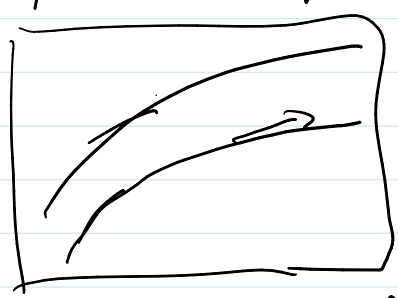
Lecture 2

Friday, October 1, 2021 7:44 AM

Periodic Toda Lattice

Integrable systems

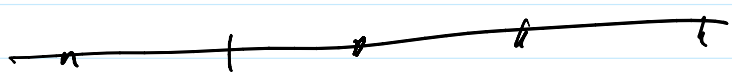
$\Rightarrow$ dynamical systems on phase space



phase space

$0+1$ finite-dimensional systems

$$\ddot{x}_i = e^{x_{i+1}-x_i} - e^{x_i - x_{i-1}}$$



Toda lattice

Calogero-Moser

$$\ddot{x}_i = -4 \sum_{i \neq j} \frac{1}{(x_i - x_j)^3}$$

$(1+1)$ $\quad 4u_t = u_{xxx} - 6uu_x$

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KdV

(2+1)

KP equation

$$3u_{yy} = (4u_t + 6uu_x - u_{xxx})_x$$

$u(x,y,t)$

2D Toda

BDHE

Q What unifies all these eq-ns?

Integrable equations are compatibility condition of overdetermined system of <u>Linear</u> problems

<u>Ex</u>

$$(L\psi)_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1}$$

$$\psi_n \longrightarrow (L\psi)_n$$

c_n, v_n – coefficients of L

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$$(L\psi)_n = E\psi_n \qquad E \text{ complex param}$$

$$\begin{pmatrix} c_n & v_n & c_{n-1} \\ & c_{n-1} & v_{n-1} \end{pmatrix} \begin{pmatrix} \psi_{n+1} \\ \psi_n \\ \psi_{n-1} \end{pmatrix} =$$

$$\begin{array}{l} E \\ - \\ \partial_t \end{array} \begin{cases} \partial_t \psi_n = \frac{c_n}{2}\psi_{n+1} - \frac{c_{n-1}}{2}\psi_{n-1} \\ E\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} \end{cases}$$

$$0 = \frac{c_n}{2}\left(\cancel{c_{n+1}\psi_{n+2}} + v_{n+1}\psi_{n+1} + c_n\psi_n\right)$$

$$- \frac{c_{n-1}}{2}\left(c_{n-1}\psi_n + v_{n-1}\psi_{n-1} + c_{n-2}\psi_{n-2}\right)$$

$$- \dot{c}_n\psi_{n+1} - c_n\left(\cancel{\frac{c_{n+1}}{2}\psi_{n+2}} - \frac{c_n}{2}\psi_n\right)$$

$$- \dot{v}_n\psi_n - v_n\left(\frac{c_n}{2}\psi_{n+1} - \frac{c_{n-1}}{2}\psi_{n-1}\right) - \cdots$$

$$\underbrace{\left(\frac{c_n}{2}v_{n+1} - \dot{c}_n - v_n\frac{c_n}{2}\right)}_{} = 0$$

$$\frac{c_n^2}{2} - \frac{c_{n-1}^2}{2} - \dot{v}_n = 0$$

$$\frac{c_n}{2} - \frac{c_{n-1}}{2} - v_n = 0$$

$$c_n = e^{x_{n+1} - x_n} \qquad v_n = \frac{\dot{x}_n}{2}$$

$$(\qquad)$$

$$\ddot{x}_n = e^{2(x_{n+1} - x_n)} - e^{2(x_n - x_{n-1})}$$

Toda lattice

Lax equation

$$\dot{L} = [A, L] = AL - LA$$

$$[\partial_t - A, L] = 0$$

$$\boxed{\text{"}L\text{"}}$$

Phase space ≡ space of Linear operators depending on a spectral parameter

Ex $\quad L(z) = u_0 + \sum_{i=1}^{N} \frac{u_i}{z - z_i}$ //

$\underline{\underline{Ex}}$ $\quad \underbrace{L(z) = u_0 + \sum_{i=1} \frac{-}{z - z_i}}$ //

