# Lecture 13

Thursday, December 3, 2020 10:00 AM

## Hamiltonian theory of integrable systems

$M^{2n}$, $\omega = h_{ij}\, dx^i \wedge dx^j$

$d\omega = 0 \qquad \omega^n \neq 0$

$H \to \partial_H \qquad \omega( \; , \partial_H) = dH( \; )$

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Poisson manifolds

$\{f,g\} = h^{ij} \frac{\partial f}{\partial x^i} \cdot \frac{\partial g}{\partial x^j}$ $\qquad \{ \; , \; \}$ satisfies Jacobi identity

For symplectic manifold

$$h^{ij} = (h_{ij})^{-1}$$

Poisson manifolds rank $h$ = const $\Longrightarrow$

$\exists \{f_i, f\} = 0 \qquad i = 1, .. \text{ corank } h$

$f_i = c_i$ - const symplectic leaf

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Liouville theorem

$\{H_i, H_j\} = 0 \qquad i = 1, \dots, n$

$H_i = c_i \qquad \approx R^n / \Lambda \qquad \Lambda$ is a lattice

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Algebraic integrability vs Liouville integrability

complex leves sets are complex abelian varieties !!!

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# Setup for Lax type integrable systems

$$"\mathcal{L}" = \left[ \begin{array}{l} \text{Phase space of operators "}L\text{" depending on} \\ \text{a spectral parameter} \end{array} \right]$$

$\delta L$ - operator valued basic one-form

$\delta = "d"$

$$L = Az + U \iff \{ a^i, U^{ij} \}$$

$$A = \mathrm{diag}(a_1, \dots, a_r)$$

$$\delta L = \delta A z + \delta U$$

$$L\Psi = \Psi k \qquad k = \mathrm{diag}(k_1(z) \dots, k_r(z))$$

$$\underline{\Psi = \Psi(L)} \qquad \delta\Psi \text{ - one form on "}\mathcal{L}\text{"}$$

- universal two form

$$\omega = -\frac{1}{2} \sum \underset{\substack{\text{at poles of } L \\ \text{and } dz}}{\mathrm{res}} \mathrm{tr}\left(\Psi^{-1} \delta L \wedge \delta\Psi\right) dz$$

$\Psi$ is defined up to normalization

$$\Psi \to \Psi h(z) \qquad h \text{ arbitrary diagonal matrix}$$

Under change of normalization

$$\omega \to \omega + \sum \mathrm{res} \; \underline{\mathrm{tr} \langle \Psi^{-1} \delta L \Psi \wedge \delta h h^{-1} \rangle} \, dz$$

$$L\Psi = \Psi k \implies \delta L \Psi = -L\delta\Psi + \delta\Psi k + \Psi\delta k$$

$$L\Psi = \Psi K \quad \Longrightarrow \quad \delta L \Psi = -L\delta\Psi + \delta\Psi K + \Psi\delta K$$

$$\mathrm{tr}(\Psi^{-1}\delta L\Psi)\,dz = \delta K\, dz \wedge \delta h$$

$\omega$ is normalization independent under restriction on a subvariety where

$\boxed{\delta K dz \text{ is holomorphic at poles of } L \text{ and } dz}$

Ex $\quad L = Az + U$

$$L\Psi = \Psi K \qquad \xi_s^{ii} = 0$$

$$\Psi = \left(1 + \sum_{s=1}^{\infty} \xi_s z^{-1}\right) \qquad K = K_{-1} z^{-1} + K_0 + K_1 z + \dots$$

$$A\xi_{s+1} + U\xi_s = \xi_{s+1} K_{-1} + \xi_s K_0 + \xi_{s-1} K_1 + \dots$$

$s = -1 \qquad K_{-1} = A \qquad \xi_0 = 1$

For each $s$ we get off-diagonal $\xi_{s+1}$ and $K_s$

$$s=0 \qquad [A, \xi_1] + U = K_0 \qquad \xi_1^{ij} = \frac{U^{ij}}{a_j - a_i}$$

$$K_0 = U^{ii}$$

Imp $\quad s = 1 \qquad \xi_2^{ij} = (\quad) \qquad K_1^i = \sum_j U^{ij} U^{ji}$

Symplectic leaves are defined by the condition:

$\boxed{\delta K dz \text{ - holomorphic}}$

$$\underbrace{\delta K_{-1} = 0 \qquad \delta K_0 = 0 \qquad \delta K_1 \neq 0}$$

$$A_i = \text{const} \qquad U^{ii} = \text{const} \qquad \sum_j U^{ij} U^{ji} = \text{const}$$

$$\dim (A, U) = r + r^2 - \#(\text{const's}) \qquad r + r + r$$

$$\omega = -\frac{1}{2} \operatorname{res}_\infty \operatorname{tr}(\Psi^{-1} \delta L \wedge \delta \Psi \cdot)\Big|_{\mathcal{L}} dz$$

$$= -\frac{1}{2} \operatorname{res} \operatorname{tr}\Big[(1 - \xi_1 z^{-1} + \quad) \underline{\underline{\delta U}} \,(\delta \xi_1 \, z^{-1})\Big] dz$$

$$= -\frac{1}{2} \operatorname{tr} \delta U \wedge \delta \xi_1 \quad = -\frac{1}{2} \sum_{ij} \frac{\delta U^{ij} \wedge \delta U^{ji}}{a_i - a_j}$$

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$\underline{\underline{Ex}}$ $\underline{\underline{1}}$ $\quad L = u_0 + \sum \dfrac{u_i}{z - z_i}$

Symplectic leaves $\delta k \, dz$ holomorphic at $z_i$ $\quad z = \infty$

$$k_j = \frac{\nu_{ji}}{z - z_i} + O(1) \qquad \nu_{ji} \text{ is eigenvalue of } u_i$$

$$u_i = g_i \nu g_i^{-1}$$

$$\delta k \, dz \text{ holomorphic at } z_i \iff \delta \nu = 0$$

in other words $\mathcal{L} = \oplus \, O_i$ - orbits of adjoint action

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$$\underline{\Psi^{-1} L = k \Psi^{-1}} \implies$$

$$\Big[\Psi^{-1} \delta L = \Psi^{-1} \delta \Psi \Psi^{-1} L + \underline{\delta k \Psi^{-1}} - k \Psi^{-1} \delta \Psi \Psi^{-1}\Big] \wedge \delta \Psi$$

$$-\frac{1}{2} \operatorname{res}_{z_i} \operatorname{tr} \langle \Psi^{-1} \delta L \, \delta \Psi \rangle dz \Big|_{\mathcal{L}} = \sum_{z_i} \operatorname{tr}\big(\underline{L \, \delta \Psi \Psi^{-1} \wedge \delta \Psi \Psi^{-1}}\big)$$

$$\omega(X,Y) = \operatorname{tr}\left(L[X,Y]\right)$$

Kostant – Lie – Kirillov form

$$\omega|_{\mathcal{I}} = \bigoplus \omega_i$$