Lecture 14

Thursday, December 10, 2020 9:59 AM

Hamiltonian theory integrable system

<u>Recall</u> On a phase space of operators "L" we defined "universal" two-form

$$\omega = -\frac{1}{2} \sum_{\substack{\text{at poles of} \\ L \text{ and } dz}} \operatorname{res} \, tr\left(\Psi^{-1} \delta L \wedge \delta \Psi\right) dz$$

operator valued basic form

$L\Psi = \Psi k$ $\quad k$ - diag.

$\Psi = \Psi(L)$ is an operator valued function on L

when normalization is chosen

- The form ω is normalization independent when restricted on $\mathcal{L} \subset$ "L" s.t

$\delta k \, dz$ $\quad k$ - eigenvalue

is holomorphic ~~at~~ at poles L and dz

<u>Ex</u> $\quad L = u_0 + \sum \frac{u_i}{z - z_i}$

at z_i $\quad k = \frac{\nu_{ij}}{z - z_i}$ $\quad \nu_{ij}$ eigenvalue of u_i

$\Rightarrow \delta \nu_{ij} = 0$ i.e. $\nu_{ij} = c_{ij}$ - const

$$\Rightarrow \quad \delta v_{ij} = 0 \qquad \text{i.e.} \quad v_{ij} = c_{ij} \ - \text{const}$$

$$\mathcal{L} = \bigoplus_i \mathcal{O}_i$$

$\omega|_{\mathcal{L}} = \oplus$ symplectic form KKBL

Lax equations are hamiltonian with respect to $\omega|_{\mathcal{L}}$ with Hamiltonian?

$$2\,\omega(\ ,\partial_t) = \delta H(\ )$$

$$\omega = -\sum \operatorname{res} \operatorname{tr}\left(\Psi^{-1}\delta L \wedge \delta\Psi\right) dz$$

$$\dot{L} = [A, L]$$

$$\dot{\Psi} = A\Psi + \underline{\Psi f} \qquad (\partial_t - A)\Psi = \Psi f$$

$$2\omega(\ ,\partial_t) = -\sum \operatorname{res} \operatorname{tr}\left(\qquad\qquad\qquad\qquad\right) dz$$

$$\Psi^{-1}(AL - LA)\delta\Psi - \Psi^{-1}\delta L A \Psi - \Psi^{-1}\delta L \Psi f$$

$$L\Psi = \Psi k \rightarrow L\delta\Psi = -\delta L \Psi + \delta\Psi k + \Psi \delta k$$

$$= \Psi^{-1}A(-\delta L\Psi + \delta\Psi k + \Psi\delta k) - k\Psi^{-1}A\delta\Psi$$

$$- \Psi^{-1}\delta L A \Psi - \Psi^{-1}\delta L\Psi f$$

$$- \Psi^{-1}\delta L\, A\, \Psi - \Psi^{-1}\delta L\, \Psi f$$

$$\Rightarrow 2\omega(\cdot, \partial_t) = -\sum \operatorname{res} \operatorname{tr}_z \Big(-2A\delta L - \underbrace{\Psi^{-1}\delta L\, \Psi}_{\delta\kappa} f\Big) dz$$

Recall $\quad A = \dfrac{L^n(\mu)}{z-\mu}$

$$= \operatorname*{res}_{z=\mu} \operatorname{tr}_z \big(2A\,\delta L + \delta\kappa\, f\big)\, dz$$

$$= \operatorname{tr}_z \big(2L^n(\mu)\,\delta L - \delta\kappa\, \kappa^n\big) = \frac{1}{n+1}\,\delta \operatorname{Tr}_z \big(L^{n+1}(\mu)\big)$$

$$\dot\Psi = A\Psi + \Psi f \qquad\qquad f = -\kappa^n$$

$$\omega(\ \ \partial_t) = \delta H(\ \ ) \qquad\qquad H = \frac{1}{n+1}\operatorname{Tr}_z\big(L^{n+1}(\mu)\big)$$

Action-angle variables

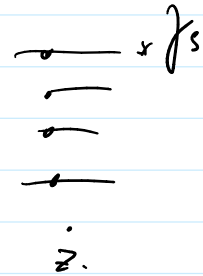
$$L\Psi = \Psi\kappa \qquad\qquad \Psi^{-1}L = \kappa\Psi^{-1}$$

A row of Ψ^{-1} can be identified with ψ^+ dual eigenvector: it has poles at branching points and zeros at poles γ_s of ψ

$$\langle \psi^+(p)\, \psi(p) \rangle = 1$$

$$\omega = -\frac{1}{2} \sum \operatorname{res} \big(\psi^+\, \delta L \wedge \delta\psi\big)\, dz$$

at preimages of poles on Γ



Th [illegible]

0 1 $\qquad\qquad$ z_i

Th $\qquad \omega = \underbrace{\sum_s \underline{\delta k_s} \wedge \underline{\delta z_s}} = \sum_{i=1}^{g} \underline{\delta A_i} \wedge \delta \varphi_i$

$z_s = z(\gamma_s) \qquad k_s = k(\gamma_s) \qquad A_i = \oint_{a_i} k\,dz$

$$\underset{\sim}{L} \to \left(\Gamma_g \text{ i.e. } R(\underset{\sim}{k}, z) = 0 \ , \ \underset{\gamma_s}{\overset{\cup}{\mathcal{D}}}\right)$$