$\underline{Ex}$ $\quad L\psi_n = c_n \psi_{n+1} + v_n \psi_n + c_{n-1} \psi_n$

$$c_n = c_{n+N} \qquad v_n = v_{n+N}$$

$$\{c_n, v_n\} \in \mathbb{C}^{2N} \Big/ \mathbb{R}^{2N}$$

$$\mathbb{R}_+^N \times \mathbb{R}^N$$

$$L\Big|_{\psi_{n+N} = \underline{\underline{z}}\, \psi_n} = L(z) \qquad z \neq 0$$

$$L(z) = \begin{array}{c} \begin{pmatrix} v_{N-1} & c_{N-1} & & & \boxed{z c_0} \\ \ddots & \ddots & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & c_n & v_n & c_{n-1} & \\ & & & \ddots & \\ \boxed{z^{-1} c_0} & & & c_0 & v_0 \end{pmatrix} \end{array} \begin{array}{c} \psi_{N-1} \\ \vdots \\ \\ \vdots \\ \\ \psi_0 \end{array}$$

Spectral transform

$$L(z) \cong \left(\text{spectral curve}, \ \text{divisor} \right)$$

$(c_n, v_n) \qquad n = 0, \dots N-1$

$$0 = \det\left(E\,\mathbb{1} - L(z)\right) = \underline{-Cz - Cz^{-1} + P_N(E)}$$

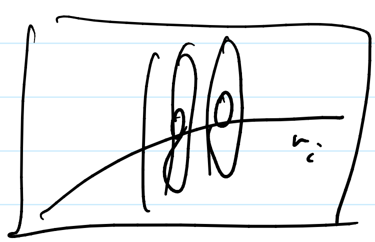
$E = E(z) \qquad C = c_0 \dots c_{N-1} = 1$

$$\begin{pmatrix} \ddots & & \boxed{z c_0} \\ & c_{n+1}\; v_n \; c_{n+1} & \\ \boxed{z^{-1} c_0} & & \ddots \end{pmatrix}$$

