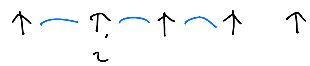
Lecture 1:
Locality & Lieb-Robinson bounds

Many-body problem:

- Consider a (quasi) local Hamiltonian, e.g. spin-$\frac{1}{2}$ ode)

σ_i^α, $\alpha = x, y, z$



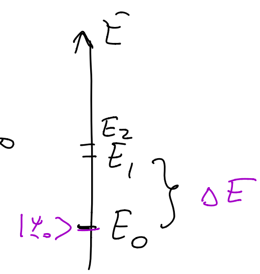
Example: transverse-field Ising model

$$H = -J \sum_{i=1}^{N-1} \underline{\sigma_i^z \sigma_{i+1}^z} + h \underline{\sum_{i=1}^{N} \sigma_i^x}$$

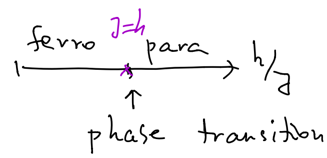
- Goal: describe ground state $|\Psi_0\rangle$, excitations above it

Often, interested in a situation when $\exists$ energy gap

$|E_1 - E_0| \geq \Delta E$, even as $N \to \infty$



- $|\Psi_0\rangle$ can be in diff. phases: in the example above, ferromagnetic or paramagnetic



Appears "just" a problem in linear algebra

Very Rich - many phases possible with simple-looking Hamiltonians, e.g.:

superconductor, ferromagnet, fractional quantum Hall states (topological order)

Challenge: Hilbert space grows exponentially with number of degrees of freedom:

$$\dim \sim 2^N$$ – Quantum very much unlike classical!

Why can we describe quantum many-body systems? despite this "exponential wall"?

Turns out that locality & entanglement properties of ground states shed light on this question

Note: Active area, far from complete understanding

Correlations:

We usually characterise state / phase by correlation functions, e.g.

$$\underbrace{\langle \Psi_0 | O_1 O_2 | \Psi_0 \rangle}_{\text{static}} \qquad \underbrace{\langle \Psi_0 | O_2(t)\, O_1(0) | \Psi_0 \rangle}_{\text{dynamical}}, \ldots$$

Our plan: 1) start from describing how correlations / information can propagate in the system (Lieb & Robinson)

2) Will derive a bound, showing that there is a finite speed at which correlations spread

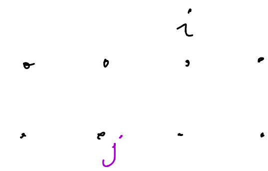
3) Remarkably, these bounds can be applied to get insights into properties of ground states

(Hastings, others..)

Some definitions:

$i, j \in \Lambda \leftarrow$ finite lattice

sites



$\hat{O}$ operator is supported in $\underline{A} \subset \Lambda$ if

$$\hat{O} = I_{\Lambda \setminus A} \otimes P \nwarrow \text{ acts on sites in } A$$

$\hat{\sigma}_i^z$ – support i $\quad \sigma_i^x \sigma_{i+1}^x \quad \{i, i+1\} \quad \bigotimes_{i \in \Lambda} H_i$

$\|O\|$ operator norm

$$\|O\| = \max_{\psi, |\psi|=1} |O\psi|$$

Distance: $\quad \mathrm{dist}(A, B) = \min_{\substack{i \in A \\ j \in B}} \mathrm{dist}(i, j)$

Diameter: $\quad \mathrm{diam}(A) = \max_{i,j \in A} \mathrm{dist}(i, j)$

Hamiltonian: $\quad H = \sum_Z H_Z$, $\quad H_Z$ supported on Z

Locality means that $\|H_Z\| \to 0$ as $\mathrm{diam}(Z) \to \infty$ and this decay is sufficiently fast. For example, an exponential decay would do.

