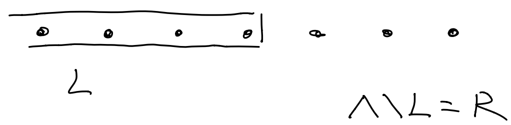
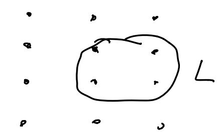
Entanglement entropy

* Ground states "simple". How to quantify?



State $|\Psi\rangle$



* How non-classical? How unlike product state?
* Bipartition: L, R parts
* $$|\Psi\rangle = \sum_{\substack{\alpha = 1 \ldots \dim(\mathcal{H}_L) \\ \beta = \ldots}} \Psi_{\alpha\beta} \, |\alpha\rangle_L \otimes |\beta\rangle_R$$
* Subsystem L in a state described by reduced density matrix:

$$\rho_L = \sum_{\beta, \alpha, \alpha'} \Psi_{\alpha\beta} \Psi^*_{\alpha'\beta} \, |\alpha\rangle_L \, {}_L\langle \alpha'|$$

* State entangled $\Rightarrow \rho_L$ mixed
* Entropy of ρ_L: measure of how entangled L is with R in state $|\Psi\rangle$

von Neumann:

$$S_{ent}(L) = -\mathrm{Tr}\,(\rho_L \log \rho_L)$$

defined: for a partition, for a given pure state $|\Psi\rangle$

* <u>Examples</u>: $|\Psi\rangle$ − product state (classical)

$$S_{ent}(L) = 0$$

* 2 spin-1/2 $\quad \frac{1}{\sqrt{2}}(\uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2)$:

$$\rho_L = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$S_{ent} = \ln 2$ – maximum possible entanglement for spin-½

* Fully random state:

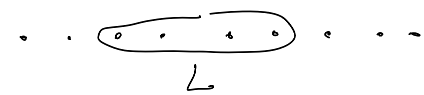
$$|\Psi\rangle = \sum_{\sigma_i = \uparrow,\downarrow} A_{\sigma_1 \ldots \sigma_N} |\sigma_1 \ldots \sigma_N\rangle$$

random Gaussian

$\sum |A_{\{\sigma\}}|^2 = 1$

2^N $\langle A^2 \rangle = \frac{1}{2^N}$

$\langle A \rangle = 0$



$$S_{ent}(L) \approx \ln 2 \cdot |L|$$ volume of L

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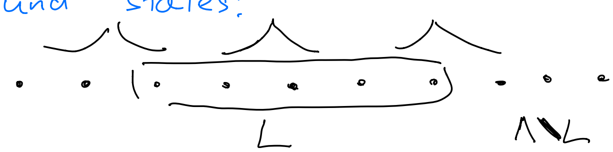
$(|\uparrow\rangle_1 + |\downarrow\rangle_1) \otimes (|\uparrow\rangle_2 + |\downarrow\rangle_2) \otimes \ldots$

* Many measures of entanglement: Renyi entropies

$$S_n(L) = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_L^n \qquad n \text{ replicas}$$

ground states:

*



Only degrees of freedom near boundary are strongly entangled with the rest

* Suggests: for ground states

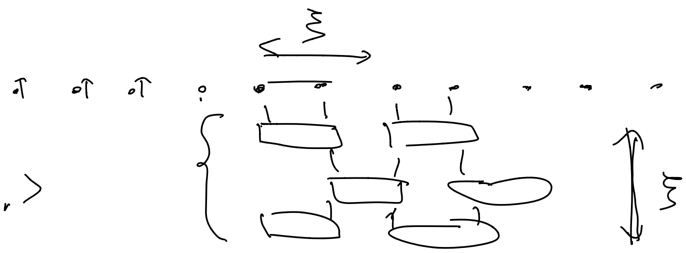
$$S_{ent}(L) \propto |\partial L|$$ boundary

"area-law"
"boundary-law"

very different from random states ($S_{ent} \propto |L|$)

* Area-law states can be efficiently "compressed"

Parameters needed $\sim \underline{\underline{\text{poly}(N)}}$, vs. $\exp(cN)$ for a random state