$$z + z^{-1} = E^N + \sum_{i=1}^{N} r_i E^{N-i}$$

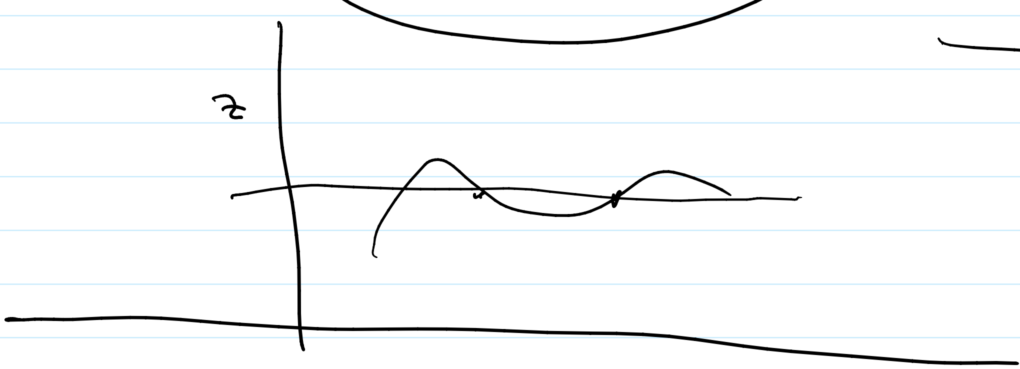
$$r_i = r_i(c_n, v_n)$$

$$(c_n, v_n) \to r_i(c_n, v_n)$$

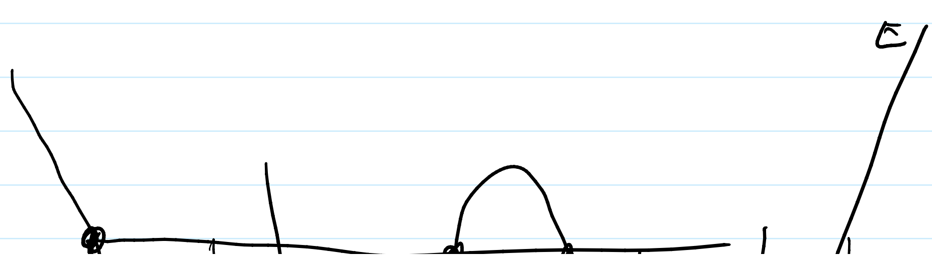


$$z^2 - 2Qz + 1 = 0$$

$$\underline{z = Q \pm \sqrt{Q^2 - 1}}$$

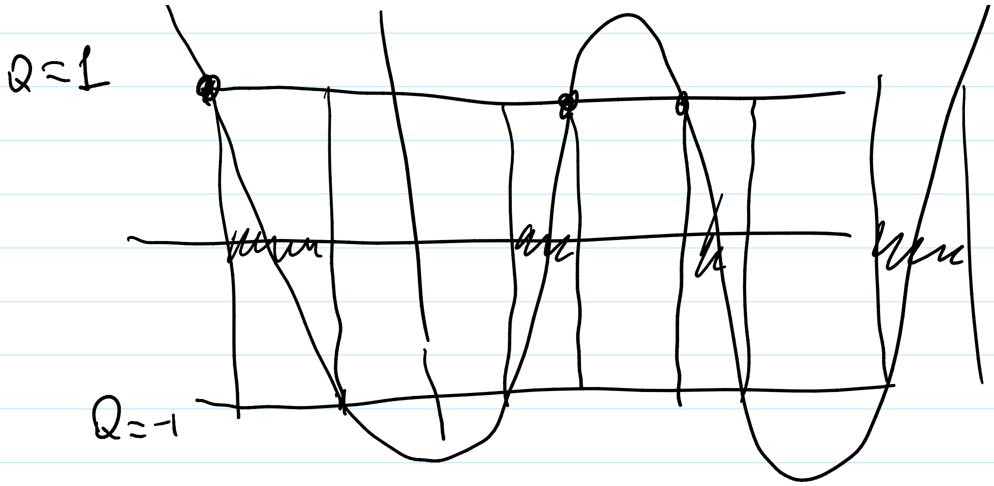


$Q = 1$



$$2Q(E) = P(E)$$

$2\alpha(E) = T(E)$



$c_n, v_n \in R$

$$Q = 1 \qquad E_0 < E_1 \le E_2 < E_3 \le \quad E$$

$$\psi_n(E) \qquad E \in [E_{2n}, E_{2n+1}]$$

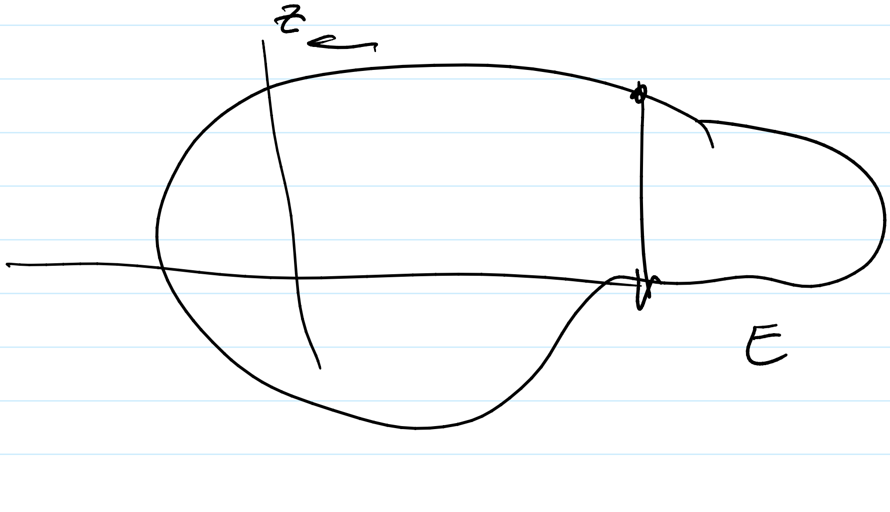
$$|z(E)| = 1$$

$$Q'(e) = 0 \qquad |Q(e)| \geq 1$$

Eigenvectors

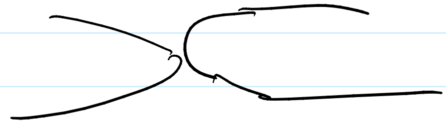
$$\psi_0^i(z) \quad \dots \quad \psi_{N-1}^i(z) \qquad i = 1, \dots, N$$

$$\psi_0(p) \quad \dots \quad \psi_{N-1}(p) \qquad p = (z, E) \in C^2$$



$$z^2 - 2Q(E)\,z + 1 = 0$$

$E \neq \infty$



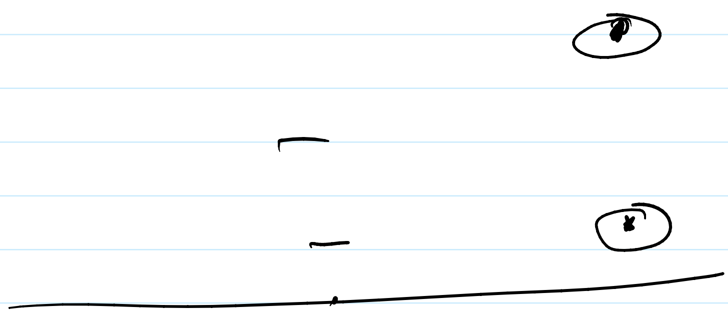
$E = \infty$

Consider in details

compactification $E =$

$$z = Q \pm \sqrt{Q^2 - 1} = Q \pm Q\sqrt{1 - \frac{1}{Q^2}}$$

$$\begin{cases} z^+ = 2Q - \dfrac{1}{2Q} + \dots \\[2ex] z^- = \dfrac{1}{2Q(E} \end{cases}$$



$E = \infty$