Often, study strictly local Hamiltonians (e.g. Ising model above)

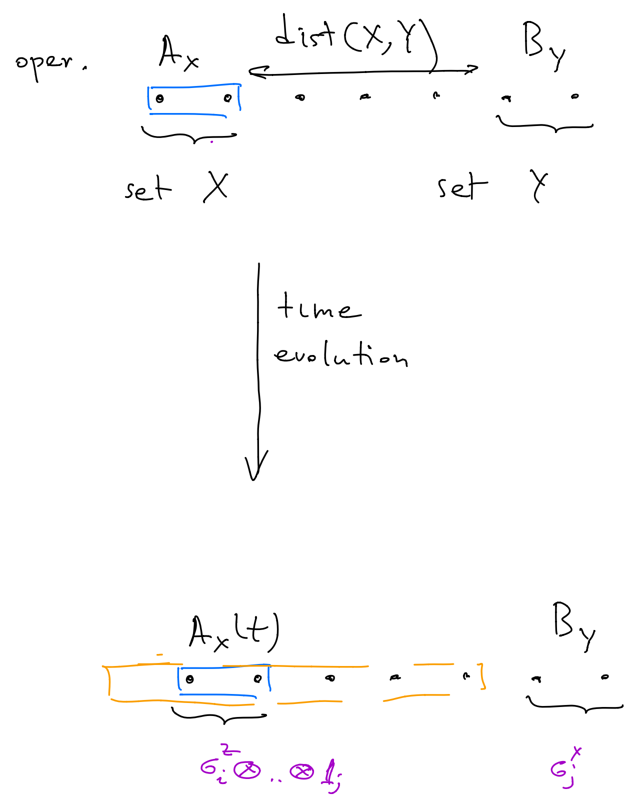
* Heisenberg-picture : time-evolved operator

$$O(t) = \exp(iHt)\, O \exp(-iHt)$$

* A way to quantify spreading of correlations :
study how initially local operators grow in real space

$[A_X, B_Y] = 0$

$[A_X(t), B_Y] \sim$?



Theorem (Lieb-Robinson bound)

$J(r) \sim e^{-\xi r}$ $\quad \mu \le \xi$ $\quad s \sim e^{*\mu}$

Suppose $\forall\, i$ $\quad \displaystyle\sum_{X \ni i} \|H_X\|\, |X| \cdot e^{\mu\, \mathrm{diam}(X)} \le s < \infty$

$\hookrightarrow$ exp. decaying interaction OK ✓

Then $\quad$ dist(X, Y) > 0

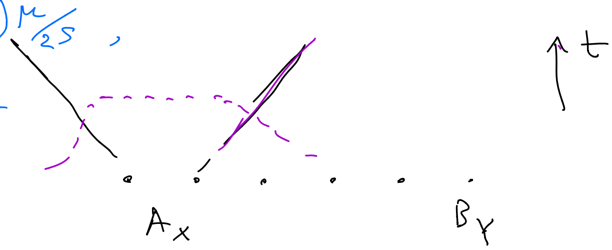
(o) $$\|[A_X(t), B_Y]\| \le 2\, \|A_X\|\, \|B_Y\|\, |X|\, e^{-\mu\, \mathrm{dist}(X,Y)} \times \left[e^{2s|t|} - 1\right]$$

$2s|t| - \mu\, \mathrm{dist}(X,Y) > 1$ $\qquad |t|_* > \frac{\mu}{2s}\, \mathrm{dist}(X,Y)$ $\qquad$ # of sites in X

× This means that $A_X(t)$ remains mostly within a "light cone" of

$$\ell(t) \sim \frac{2st}{\mu}$$

Only when $t \sim \mathrm{dist}(X,Y)\,\mu/2s$, can $[A_X(t), B_Y]$ become sizeable



Proof

Lieb, Robinson, Comm Math Phys 28, 251

* $\epsilon = t/N$, N-large,

$$t_n = \frac{t}{N} n \qquad n = 0 \ldots N$$

$$\|[A(t),B]\| - \|[A(0),B]\| = \sum_{n=0}^{N-1} \epsilon \times \frac{\|[A(t_{n+1}),B]\| - \|[A(t_n),B]\|}{\epsilon}$$

* Will use $\|U^\dagger O U\| = \|O\|$

$$A(\epsilon) = e^{i\epsilon H} A e^{-i\epsilon H} = A + i\epsilon [H, A] + O(\epsilon^2)$$

* $$\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| = \|[A(\epsilon), B(-t_n)]\| - \|[A, B(-t_n)]\| \leq \|[A + i\epsilon [I_X, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + O(\epsilon^2) \quad (1)$$

⇒ where $I_X = \sum_{Z: Z\cap X \neq \emptyset} H_Z$ – part of H which has support on X

