

# Quantum integrable systems – 2021

## Problems

1. Find the spectrum of the  $XYZ$  Heisenberg model for  $N = 2$ :

$$H = J_x \sigma_x^{(1)} \sigma_x^{(2)} + J_y \sigma_y^{(1)} \sigma_y^{(2)} + J_z \sigma_z^{(1)} \sigma_z^{(2)}.$$

Consider the cases of the  $XXX$  ( $J_x = J_y = J_z$ ) and  $XXZ$  ( $J_x = J_y \neq J_z$ ) models and compare with the solution obtained through the Bethe ansatz.

2. Find the spectrum of the  $XXX$  model on 3 sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^3 \left( \vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \quad \vec{\sigma}^{(4)} \equiv \vec{\sigma}^{(1)}.$$

Find multiplicity of the energy levels.

3. Consider the  $XXX$  model on  $N$  sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^N \left( \vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \quad \vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}.$$

Let  $\vec{S}$  be the vector of the total spin:  $\vec{S} = \sum_j \vec{\sigma}^{(j)}$ . Let  $|\Psi_m\rangle$  be Bethe states with  $m$  magnons (i.e.  $S_z |\Psi_m\rangle = (N - 2m) |\Psi_m\rangle$ ). Prove that  $S_+ |\Psi_m\rangle = 0$  at  $m = 1, 2$ , where  $|\Psi_m\rangle$  are Bethe states with  $m$  magnons (such that the corresponding Bethe roots  $\lambda_i < \infty$ ).

4. Find the Yang function for the  $XXX$  spin chain with spin  $\frac{1}{2}$ .
5. Find eigenstates and the energy spectrum of one and two Bose particles with the Hamiltonian

$$\hat{H}_N = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k) \quad (N = 1, 2)$$

on the segment  $[0, L]$  with impenetrable walls (i.e. such that the wave function vanishes if at least one particle is at the endpoints of the segment).

6. For the system of three identical Bose particles with the Hamiltonian

$$\hat{H}_3 = -\sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq 3} \delta(x_j - x_k)$$

- a) construct common eigenstates of the Hamiltonian and the total momentum

$$\hat{P} = -i \sum_{j=1}^3 \frac{\partial}{\partial x_j}$$

- b) impose periodic boundary conditions on the segment  $[0, L]$  and obtain the system of Bethe equations.
7. Consider the Bethe equations for the model of the Bose gas with  $N$  particles on the segment  $[0, L]$  with periodic boundary conditions:

$$L\lambda_j + 2 \sum_{k=1}^N \operatorname{arctg} \frac{\lambda_j - \lambda_k}{c} = 2\pi n_j$$

Prove that in the state which is characterized by integer or half-integer numbers  $n_j$  the total momentum of the system is  $P = \frac{2\pi}{L} \sum_{j=1}^N n_j$ .

8. Prove that  $|+++ \dots +\rangle$  is an eigenvector for the transfer matrix of the 6-vertex model and find the corresponding eigenvalue.
9. Consider the 6-vertex model on the  $N \times N$  lattice with the  $R$ -matrix

$$R = R(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

where  $a = \sinh(u + \eta)$ ,  $b = \sinh u$ ,  $c = \sinh \eta$  and the quantum monodromy matrix

$$\mathcal{T}(u) = R_{10}(u)R_{20}(u) \dots R_{N0}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Find the scalar product  $\langle \Omega | C(v)B(u) | \Omega \rangle$ . Here  $|\Omega\rangle = |+++ \dots +\rangle$  is the generating vector.

10. Let  $R_{0j}(u) = \mathbf{1}u + P_{0j}$  be the quantum  $R$ -matrix of the  $XXX$  model ( $P_{0j}$  is the permutation operator). Consider the transfer matrix of the  $XXX$  model

$$T(u) = \operatorname{tr}_0 (R_{01}(u)R_{02}(u) \dots R_{0N}(u)).$$

Find  $T(0)$ ,  $\partial_u \log T(u)|_{u=0}$ ,  $\partial_u^2 \log T(u)|_{u=0}$ .

11. Find the commutation relations between the operators  $A(u)$ ,  $B(u)$ ,  $C(u)$ ,  $D(u)$  which are encoded in the intertwining relation  $R_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R_{12}(u-v)$  with the 6-vertex  $R$ -matrix.
12. Prove that the quantity

$$\det_q \mathcal{T}(u) = A(u + \eta)D(u) - B(u + \eta)C(u)$$

is a central element of the algebra generated by elements of the quantum monodromy matrix, i.e., that  $\det_q \mathcal{T}(u')$  (the quantum determinant) commutes with  $A(u)$ ,  $B(u)$ ,  $C(u)$ ,  $D(u)$  for all  $u, u'$ .

13. Let  $R(u) = \sum_{a=0}^3 W_a(u + \eta) \sigma_a \otimes \sigma_a$  be the  $R$ -matrix of the 8-vertex model (here  $W_a(u) = \theta_{a+1}(u)/\theta_{a+1}(\eta)$ ) and  $L(u) = \sum_{a=0}^3 W_a(u) \sigma_a \otimes S_a$  be the  $L$ -operator. Find the commutation relations between the operators  $S_a$  which are equivalent to the intertwining relation  $R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v)$  (these are the commutation relations of the Sklyanin algebra).
14. The Sklyanin algebra is the quadratic algebra with the generators  $S_a$ ,  $a = 0, \dots, 3$  and the six commutation relations

$$[S_0, S_\alpha]_- = iJ_{\beta\gamma}[S_\beta, S_\gamma]_+, \quad [S_\alpha, S_\beta]_- = i[S_0, S_\gamma],$$

where  $[\ , \ ]_{\mp}$  is the commutator (anticommutator) and  $(\alpha\beta\gamma)$  means any cyclic permutation of the indices (123). The structure constants  $J_{\alpha\beta}$  are

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma},$$

where  $J_\alpha$ ,  $\alpha = 1, 2, 3$  are free parameters. Prove that

$$\Omega_1 = S_0^2 + S_1^2 + S_2^2 + S_3^2, \quad \Omega_2 = J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2$$

are central elements of the Sklyanin algebra.