Skoltech 2021
Differential Geometry of connections
16.12.2021
Final examination problems
Please send solutions to *kazarian@mccme.ru* by 12:00 p.m. 19th December 2021

Basic problems

1. Let γ be a convex smooth plane curve of length ℓ. Compute the area of the strip swept by the outer normal segments of length r to γ and the length of the outer boundary oval of the strip.

2. Let $\gamma \subset \mathbb{R}^3$ be a spatial curve, S be the union of its tangent lines.
(a) Parametrize the surface S, compute the first and the second quadratic forms for the chosen coordinates at the smooth points of S.
(b) Show that the Gaussian curvature of S is identically equal to zero.
(c) Find Euclidean coordinates on S for the case when γ is the winding curve $(\cos t, \sin t, t)$.

3. Sections of the trivial vector bundle with the fiber $\mathbb{R}^2 = \mathbb{C}$ over the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$ can be considered as the complex-valued functions. Consider the connection in this bundle given by the formula
$$\nabla_\xi u = \partial_\xi u + a \frac{\xi_1 + i\xi_2}{x_1 + ix_2} u,$$
where $\xi = \xi_1 \partial_{x_1} + \xi_2 \partial_{x_2}$ and $a \in \mathbb{C}$ is a constant. Find the parallel transform along the circle of radius 2 centered at $(0,1)$.

Additional (advanced) problems

4. The geodesic flow on a surface S can be regarded as the vector field V on the 3-dimensional manifold M formed by unit tangent vectors. Let $J : M \to M$ be the diffeomorphism defined by rotation to the angle $\pi/2$ in each tangent plane. Prove that the commutator $[V, J_* V]$ is tangent to the fibers of the projection $M \to S$. Express this commutator in terms of the Gaussian curvature.

   Hint: here is a possible way of solving:
   a) Parametrize M assuming that the surface is $\mathbb{R}^2$ with coordinates x and y and the metric is $g = u_1^2 + u_2^2$ for some 1-forms u_1 and u_2.
   b) Find the coordinate presentation for V and for $J_* V$.
   c) Compute the commutator $[V, J_* V]$ using this coordinate presentation.

5. Compute the structure of Frobenius manifold[1] on the 3-dimensional space of polynomials
$$\{f = x^4 + ax^2 + bx + c\}$$
with distinct critical values. Namely:
(a) compute the metric and the multiplication table in the basis of coordinate vector fields for the coordinate system (a, b, c). (Caution: this coordinate system is not flat so that the matrix $\eta_{i,j}$ is not constant for this coordinate system!);
(b) find the system of flat coordinates t^1, t^2, t^3 such that the metric coefficients become constant in these coordinates;
(c) compute the Frobenius potential for the obtained flat coordinate system.

[1]Some comments about Frobenius manifolds are given below. While solving this problem it is allowed to use available literature with the following requirement: if some source is used, an explicit reference should be provided.

Frobenius manifolds

Frobenius potential is a (holomorphic) function in n variables $F(t^1,\dots,t^n)$ satisfying the following properties:

- The square symmetric matrix with the entries
$$\eta_{i,j}=\frac{\partial^3}{\partial t^i\partial t^j\partial t^1}$$
is constant (independent t) and nondegenerate. We denote by $\eta^{i,j}$ the entries of the inverse matrix.

- For any tuple of indices (a,b,c,d) the following equation holds true (it is called *associativity* or WDVV equation):
$$\frac{\partial^3 F}{\partial t^a\partial t^b\partial t^i}\eta^{i,j}\frac{\partial^3 F}{\partial t^j\partial t^c\partial t^d}=\frac{\partial^3 F}{\partial t^a\partial t^c\partial t^i}\eta^{i,j}\frac{\partial^3 F}{\partial t^j\partial t^b\partial t^d}.$$

