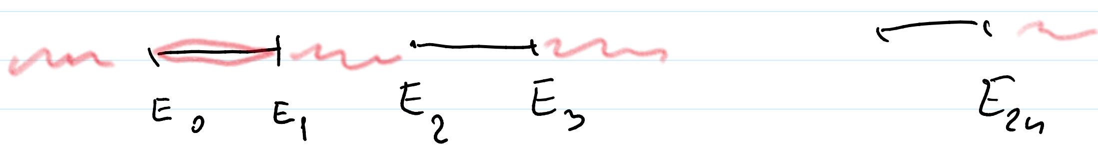
$$Q(E) = 1 + \alpha(E - e) + \dots$$

$$z = Q(E) \pm \sqrt{2\alpha(E - e) + \dots}$$

$$z = \sqrt{2\alpha}\, w + \dots \qquad w = \sqrt{E - e}$$

$E_0 \quad E_1$

$$E < E_1 \leq E_2 < E_3 \dots$$



$E = \infty$

$z = \infty$

$\overset{\cdot}{-}$ $E=\infty$

$z=0$

$z(p) = z_0$

# of solutions $= d = \deg z(p)$

$$z = E^N + \dots = (E^{-1})^{-N} + \dots$$

$$p=(E,z) \quad : \quad (\qquad\qquad) = 0$$

$$\begin{cases} L\psi = E\psi \\ \underbrace{T\psi = z\psi}_{[T,L]=0} \end{cases} \qquad T: \psi_n \to \psi_{n+N}$$

$$L\psi = E\psi$$

$$\underline{c_n \psi_{n+1}} + v_n \psi_n + c_{n-1}\psi_{n-1} = E\psi_n$$

$$\varphi_0 = 1 \qquad \theta_0 = 0$$

$$\varphi_1 = 0 \qquad \theta_1 = 1$$

$$\varphi_n(E) = \frac{c_0}{c_1 \cdots c_{n-1}}\left(E^{n-2} - \left(\sum_{k=2}^{n-1} v_k\right)E^{n-3} + \dots\right)$$

$$\theta_n(E) = \frac{1}{c_1 \cdots c_{n-1}}\left(E^{n-1} - \left(\sum_{k=1}^{n-1} v_k\right)E^{n-2} + \dots\right)$$

$$T(E) : \Big|_{T/\Lambda(E)}$$

$$T(E) \quad \begin{pmatrix} \varphi_N(E) & \theta_N(E) \end{pmatrix}$$

$$T(E) = \begin{pmatrix} \underline{\varphi_N(E)} & \theta_N(E) \\ \varphi_{N+1}(E) & \theta_{N+1}(E) \end{pmatrix}$$

$$z^2 - 2Qz + 1 = \det\left(z \cdot 1 - T(E)\right)$$

$$\underline{2Q = \varphi_N(E) + \theta_{N+1}(E)}$$

$$\varphi_N \theta_{N+1} - \varphi_{N+1} \theta_N = 1$$

$$(c_n, v_n) \longrightarrow Q_N(E)$$

$$\begin{cases} \underline{L \ \psi_n}(p) = E(p)\, \psi_n(p) \\ \underline{T \ \psi_n}(p) = z(p)\, \psi_n(p) \end{cases}$$

$$\psi_0 = 1 \qquad \psi_i$$

$$L(\underline{z})\psi = \left(\quad \right) \psi = \underline{E}\, \psi$$

$$\psi_1 = 1 \qquad \psi_i = \frac{\Delta_i}{\Delta_1}$$

$$\psi_n = \underline{\varphi_n}(E) + \qquad \theta_n(E) \qquad \underline{\psi_0 = 1}$$

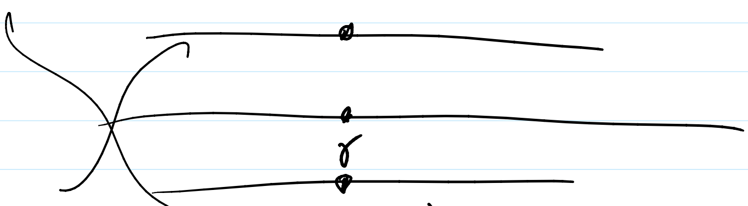
$$\begin{pmatrix} \varphi_N & \underline{\theta_N} \\ & \end{pmatrix} 1 = z \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \varphi_N & \theta_N \\ \varphi_{N+1} & \theta_{N+1} \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = z \begin{pmatrix} 1 \\ y \end{pmatrix}$$

$$\varphi_N + y\,\theta_N = z$$

$$\psi_n = \varphi_n(E) + \frac{z-\varphi_N}{\theta_N}\,\theta_n(E) \qquad y = \theta_N \varphi_n + (z-\varphi_N)\,\theta_n$$

poles ψ_n = zeros θ_N



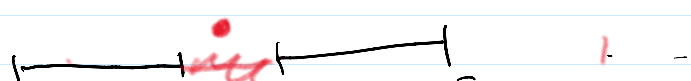
E

$\theta_N(y) = 0$

poles of ψ_n on (Γ, ∞) $= N-1$

$$\det \begin{pmatrix} \varphi_N - z & \theta_N \\ \varphi_{N+1} & \theta_{N+1} - z \end{pmatrix} = 0$$

$\theta_N(y) = 0 \qquad |Q(y)| \geq 1$



$E_0 \quad E_1 \quad E_2 \quad E_3$

$$Q_N = \varphi_N + \theta_{N+1}$$

$$\begin{pmatrix} \varphi_N & \theta_N \\ \varphi_{N+1} & \theta_{N+1} \end{pmatrix}$$

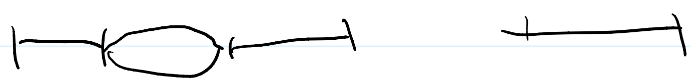
$$\det T = \varphi_N \theta_{N+1} = 1$$

$$\text{if } \theta_N = 0$$

$$\underline{E_0} < \underline{E_1} \le \gamma_1 \le E_2 < E_3$$

$$(c_n, v_n) \to \underline{\underline{Q(E)}}$$

level set



Spectral meaning of γ_s $\quad s = 1, .. N-1$

Eigenvalues of Dirichlet problem

$$\left\| \begin{array}{l} Ly = \gamma y \\ y_0 = 0 \qquad y_N = 0 \end{array} \right.$$