Lecture 11

Thursday, November 19, 2020 10:00 AM

Recall Universal soliton hierarchy

- Times $t_{\alpha,i}$ $\alpha = 1, \dots N$
  $i = 0, \dots \infty$
  $\sum_\alpha t_{\alpha,0} = 0$ $t_{\alpha,0} \in \mathbb{Z}$

- Universal system of linear problems

$N=1$ KP hierarchy

$\forall \alpha$
$$\underbrace{\left(\partial_{\alpha,n} - L_{n,\alpha}\right)} \psi = 0$$

$$\partial_{\alpha,n} = \frac{\partial}{\partial t_{\alpha,n}} \qquad L_{n,\alpha} = \partial_{\alpha,1}^n + \sum u_i^{(n,\alpha)}(t) \partial_{\alpha,1}^i$$

$$\Rightarrow \left[\partial_{\alpha,n} - L_{n,\alpha}, \partial_{\alpha,m} - L_{m,\alpha}\right] = 0$$

$$\underbrace{\partial_{\alpha,m} L_{n,\alpha} - \partial_{\alpha,n} L_{m,\alpha} + [L_{n,\alpha}, L_{m,\alpha}] = 0} \qquad (*)$$

Zakharov - Shabat representation

$\forall n, m$ equation $(*)$ is well-defined system in the sense that

\# of equations $n+m$

\# of unknowns $n+m$

$N=2$ New discrete variables

$$T_{\alpha\beta} \quad \begin{cases} t_{\alpha,0} \to t_{\alpha,0} + 1 \\ t_{\beta,0} \to t_{\beta,0} - 1 \end{cases}$$

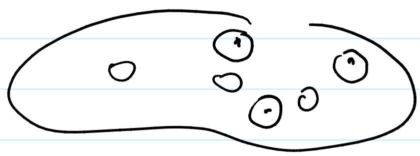
$$t_{\beta,0} \to t_{\beta,0} - 1$$

$$\left(T_{\alpha\beta} - c_{\alpha,\beta}(t)\partial_{\alpha,1} - v_{\alpha\beta}(t)\right)\psi = 0$$

$$\left(T^{-1}_{\alpha,\beta} - d_{\alpha\beta}(t)\partial_{\beta,1} - w_{\alpha\beta}(t)\right)\psi = 0$$

$N=3$ $\quad \underline{\psi_{n+1,m}} = \psi_{n,m+1} + v_{n,m}\underline{\psi_{n,m}} \quad \psi_{n,m}(k)$

$$\psi \to f\psi$$



$P_0 \quad \psi_{n,m} \sim k^{n+m}\left(\underline{\underline{1}} + O(k^{-1})\right)$

$k = z^{-1} \qquad P_1 \quad \psi \sim k^{-n}\left(\xi_{01} + O(k^{-1})\right) \quad P_2 = k^{-m}\left(\xi_{02} + O(k^{-1})\right)$

$$\boxed{v_{n,m} = \frac{\xi_{02}(n+1,m)}{\xi_{02}(n,m)} = -\frac{\xi_{01}(n,m+1)}{\xi_{01}(n,m)}}$$

$$\psi_{n,m} = \frac{\theta(A(p) + Un + Vm + Z)}{\theta(A(p) + Z)} \frac{\theta(A(p_1) + Z)}{\theta(A(p_1) + Un + Vm + Z)}$$

$$\times\, e^{n\Omega_{01} + m\Omega_{02}}$$

$$U = A(p_1) - A(p_0)$$

$$V = A(p_2) - A(p_0) \qquad A(p_0) = 0$$

$$v_{n,m} = c\,\frac{\theta((n+1)U + (m+1)V + Z)}{\theta(U(n+1) + mV + Z)} \frac{\theta(Un + Vm + Z)}{\theta(Un + V(m+1) + Z)}$$