* Next, use:

$$A + i\epsilon [I_X, A] = e^{i\epsilon I_X} A\, e^{-i\epsilon I_X} + \mathcal{O}(\epsilon^2)$$

$$\| [A + i\epsilon [I_X, A], B(-t_n)] \| \leq \| [e^{i\epsilon I_X} A\, e^{-i\epsilon I_X}, B(-t_n)] \| + \mathcal{O}(\epsilon^2) =$$

$$= \| [A, e^{-i\epsilon I_X} B(-t_n)\, e^{i\epsilon I_X}] \| + \mathcal{O}(\epsilon^2) \leq$$

$$\leq \| [A, B(-t_n)] \| + \epsilon \| [A, [I_X, B(-t_n)]] \| + \mathcal{O}(\epsilon^2)$$

* Substitute to (1):

$$\| [A(t_{n+1}), B] \| - \| [A(t_n), B] \| \leq \epsilon \| [A, [I_X, B(-t_n)]] \| + \mathcal{O}(\epsilon^2)$$

$$\leq 2\epsilon \|A\| \, \| [I_X(t_n), B] \| + \mathcal{O}(\epsilon^2)$$

* Then:

$$\| [A(t), B] \| - \| [A(0), B] \| \leq 2 \|A\| \sum_{n=0}^{N-1} \epsilon * \| [I_X(t_n), B] \| + \mathcal{O}(\epsilon) \leq$$

$$\leq 2 \|A\| \sum_{Z : Z \cap X \neq \emptyset} \sum_{n=0}^{N-1} \epsilon \times \| [H_Z(t_n), B] \| + \mathcal{O}(\epsilon)$$

* Converge to integral:

$$(2) \qquad \| [A(t), B] \| - \underbrace{\| [A(0), B] \|}_{\overset{\shortparallel}{0}} \leq 2 \|A\| \sum_{Z : Z \cap X \neq \emptyset} \int_0^t ds\, \| [H_Z(s), B] \|$$

$$\frac{\|[H_Z(s),B]\|}{\|H_Z\|} \leq C_B(Z,s)$$

$$\frac{\|[A(t),B]\|}{\|A\|} \leq C_B(X,t)$$

* Next, define: set $\neq$ support of B

$$C_B(X,t) = \sup_{A \in \mathcal{A}_X} \frac{\|[A(t),B]\|}{\|A\|}$$

operator

$\mathcal{A}_X$ – set of observables supported on set X

* Then (2) gives:

$$(3) \qquad C_B(X,t) \leq \underset{0}{C_B(X,0)} + 2 \sum_{Z:Z\cap X \neq \emptyset} \|H_Z\| \cdot \int_0^t ds\, C_B(Z,s)$$

* Next step: iterate (3) $\qquad C_B(Z_1,s) \overset{(3)}{\leq} C_B(Z_1,0) + 2 \sum_{Z_2: Z_2\cap Z_1 \neq \emptyset} \|H_{Z_2}\| \int \ldots$

$$C_B(X,t) \leq 2 \sum_{Z_1: Z_1\cap X \neq \emptyset} \|H_{Z_1}\| \int_0^t ds\, C_B(Z_1,s) \leq$$

use (3) for r-h.s

$$\leq 2 \cdot \sum_{Z_1:Z_1\cap X \neq \emptyset} \|H_{Z_1}\| \int_0^t ds_1\, C_B(Z_1,0) +$$

$$+ 2^2 \cdot \sum_{Z_1: Z_1\cap X \neq \emptyset} \|H_{Z_1}\| \cdot \sum_{Z_2: Z_2\cap Z_1 \neq \emptyset} \|H_{Z_2}\| \int_0^t ds_1 \int_0^{s_1} ds_2\, C_B(Z_2,s_2) \leq \ldots$$

$$\leq 2\|B\|\,(2|t|) \sum_{\substack{Z_1: Z_1\cap X \neq \emptyset \\ Z_1 \cap Y \neq \emptyset}} \|H_{Z_1}\| + \ldots$$

← this appears because of $C_B(Z_1,0)$

The nth term will be bounded by:

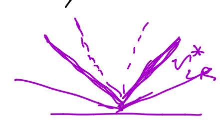
$$2\|B\| \cdot \exp(-\mu\, \mathrm{dist}(X,Y)) \cdot \frac{(2 s |t|)^n}{n!}$$

