Lecture 15

Thursday, December 17, 2020 7:23 AM

[Integrable systems. I](http://www.math.columbia.edu/~krichev/research.html). (PDF: Russian)
Itogi Nauki i Tekhniki, Akad. Nauk SSSR, VINITI, **Dynamical Systems 4** (**1985**), 179–277 (with B.A. Dubrovin, S.P. Novikov). ✓

From <http://www.math.columbia.edu/~krichev/research.html>

Vector bundles and Lax equations on algebraic curves. (PDF: English, arXiv: hep-th/0108110)
Comm. Math. Phys. **229:2** (**2002**), 229–269 . ✓

From <http://www.math.columbia.edu/~krichev/research.html#00s>

Krichever, I.; Shiota, T. Soliton equations and the Riemann-Schottky problem. *Handbook of moduli. Vol. II,* 205–258, Adv. Lect. Math. (ALM), 25, *Int. Press, Somerville, MA,* 2013. ✓
>

Road map

1. Phase space $\equiv$ "L" operatorovs depending on spectral parameter

a) $L = u_0 + \sum \frac{u_i}{z - z_i}$

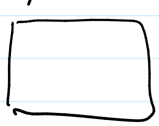
more general Higgs field $\Phi = L\,dz$

generalizations to field analog

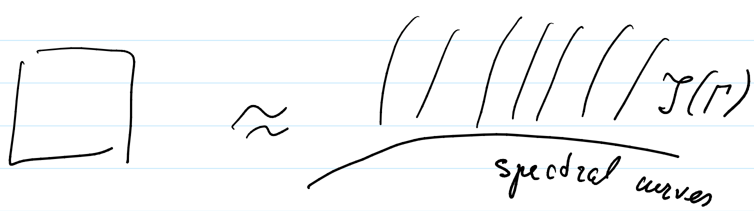
b) $L = u_n \partial_x^n + \sum_{i=0}^{n-1} u_i(x)\,\partial_x^i$

$u_n^{ij} = u_n^i \delta^{ij}$

Spectral transform

 $\underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}}$ spectral curve, line bundle

analogs of eigenvalues and eigenvectors



In particular examples curves come with additional structures like involution

$J(\Gamma)$ is replaced by $Prim(\Gamma)$

- Spectral transform solves Lax equations

• Spectral transform "solves" Lax equations

" $\dot{L} = [A, L]$ " Lax equation

defines $A(L)$

$$A_\alpha = \frac{v_\alpha}{z-\mu} \qquad v = L^n(\mu)$$

$\alpha = (n, \mu)$

!!! ∂_α : $[A_\alpha, L]$ - vector field tangent to "L"

$$[\partial_a, \partial_b] = 0$$

• $\partial_a(\Gamma) = 0$ on $J(\Gamma)$ ∂_b is constant

$\Rightarrow$ gauge transformed eigenvector ψ

$$\psi \to e^{-\int f} \psi$$

is the Baker-Akiezer

$L \longrightarrow \Psi$ Baker-Akhiezer function

NLPDE $\to$ L $\longrightarrow \Psi$

BA function

Starting point Ψ BA function

is defined $\Gamma, \mathcal{D} \to \Psi$

• $\Psi(t, p)$ is a solution of

linear equations

Linear equations

$$\underbrace{(\partial_a - L_a)\psi = 0}_{L_a \text{ is independent of } p \in \Gamma}$$

$$[\partial_a - L_a, \partial_b - L_b] = \mathcal{D} H_{\alpha\beta} \qquad a = (\alpha, i)$$

"Universal soliton hierarchy"

Each equation is $(2+1)$ system

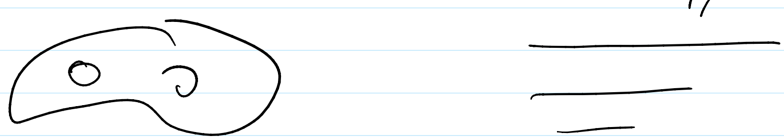
L_α is a differential (difference) operator in $t_{\alpha,1}$ $(t_{\alpha,0})$

Reduction to $\Big|_{\partial_a = 0}$ $(1+1)$ systems

Corresponds to a pair

(abstract curve) $\rightarrow$ curves with function

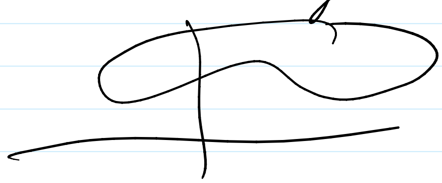
$\parallel$



Covers of $\mathbb{CP}^1$

Reduction $\partial_a = 0$, $\partial_b = 0$

is $(0+1)$ systems



Hamiltonian theory

- Universal two form ω

Lax equation are Hamiltonian

Lax equation are Hamiltonian

$$\omega(\ \cdot\ ,\partial_{t_a}) = dH_a(\ \cdot\ )$$

$\Rightarrow$ "L" is foliated by symplectic leaves

$$\omega = \operatorname{res}\operatorname{tr}\left(\Psi^{-1}\delta L\wedge\delta\Psi\right)dz$$

$$\omega^1 = \operatorname{res}\operatorname{tr}\,\Psi^{-1}L^{-1}\delta L\wedge\delta\Psi\, dz \quad \Big| = \sum \delta k_s\wedge\delta z_s$$

$$\downarrow$$

$$\sum \delta\ln k_s\wedge\delta z_s$$

Bihamiltonian systems

$$H = \frac{1}{n+1}\operatorname{tr} L^{n+1}(\mu)$$

$$H_1 = \frac{1}{n}\operatorname{tr}\left(L^{n}(\mu)\right)$$

What is missing

Reality conditions? curves, divisors are real

How to single out "regular" (smooth) solutions

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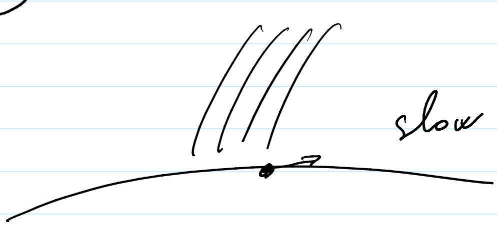
(1) Whitham perturbation theory



slow motion of curves

τ-function of universal Whitham hierarchy

τ-function of universal Whitham hierarchy

Applications:

TQFT

SW

Laplac growth (Riemann map theorem)

2 Spectral theory of 2D Schrödinger equ and sigma models

3 Linear operator with self-consistent potentials

Let V rang r degree rg

$$\begin{array}{c} V \\ \downarrow \\ \Gamma \end{array} \qquad \begin{array}{l} h^0(V) = r \\ s_1, \dots, s_r \end{array}$$

$$\sum \alpha_{ij} \, s_i(\gamma_j) = 0$$

$$\mathcal{M} \simeq \Big(\gamma_1 \dots, \gamma_{rg}, \; \alpha_1 \dots \in \mathbb{C}P^{r-1} \Big)$$

$$L(z)$$