$$c_1 \overbrace{\frac{\theta(A(p) + U + Z_{n,m})}{\theta(U + Z_{n,m})}}^{\psi_{n+1,m}} = c_2 \overbrace{\frac{\theta(A(p) + V + Z_{n,m})}{\theta(\ldots)}}^{\psi_{n,m+1}} +$$

$$c_1 \frac{\theta(\dots)}{\theta(U + Z_{n,m})} = c_2 \frac{\theta(A(p)+V+Z_{n,m})}{\theta(V+Z_{n,m})} +$$

$$Z_{n,m} = Z + V_n + V_m$$

$$+ c_3 \frac{\theta(U+V+Z_{n,m})\,\cancel{\theta(Z_{n,m})}}{\theta(U+Z_{n,m})\,\theta(V+Z_{n,m})} \quad \frac{\theta(A(p)+Z_{n,m})}{\cancel{\theta(Z_{n,m})}}$$

$$\underline{c_1\, \theta(A+U+Z_{n,m})\,\theta(V+Z_{n,m})} =$$

$$\underline{= c\, \theta(A+V+Z_{n,m})\,\theta(U+Z_{n,m}) + c_3\, \theta(U+V+Z_{n,m})\,\theta(A+Z_{n,m})}$$

Fay trisecant identity

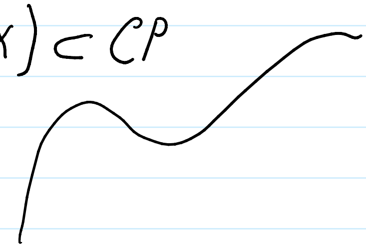
Detour $\quad B \to \theta(z, B)$

Addition formula

$$\boxed{\theta(z_1)\,\theta(z_2) = \sum_{\varepsilon = (0,\frac{1}{2})^g} \hat{\theta}_\varepsilon\left(\frac{z_1+z_2}{2}\right) \hat{\theta}_\varepsilon\left(\frac{z_1-z_2}{2}\right)}$$

Theta functions of level 2 with characteristics

$$\underbrace{\hat{\theta}_\varepsilon(z)} : \mathbb{C}^g / \Lambda + m B \longrightarrow K(X) \subset \mathbb{CP}^{4^g-1}$$



$$X \longrightarrow \mathbb{CP}^{4^g-1}$$

ppav

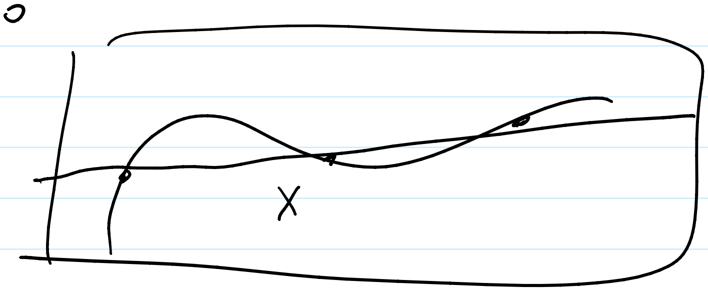
$$\sum_\varepsilon \left[c_1 \hat{\theta}_\varepsilon\left(\frac{A+U-V}{2}\right) + c_2 \hat{\theta}_\varepsilon\left(\frac{A+V-U}{2}\right) + c_3 \hat{\theta}_\varepsilon\left(\frac{U+V-A}{2}\right) \right] \times$$

$$\hat{\theta}_\varepsilon\left(\frac{A+U+V+Z}{2}\right) = 0$$

- Fact $\hat{\theta}_\varepsilon(Z)$ linear independent function

$$\Rightarrow c_1 \hat{\theta}_\varepsilon\left(\frac{A+U-V}{2}\right) + c_2 \hat{\theta}_\varepsilon\left(\frac{A+V-U}{2}\right) + c_3 \hat{\theta}_\varepsilon\left(\frac{U+V-A}{2}\right)$$