Add, get the desired bound (o)

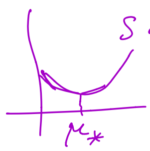
Lieb-Robinson velocity

Lieb Robinson (LR) bound implies that correlations spread at a finite velocity, in the following sense:

For $v_{LR} = 4\boxed{s}/\mu$, $t \le \dfrac{dist(X,Y)=\ell}{v_{LR}}$,



$s \sim e^{\mu}$ $\mu \to \infty$



$t \to 0$

$$\|[A_X(t), B_Y]\| \le \boxed{\frac{v_{LR}\cdot t}{\ell}} \cdot \underline{g(\ell)} \cdot |X| \cdot \|A_X\| \|B_Y\|,$$

where $g(\ell)$ decays exponentially with ℓ $e^{-\frac{\mu\ell}{2}}$

* Also, $\underline{A_X(t)}$ can be approximated by an operator with a support in $\overset{\text{site distance}}{\vee}$ $\ell = v_{LR}\, t$ away from X, with an exponentially small error

<u>Note</u>: We considered exp. decaying interactions. Other kinds (e.g. <u>strictly local</u>) can be considered similarly, producing a slightly stronger bound:

$$g(\ell) \propto \exp(-\text{const}\cdot \ell^2)$$

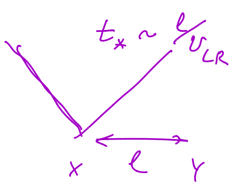
* Also: power-law extensions

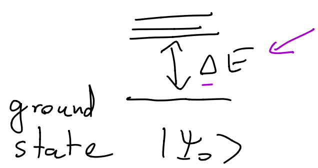
Application: correlation decay
From dynamics to statics

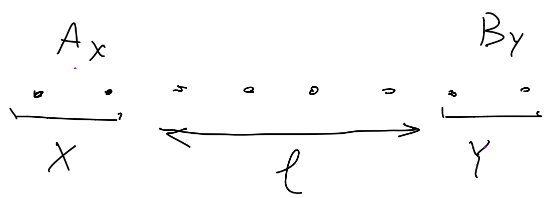
* Apply LR to describe static properties of ground states

* We take it for granted that if there is a gap, correlations decay exponentially. LR bounds help to prove it

Theorem (Hastings)







$$\rightarrow \left| \langle \Psi_0 | \underline{A_X B_Y} | \Psi_0 \rangle - \langle \Psi_0 | A_X | \Psi_0 \rangle \langle \Psi_0 | B_Y | \Psi_0 \rangle \right| \lesssim$$

$-\frac{\ell \Delta E}{2 v_{LR}}$ $\quad -\frac{\mu}{2}\ell \quad \Delta E \lesssim v_{LR}\mu \sim s$

$$\leq C \left\{ \exp\left(-\ell \frac{\overset{\text{gap}}{\Delta E}}{\underbrace{2 v_{LR}}_{\sim \text{ Lieb-Robinson velocity}}}\right) + \min(|X|, |Y|)\, \underline{g(\ell)} \right\} \|A_X\| \, \|B_Y\|$$

Idea: go from time domain (LR bounds) to frequency domain – operators, matrix elements etc

* More concretely: express action of $\underline{B_Y}$ via Fourier of $\tilde{B}_Y(t)$, then use LR bounds.

<u>Challenge</u>: naively, $t \to \infty$ enters Fourier.

Thus, need a good approximation which would cut off long-time contribution

* 1st step: Introduce

$$(B_Y^+)_{ij} = (B_Y)_{ij}\, \theta(E_i - E_j) \quad - \text{ "raising" part of } \tilde{B}$$

ground state

* $\langle \Psi_0 | A_X B_Y | \Psi_0 \rangle = \langle \Psi_0 | A_X B_Y^+ | \Psi_0 \rangle = \langle \Psi_0 | [A_X, B_Y^+] | \Psi_0 \rangle$

$\langle \Psi_0 | A_X | \Psi_0 \rangle \langle \Psi_0 | B | \Psi_0 \rangle$

* 2nd step: approximate B_Y^+ with $\tilde{B}_Y^+$ which would be boundable by LR bounds

Define:

