Lecture 12

Thursday, November 26, 2020 10:00 AM

Universal soliton hierarchy

$t = (t_{\alpha,i})$ $\quad \alpha = 1, \dots, N \quad i = 0, 1, \dots$

$$\sum t_{\alpha,0} = 0 \qquad t_{\alpha,0} \in \mathbb{Z}$$

$\psi(t, p)$

Linear probleme

$$\forall_{\alpha,n} \ \exists \ L_{\alpha,n} = \partial_{\alpha,1}^{n} + \sum u_i^{(\alpha,n)} \partial_{\alpha,1}^{i}$$

1. $(\partial_{\alpha,n} - L_{\alpha,n})\psi = 0$ (KP hierarchy)

Remark $t_{\alpha,0}$, $t_{\alpha,1}$ are special

2. $T_{\alpha,\beta}$ shift $t_{\alpha,0} \to t_{\alpha,0} + 1$ $\quad t_{\beta,0} \to t_{\beta,0} - 1$

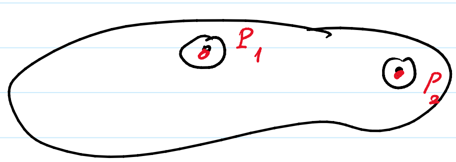
$$\begin{cases} (T_{\alpha,\beta} + v_{\alpha\beta}\partial_{\alpha,1} + w_{\alpha,\beta})\psi = 0 \\ (T_{\alpha,\beta}^{-1} + \tilde{v}_{\alpha\beta}\partial_{\beta,1} + \tilde{w}_{\alpha\beta})\psi = 0 \end{cases} \qquad \text{(2D Toda hierarchy)}$$

3. $(T_{\alpha\beta} + h_{\alpha\beta} T_{\alpha\gamma} + f_{\alpha\beta})\psi = 0$ (BDHE)

4. Two-dimensional Shrödinger operator in magnetic field

$z = t_{\alpha,1}$ $\qquad \bar{z} = t_{\beta,1}$

Two point BA function



Γ smooth genus g curve with $P_{1,2}$

Let $\psi(z, \bar{z}, p)$ be the BA function $p \in \Gamma$

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① ψ on $\Gamma \setminus \{P_1,P_2\}$ is meromorphic with poles at $\mathcal{D}=\gamma_1+\dots+\gamma_g$

②
$$\psi = e^{\varphi_+} e^{k_1 z}\left(1+\sum \xi_1^+(z,\bar z)k_1^{-s}\right) \qquad k_{1,2}^{-1}=z_{1,2}$$
$$\psi = e^{\varphi_1} e^{k_2 \bar z}\left(1+\sum \xi_1^-(z,\bar z)k_2^{-s}\right)$$
$$\varphi_\pm=\varphi_\pm(z,\bar z) \longrightarrow \varphi_\pm(z,\bar z) \ne c(z,\bar z)$$

Th ψ satisfies the eq-n

$$\left(\partial_z\partial_{\bar z}+A_z\partial_z+A_{\bar z}\partial_{\bar z}+u\right)\psi=0$$

$$\partial_z\partial_{\bar z}\psi = \varphi_{+\bar z}e^{\varphi_+}e^{k_1 z}(k+\dots) \qquad \text{near } P_1$$
$$A_z\partial_z\psi = A_z e^{\varphi_+}e^{k_1 z}(k \qquad)$$

$$A_z=-\varphi_{+\bar z} \qquad A_{\bar z}=-\varphi_{-\bar z}$$

Ex Find the expression for $u=$?

Lemma Let $\mathcal{D}$ be a differential operator in $z,\bar z$ s.t

$$\mathcal{D}\psi=0$$

Then $\exists$ diff operator $\mathcal{D}_1$ s.t

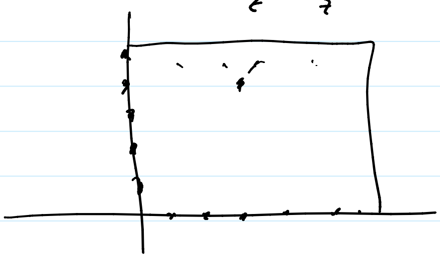
$$\mathcal{D}=\mathcal{D}_1 H$$

In other words the idal of operators that annigilate ψ is principal

Idea of the proof : division with (остатком)

$$\partial_z^i \partial_{\bar z}^j = \qquad \underset{\sim}{i \geq 1} \quad j \underset{\sim}{\geq 1}$$

$$\partial_z^{i-1} \partial_{\bar z}^{j-1} H \quad + \text{ of order } \leq i+j-1$$



$$\mathcal{D} = \mathcal{D}_1 H + \delta_1 + \delta_2$$

δ_1 is a diff. in z only

δ_2 is a diff in $\bar z$ only

$$\mathcal{D}\psi = 0 \Rightarrow \delta_1 \psi + \delta_2 \psi = 0$$

$$\delta_1 = 0 \qquad \delta_2 = 0$$

$$\underbrace{\left[\partial_{\alpha,n} - \underset{\sim}{L_{\alpha,n}}, \partial_{\beta,m} - \underset{\sim}{L_{\beta,m}}\right]}_{\text{differential operator in } t_{\alpha,1},\, t_{\beta,1}} \psi = 0$$

$$\Rightarrow \underline{\left[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\beta,m} - L_{\beta,m}\right] = \mathcal{D}_{\alpha\beta} H_{\alpha\beta}}$$