$= 0$



$\mathbb{C}P^{4^g-1}$

<u>Welder's trisecant conjecture</u>

ppav is the Jacobian iff $\exists$ a trisecant of its Kummer variety

---

$$\begin{cases} \underline{\psi_{n+1,m,k}} = \psi_{n,m+1,k} + v_{n,m,k}\,\psi_{n,m,k} \\ \psi_{n+1,m,k} = \psi_{n,m,k+1} + u_{n,m,k}\,\psi_{n,m,k} \end{cases}$$

$$\begin{cases} v_{n,m,k} = \dfrac{\tau_{n+1,m+1,k}\ \tau_{n,m,k}}{\tau_{n+1,m,k}\ \tau_{n,m+1,k}} \\ u_{n,m,k} = \dfrac{\tau_{n+1,m,k+1}\ \tau_{n,m,k}}{\tau_{n+1,m,k}\ \tau_{n,m,k+1}} \end{cases}$$

$v_{n} \quad - u \quad = - w$

$$v_{n,m,k} - u_{n,m,k} = - w_{n,m,k}$$

$$\psi_{n,m+1,k} = \psi_{n,m,k+1} + w_{n,m,k}\, \psi_{n,m,k}$$

$$\boxed{\frac{\tau_{n+1,m+1,k}\ \tau_{n,m,k}}{\tau_{n+1,m,k}\ \tau_{n,m+1,k}} - \frac{\tau_{n+1,m,k+1}\ \tau_{n,m,k}}{\tau_{n+1,m,k}\ \tau_{n,m,k+1}} = - \frac{\tau_{n,m+1,k+1}\ \tau_{n,m,k}}{\tau_{n,m+1,k}\ \tau_{n,m,k+1}}}$$

$$\tau_{n+1,m+1,k}\ \tau_{n,m,k+1} + \tau_{n,m+1,k+1}\ \tau_{n+1,m,k} =$$

$$= \tau_{n+1,m,k+1}\ \tau_{n,m+1,k} \qquad (**)$$

$(**)$ is BDHE (bilinear discrect Hirota equation)

So far : Universal hierachy of linear problems

Compatibelity conditions – soliton equations

$$\left[\partial_{\alpha,n} - L_{\alpha,n},\ \partial_{\beta,m} - L_{\beta,m}\right] = D_{\alpha,\beta}\ H_{\alpha,\beta} \qquad (*)$$

$$\left[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\beta,m} - L_{\beta,m}\right] = \mathcal{D}_{\alpha,\beta}\, \underset{\sim}{H_{\alpha,\beta}} \qquad (*)$$

$$H_{\alpha,\beta} = \underline{\partial_{\alpha 1}\partial_{\beta 1}} + v_{\alpha\beta}\partial_{\alpha,1} + w_{\alpha\beta}\partial_{\beta,1} + u_{\alpha\beta}$$

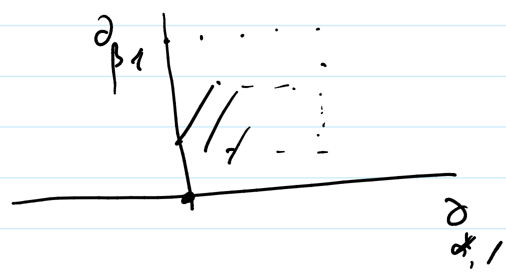
- The meaning of $(*)$
- $(*)$ is well defined system i.e

\# of equations = \# of uknowns

(coeff $L_{\alpha,n}$, $L_{\beta,m}$, $H_{\alpha,\beta}$)

is a differential operator in $t_{\alpha,1}$ $t_{\beta,1}$

$$\sum_{\substack{i \le n-1 \\ j \le m-1}} \partial_{\alpha,1}^{i}\, \partial_{\beta,1}^{j}$$



Any operator $\mathcal{D}$ has the unique representation

$$\underset{\sim}{\hat{\mathcal{D}}} = \underset{\sim}{\mathcal{D}} H_{\alpha,\beta} + d_1 + d_2$$

d_1 is diff in $\alpha,1$

d_2 is diff in $\beta,1$