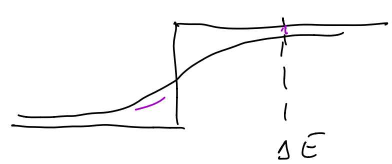
$\sum |i\rangle\langle j|\, (B_Y)_{ij}\, e^{-i(E_i - E_j)t}$

$$\tilde{B}_Y^+ = \frac{1}{2\pi} \int dt\, B_Y(t) \cdot \frac{1}{it + \epsilon}\, e^{-\frac{(t\Delta E)^2}{2q}}$$

q-parameter, choose later

If we had no $e^{-\frac{(t\Delta E)^2}{2q}}$, this would give B_Y^+. The gaussian term allows to cut off contribution of $t \to \infty$, while still well-approximating considered correlator

* Effectively, this trick allows to approximate $\theta(E_i - E_j)$ by a smoother function



Precisely,

$$\left| B_Y^+ |\Psi_0\rangle - \tilde{B}_Y^+ |\Psi_0\rangle \right| \leq C\, e^{-q/2}\, \|B_Y\|$$

* Second,

$$\| [A_x, \tilde{B}_y^t] \| \leq \frac{1}{2\pi} \int dt \, \underbrace{\| [A_x, B_y(t)] \|}_{\text{suppresied short-time contribution}} \frac{1}{|it+\epsilon|} \cdot \underbrace{e^{-\frac{(t\Delta E)^2}{2q}}}_{\text{long-time contribution}}$$

For $t < \frac{\ell}{v_{LR}}$, we use LR bound $\leq g(\ell)\, |X| \cdot \|A_x\| \, \|B_y\|$

For $t > \ell / v_{LR}$, use naive operator bound (there Gaussian would be already small)

This gives:

$$\# \; \underbrace{g(\ell)\, |X| \cdot \|A_x\| \, \|B_y\|}_{\text{short times}} + \underbrace{\|A_x\| \, \|B_y\|}_{\text{long times}} \left(e^{-\frac{(\ell \Delta E / v_{LR})^2}{2q}} \right) \quad e^{-q/2}$$

Finally, we see that q should be chosen

$$\underline{\underline{q_*}} = \frac{\ell \Delta E}{v_{LR}}$$

(minimize errors from approximation of $\tilde{B}_y^t$ and long-time contribution

Spectral gap $\Rightarrow$ exponential decay of correlations. In other words, correlations are short-ranged.

Lets explore some consequences for complexity of describing quantum many-body ground states

<u>Note</u>: many other interesting applications of LR bounds. Will return to them.

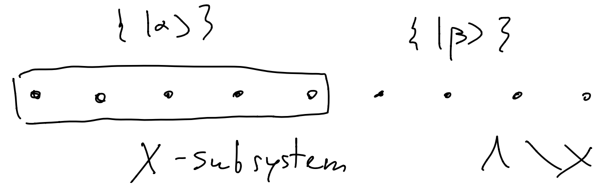
Entanglement: intuition

$|\Psi_0\rangle$

Exponential decay of correlations $\sim e^{-|i-j|/\xi}$

Presumably, remote degrees of freedom are (almost) independent. What does it mean for many-body wave function? How to quantify it?

Entanglement:



$$|\Psi\rangle = \sum \Psi_{\alpha\beta} \, |\alpha\rangle \otimes |\beta\rangle$$

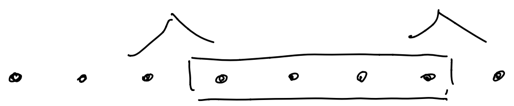
Trace out complement, look at the state of the subsystem X

$$\rho_X(|\Psi\rangle) = \sum_{\beta} \Psi_{\alpha\beta} \Psi^*_{\alpha'\beta} \, |\alpha\rangle\langle\alpha'|$$

The entropy of ρ_X gives a measure of how (non-classical) or entangled the state is.

In a certain sense, gives a measure of how difficult it is to describe a quantum state $|\Psi\rangle$ (its complexity)

$$S_{ent}(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$$



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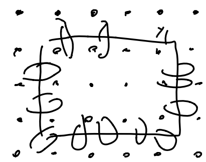

X

Exp. decay of correlations suggests that only degrees of freedom near the boundary of a region would be strongly entagled with the rest of system