$H_{\alpha\beta}$ – Shrodinger operator in $t_{\alpha,1}$ $t_{\beta,1}$

That is Zakharov - Shabat form of the soliton hierarchy in continuous "times"

Ex Look at Veselov - Novikov eq-n

$$H = \partial\bar\partial + u$$

Missing Phase space for universal hierarchy

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for KP $\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + \dots$

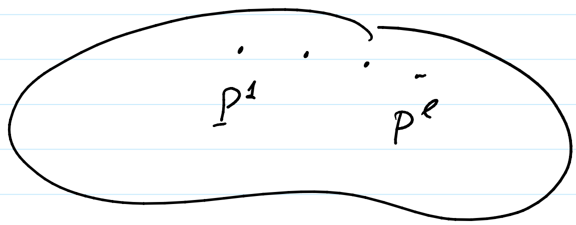
2D Toda two pseudo-difference operators

$$\mathcal{L}_{\pm} = a T^{\pm} + \sum_{s=0}^{\infty} a_s^{\pm} T^{\mp s}$$

Matrix case

Ex Matrix KP.

ψ - vector BA function



$$\psi = \begin{pmatrix} \psi_1 \\ \psi_l \end{pmatrix}$$

$$\mathcal{D} = \gamma_1 \dots + \gamma_{g+l-1}$$

$t_i^{(l)}$

$= u^{(l)} t_i$

ψ_i has form

$$\psi \sim e^{\sum t_i u_i^{(j)} k^i} (\text{regular}) \quad \text{near } P^{(j)}$$

Ex Show that $\exists\ L_n = u_n \partial_{t_1}^n + \dots$

$$(\partial_{t_n} - L_n)\psi = 0 \qquad u_n - \operatorname{diag}(u_n^j)$$

Reductions

$(2+1) \longrightarrow (1+1) \longrightarrow (0+1)$

Universal soliton hierarchy

Stationary of t_1 for some $(d\,n)$

Stationary points of $t_{d,n}$ for some (d,n)

<u>Ex</u> KP hierarchy

Suppose that on Γ $\exists$ meromorphic function with pole of the form

$$E = k^n + O(k^{-1}) \quad \text{at the marked point}$$

P_1

i.e $\Omega_n = E$ $\qquad d\Omega_n = dk^n + O(k^{-1})$

$$\psi(t_1, \dots, t_{n-1}, t_n, \dots) = e^{t_n E}\tilde{\psi}(t_1, \dots, t_{n-1}, 0, t_{n+\dots})$$

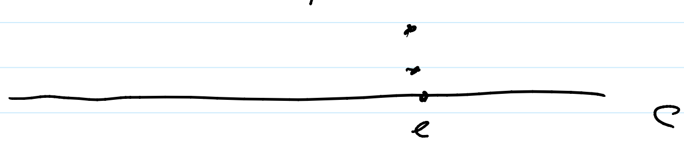
$$(\partial_{t_n} - L_n)\psi = 0 \iff L_n\tilde{\psi} = E\tilde{\psi}$$

$$\begin{cases} L_n\psi = E\psi \\ (\partial_{t_m} - L_m)\psi = 0 \end{cases}$$

$$[\partial_{t_n} - L_n, \partial_{t_m} - L_m] = 0 \iff \partial_{t_m} L_n = [L_m, L_n]$$

<u>Ex</u> $n=2$

$$E(p) : \Gamma \to \mathbb{C} \qquad E(p) = e$$



Γ: $y^2 = P_{2g+1}(E)$ hipelliptic curve

$$\cancel{\tfrac{3}{4}u_{yy}} = \left(\partial_t u - \tfrac{3}{2}uu_x + \tfrac{1}{4}u_{xxx}\right)_x \qquad \partial_y u = 0 \qquad t_2 = y$$

$$\cancel{\tfrac{3}{4} u_{yy}} = \left(\partial_t u - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} \right)_x \qquad \partial_y u = 0 \qquad \tau_2 = y$$

$$u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} = 0 \qquad \text{KdV}$$

$$u = 2 \partial_x^2 \ln \theta(Ux + \underbrace{Wt}_{Vy = 0} + Z)$$

$n = 3$ Boussinesque

$$\frac{3}{4} u_{yy} = \left(- \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} \right)_x$$

(2+1) system $\longrightarrow$ (1+1) $\longrightarrow$ (0+1)

abstract curves — covers of complex plane

Ex Reduction to stationary points of two flows

$$\partial_{t_n} = 0 \qquad \partial_{t_m} = 0$$

$$[\partial_{t_n} - L_n, \partial_{t_m} - L_m] = 0 \implies [L_n, L_m] = 0$$

commuting difference operators

$$R(L_n, L_m) = 0 \qquad (\text{Burchnall-Chaundy})$$

Let E, H be two meromorphic function on Γ $\exists$ $R(E, H) = 0$

(2+1) $\longrightarrow$ (1+1) $\longrightarrow$ (0+1)

abstract alg. — covers — plane (may be singular) curves abstract alg. covers plane curves (may be singular)

Next time Hamiltonian theory