$|\psi_{appr}\rangle$ $\quad S_{ent} < \text{const}$

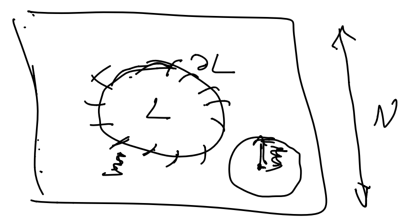
$S_{ent} \sim \xi \cdot \ln 2$

$\xi \quad \sim e^{c\xi}$ parameters

N/ξ parts

$\sim e^{c\xi} \cdot N/\xi \sim e^{c \cdot S_{ent}}$

$S_{ent} \sim N$



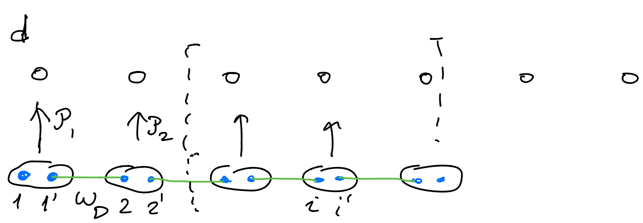
$e^{c\xi^2}$

* $|\langle \psi_{appr}| \hat{O} | \psi_{appr}\rangle - \langle \psi | \hat{O} | \psi \rangle| \leq \varepsilon$

* $\underline{||\psi\rangle - |\psi_{appr}\rangle|^2} \sim \varepsilon N$

* Note: Argued for area-law in ground states of gapped systems. 1d: possible to prove (Hastings '07)

Matrix-product states (MPS)
Tensor networks



1) Introduce auxiliary spins: 2 per site, $|\alpha\rangle = 1 \dots D$

Entangle neighboring auxiliary spins:

$$|\omega_D\rangle_i = \sum_{\alpha=1}^{D} |\alpha\rangle_i \otimes |\alpha\rangle_{i+1} \qquad \underline{\underline{\sim \ln D}}$$

sets maximum entanglement

D - "bond dimension"

area-law

2) Act with a linear operator:

$$\mathcal{P}_i : |\alpha\rangle_i \otimes |\beta\rangle_{i'} \to |\sigma\rangle$$

$$\mathcal{P}_i = \sum_{\sigma, \alpha\beta} A^{[i]}_{\sigma, \alpha\beta} \cdot |\sigma\rangle \langle \alpha, \beta|$$

physical auxiliary

Consider state:

$$|\Psi\rangle = (\mathcal{P}_1 \otimes \ldots \otimes \mathcal{P}_N) \, |\omega_D\rangle^{\otimes N} \qquad (1)$$

* Explicit form: $\quad |\sigma_1 \ldots \sigma_N\rangle = |\sigma_1\rangle \otimes \ldots |\sigma_N\rangle$

$$|\Psi\rangle = \sum_{\{\sigma\},\{\alpha\}} A^{[1]}_{\sigma_1, \alpha_1\alpha_2} A^{[2]}_{\sigma_2, \alpha_2\alpha_3} \ldots A^{[N]}_{\sigma_N, \alpha_N\alpha_1} |\sigma_1 \ldots \sigma_N\rangle =$$

(periodic boundary conditions)

approximate ground state

$$= \sum_{\{\sigma\}} \mathrm{Tr}\left[A^{[1]}_{\sigma_1} \cdot \underline{A^{[2]}_{\sigma_2}} \ldots A^{[N]}_{\sigma_N} \right] |\sigma_1 \ldots \sigma_N\rangle$$

For each site i : σ_i : $A^{[i]}_{\sigma_i}$

* Matrices have dimension $\underline{D} \times D$: d

* Maximum entanglement: $\propto \log D$ $\qquad$ Area-law

* Parameters needed: $\leq D^2 \cdot d \cdot N$

$A \to V^{-1} A V$ $\qquad$ redundancy

* $\log D \sim |S_{ent}|$

MPS

approximate ground states

Density-matrix renormalization group (DMRG)

construct Hamiltonians with MPS ground state

Affleck-Lieb-Tasaki-Kennedy

AKLT

$\underline{D} = 2$

Haldane

Symmetry, classification of ground states

symmetry-protected topological phases

$\underline{\text{Spin-1}}$

$H \qquad |\Phi_D\rangle \qquad \min \dfrac{\langle \Phi_D | H | \Phi_D \rangle}{\langle \Phi_D | \Phi_D \rangle} \qquad N \sim 30\text{-}40$

- Complexity, representation power of MPS?
- Algorithms: ground states, evolutions...
- Construct solvable models using MPS (AKLT)
- Symmetry transformation of MPS

$H = H_{XXZ} + V$

$\left\{ \begin{array}{l} \text{Ex. Excited state: Volume-law entanglement} \\ \text{Thermalization} \\ \text{Eigenstate thermalization hypothesis} \end{array} \right.$

Many-body localization:

Excited states: area-law