Ex $\qquad L(z) = u_0 + \sum \frac{u_i}{z - z_i} \qquad$ rank $r = 2$

$$L\psi = k\psi \qquad \psi = \begin{pmatrix} 1 \\ \chi \end{pmatrix}$$

$$(k - L_{11}) + L_{12}\,\chi = 0$$

$$\chi = \frac{L_{11} - k}{L_{12}}$$

poles of ψ are zeros L_{12}

z_s are zeros of $L_{12}(z)$

$k_s = k(z_s)$

$$d\Omega = (\psi^* \underset{\sim}{\delta L} \wedge \delta\psi)\, \underset{\sim}{dz}$$

$$\omega = \frac{1}{2} \sum_{\substack{\text{all} \\ \text{other poles}}} \operatorname{res} d\Omega$$

$d\Omega$ has poles at γ_s and branching points

$d\Omega$ has pole at γ_s and branching points of Γ

$$\psi = \frac{\varphi_s}{z - z_s} + \cdots \qquad \psi^+ = \varphi_s^+ (z - z_s) + O((z-z_s)^2)$$

$$(\varphi_s^+ \varphi_s) = 1$$

$$\delta\psi = \delta z_s \frac{\varphi_s}{(z-z_s)^2} + O((z-z_s)^{-1})$$

$$\boxed{\underbrace{\operatorname{res}_{\gamma_s} d\Omega}\ = (\varphi_s^+ \delta L \varphi_s) \wedge \delta z_s = \underline{\delta \kappa_s \wedge \delta z_s}}$$

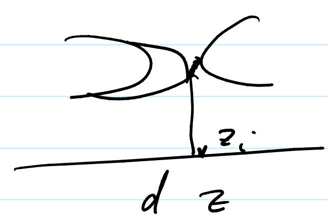
$$\varphi_s^+ \delta L \varphi_s = \delta \kappa_s \qquad (L - \kappa_s)\varphi_s = 0$$

$$\varphi^+ (L - \kappa_s) = 0$$

$$(\varphi^+ \varphi) = 1$$

$$\varphi_s^+ (\delta L - \delta\kappa_s)\varphi_s = -\varphi_s^+ (L - \kappa)\delta\varphi_s = 0$$

$$\varphi_s^+ \delta L \varphi_s = \underset{1}{\underset{\shortparallel}{(\varphi_s^+ \varphi_s)}} \delta \kappa_s$$



$$\operatorname{res}_{q_i} (\psi^+ \delta L \wedge \delta\psi) dz$$

has pole at q_i

- $\delta\psi$ has pole at q_i

In the neighborhood of q_i local coordinate

$$w = \sqrt{z - z_i} \qquad z_i = z(q_i)$$

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$$\psi = a + b\,(z - z_i)^{1/2} + O(z - z_i)$$

$$\delta\psi = \frac{b}{2\sqrt{z - z_i}}(-\delta z_i) + O(1)$$

$$\boxed{\delta\psi = -\frac{d\psi}{dz}\,\delta z_i + O(1)}$$

$$\boxed{\delta k = -\frac{dk}{dz}\,\delta z_i}$$

$$\operatorname{res}_{q_i} d\Omega = \operatorname{res}_{q_i}\left(\psi^+ \delta L \frac{d\psi}{dk} \wedge \delta k\right) dz$$

$$= \operatorname{res}_{q_i}\left(\psi^+ (\delta L - \delta k) \frac{d\psi}{dk} \wedge \delta k\right) dz$$

$$= -\operatorname{res}_{q_i} \delta\psi^+ (L - k) \frac{d\psi}{dk} \wedge \delta k$$

$$= \operatorname{res}_{q_i}\left(\delta\psi^+ \left(\frac{\cancel{dL} - dk}{dk}\right) \psi \wedge \delta k\right) dz$$

$L\psi = k\psi$

$\delta L\,\psi + L\delta\psi$

$= k\delta\psi + \delta k\,\psi$

dL has zero at q_i

$$= \operatorname{res}_{q_i} (\delta\psi^+ \psi)\, \delta k\, dz$$

$$\operatorname{res}_{q_i} (\psi^+ \delta\psi)\, \delta k\, dz$$

$$\overset{?}{=} \sum \operatorname{res}_{q_i} \left(\qquad \right) \overset{?}{=} \sum \underset{\text{at other}}{\operatorname{res}} \left(\psi^+ \delta\psi\right) \delta k\, dz$$

$= \text{res } \sum \text{res}_{q_i} ($ $\qquad) = \sum \text{res} (\psi^+ \delta\psi) \delta\kappa\, dz$

at other poles
$\|$
γ_s

$$= \sum \delta\kappa_s \wedge \delta z_s$$

$$\psi^+ = \underline{\varphi_s^+} (z - z_s)$$

$$\psi = \varphi_s \frac{1}{(z - z_s)}$$

$$\psi^+ \delta\psi = \frac{\varphi_s^+ \varphi_s}{(z - z_s)} \delta z_s$$

$$\omega = \sum_{\gamma_s} \delta\kappa_s \wedge \delta z_s$$

$$\omega = \delta S$$

$$S = \underbrace{\sum \int^{\gamma_s} \delta\kappa\, dz}_{\text{un "L"}} \quad \text{- one form}$$

$$\delta S = \delta\kappa_s \wedge \delta z_s$$

$\delta\kappa\, dz$ is a holomorphic diff on Γ

$$\Rightarrow \delta\kappa\, dz = \underbrace{\delta A_i}\ \omega_i$$

$$\boxed{A_i = \oint_{a_i} \kappa\, dz}$$

$$\sum_s \int^{\gamma_s} \omega_i = \varphi_i$$

$$S = \sum_i \delta A_i\ \varphi_i$$

$$\omega = \sum_i \delta A_i \wedge \delta\varphi_i$$

$$\omega = \sum_i \delta A_i \wedge \delta \varphi_i$$