Frobenius manifold is an open domain $M\subset\mathbb{C}^n$ with distinguished coordinate system $(t^1,\dots,t^n)$ and a Frobenius potential on it. The Frobenius manifold is equipped with the metric $\eta_{i,j}dt^idt^j$. Besides, the tangent space T_tM at each point $t\in M$ is equipped with the structure of an algebra, where the multiplication is defined by
$$(\partial_i\circ\partial_j,\partial_k)=\frac{\partial^3 F}{\partial t^i\partial t^j\partial^k},$$
where $\partial_k=\frac{\partial}{\partial t^k}\in T_tM$. More explicitly, we have $\partial_i\circ\partial_j=\frac{\partial^3 F}{\partial t^i\partial t^j\partial^l}\eta^{l,k}\partial_k$. This algebra is automatically commutative (due to the equality of mixed partial derivatives) and has the unit $1=\partial_1$. Besides, the associativity equations are equivalent to the associativity of the algebra structure.
Denote by A_k the linear operator acting on the tangent space T_tM defined by the multiplication by the vector ∂_k: $A_k(v)=\partial_k\circ v$. Define the connection on the tangent bundle TM depending on an additional parameter z and defined by
$$\nabla_k=\frac{\partial}{\partial t^k}+zA_k.$$
Then this connection is flat for any value of z which is equivalent to the equalities
$$dA=A\wedge A=0$$
where $A=A_kdt^k$ is the corresponding matrix of 1-forms.
In an equivalent invariant approach a Frobenius manifold is a manifold M equipped with a metric (a nondegenerate symmetric bilinear form) and a structure of commutative associative algebra with unit on each tangent space satisfying the following conditions:

- The multiplication is compatible with the metric in a sense that
$$(u\circ v,w)=(u,v\circ w)=(u\circ v\circ w,\mathbf{1})$$
for any triple of vectors $u,v,w\in T_tM$.

- The vector field of units $\mathbf{1}$ is covariantly constant.

- The family of connections defined by
$$\nabla^z_v=\nabla^0_v+z\,v\circ$$
is flat for each parameter value $z\in\mathbb{C}$, where ∇^0 is the Levi-Civita connection and $v\circ$ is the operator of multiplication by $v\in T_tM$. In particular, the metric itself is flat.

Example. Denote by $\mathbb{C}^n$ the space of polynomials of the form

$$f = x^{n+1} + a_2 x^{n-1} + a_3 x^{n-2} + \cdots + a_{n+1}.$$

The coefficients $a_2, \ldots, a_{n+1}$ form a coordinate system on $\mathbb{C}^n$. Denote by $M \subset \mathbb{C}^n$ the open subspace formed by polynomials with only simple critical points that is by those polynomials whose derivative f' has no multiple roots. The natural structure of Frobenius manifold on M introduced below is one of the basic examples of Frobenius manifolds.
For $f \in M$ denote

$$Q_f = \mathbb{C}[x]/(f'(x)),$$

the quotient algebra of polynomials in x over the ideal generated by the derivative $f'(x)$. Q_f is a vector space of dimension n. For a basis one can take, for example, the polynomials $1, x, \ldots, x^{n-1}$. Indeed, every polynomial is equivalent in Q_f to the remainder of its polynomial division by f', and this remainder is a polynomial of degree at most $n-1$.
We identify $T_f M$ with Q_f by associating the tangent vector $\partial_{a_i} \in T_f M$ with the class of the polynomial $\frac{\partial f}{\partial a_i} = x^{n+1-i}$. This identification provides the structure of associative commutative algebra with unit on $T_f M$ since Q_f is an algebra.
The missing yet ingredient of Frobenius manifold is the metric. It is defined as follows. Denote by $x_1, \ldots x_n$ the critical points of f (that is, the roots of f'). For given two elements $p, q \in Q_f$ we set

$$(p, q) = \sum_{i=1}^{n} \frac{p(x_i) q(x_i)}{f''(x_i)}.$$

This formula makes sense since $f''(x_i) \neq 0$ and the right hand side does not depend on a choice of polynomials representing given elements of Q_f. For practical computations it is more advised to use the equality

$$(p, q) = \sum_{i=1}^{n} \operatorname*{res}_{x=x_i} \frac{p(x) q(x)}{f'(x)} dx = - \operatorname*{res}_{x=\infty} \frac{p(x) q(x)}{f'(x)} dx$$

The last expression does not require an explicit computation of roots of f'. Thus, both the metric and the multiplication on $T_f M$ are defined. The fact that they satisfy the axioms of Frobenius manifolds is checked by direct computations.