Lecture 10

Thursday, November 12, 2020 8:54 AM

Recall:

	Operators	
$0+1$	$L(z)=\sum \frac{u_i}{z-z_i}$	$\longrightarrow (\Gamma, \mathcal{D}, .)$
$(1+1)$	$L = u_n \partial_x^n + \sum_{j=0}^{n-1} v_n(x)\partial_x^j$	$\longrightarrow$ formal BA functions
$(2+1)$	$L = \partial_y - \partial_x^2 + u(x,y)$	KP
	$L = \partial_x^2 + \partial_y^2 + u(x,y)$	NV

Last time we constructed solutions to the KP "hierarchy"

$$u(x,y,t) = -2\partial_x^2 \ln \Theta(Ux + Vy + Wt + Z) + \text{const}$$

KP $\qquad \sum t_i V_i$ hierarchy

"Universal soliton hierarchy"

- "Times" $\quad t_{\alpha,i} \quad \alpha = 1,\dots,N \quad i = 0,1,\dots$

$$\sum t_{\alpha,0} = 0 \qquad t_{\alpha,0} \in \mathbb{Z}$$

- Universal (scalar) Baker-Akhiezer function

Given $\Gamma_g, P_\alpha, z_\alpha(p), \mathcal{D} = \gamma_1 \dots \gamma_g \quad \exists!$

① $\psi(t,p) = z_\alpha^{t_{\alpha 0}} \exp\left(\sum_{i=1}^{\infty} t_{\alpha i} z_\alpha^{-i}\right) \underbrace{\left(\sum_{s=0}^{\infty} \xi_{s\alpha}(t) z_\alpha^{s}\right)}$

$p \in \Gamma$

② ψ is meromorphic on $\Gamma \setminus \{P_\alpha\}$ with divisor of poles $\mathcal{D}$

(2) ψ is meromorphic on $\Gamma \setminus \{P_\alpha\}$ with divisor of poles D

up to multiplication $\psi \to r(t)\,\psi(t,p)$

Exact theta functional formula

$$\psi(t,p) = r(t)\exp\Big(\sum_{\alpha,i} t_{\alpha i}\,\Omega_{\alpha i}(p)\Big)\cdot \frac{\theta(A(p)+(U,t)+Z)}{\theta(A(p)+Z)}$$

Z - arbitrary $\qquad (U,t) = \sum_{\alpha,i} t_{\alpha i} U_{\alpha i}$

$U_{\alpha i}$ is the vector of b-periods of $d\Omega_{\alpha,i}$

$N=1 \qquad (\partial_{t_n} - L_n)\psi = 0 \qquad L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(t)\,\partial_x^i$

$n=2 \qquad (\partial_2 - \partial_x^2 - u)\psi = 0 \qquad x = t_1$

$$\psi = \underbrace{e^{kx+k^2y+\dots}\Big(1+\sum_{s=1}^{\infty}\xi_s k^{-s}\Big)} \qquad k = z^{-1}$$

L_n is defined by the congruence

$$(\partial_{t_n} - L_n)\psi = O(k^{-1})\exp(\quad)$$

$k^n \qquad L_n\psi = k^n\psi + O(k^{-1})\exp$

!!! $\quad (\partial_{t_n} - L_n)\psi \equiv 0$

$$u(x,y,t) = 2\partial_x^2 \ln\theta(Ux+Vy+Wt+Z)+c \qquad x=t_1,\ y=t_2,\ t_3=t$$

solves the KP

For universal BA function $\exists!$

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$$\left(\partial_{t_{n,\alpha}} - L_{n,\alpha}\right)\psi = 0 \qquad \alpha = 1, \dots$$

$$L_{n,\alpha} = \left(\partial^n_{t_{\alpha,1}} + \sum_{i=0}^{n-1} u_{i,\alpha}(t)\, \partial^i_{t_{\alpha,1}}\right)$$

$$\Rightarrow \underbrace{\left[\partial_{t_{n,\alpha}} - L_{n,\alpha},\ \partial_{t_{m,\alpha}} - L_{m,\alpha}\right] = 0}_{} \qquad KP_\alpha \text{ hierarchy}$$

$\underline{N=2}$ 2D Toda

P_1, P_2

$$\psi_n(\xi,\eta,p) = \underset{\sim}{k}^n e^{k\xi}\left(1+O(k^{-1})\right) \qquad \text{near } P_1 =$$

$$= e^{\underline{\varphi_n(t)}} \underset{\sim}{k}^{-n} e^{k\eta}\left(\underset{\sim}{1+O(k^{-1})}\right) \qquad P_2$$

$$\left\{\begin{array}{l} \underline{\partial_\xi \psi_n(\xi,\eta,p) = \psi_{n+1}(\xi,\eta,p) + v_n(\xi,\eta)\,\psi_n(\xi,\eta,p)} \\ \qquad \underline{v_n(\xi,\eta)} = \varphi_{n\xi} \\ \partial_\eta \psi_n(\xi,\eta,p) = c_n(\xi,\eta)\,\psi_{n-1}(\xi,\eta,p) \\ \qquad \underline{c_n = e^{\varphi_n - \varphi_{n-1}}} \end{array}\right.$$

$$\varphi_{n\xi\eta} = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}$$

$$\psi_n(\xi,\eta) = \frac{\theta\left(A(p) + n\underset{\sim}{U} + \xi V + \eta W + Z\right)\theta\left(A(p_1)+Z\right)}{\theta\left(A(p_1) + nU + \xi V + \eta W + Z\right)\theta\left(A(p+Z)\right)} e^{n\Omega_0 + \xi\Omega_{1,1} + \eta\Omega_{2,1}}$$

$$2\pi i\, V = \oint_b \Omega_{1,1} \qquad 2\pi i\, W = \oint_b \Omega_{2,1}$$

$$\oint_a d\Omega_{\alpha,0} = 0 \qquad d\Omega_{\alpha,0} \text{ has simple pole at}$$

P with residue 1 and

$\int_a \Omega_{\alpha,0}$ $\quad \Omega_{\alpha,0}$ has simple pole at

p with residue 1 and
simple pole at q_0 with residue -1

$$n\,\Omega_{1,0} - n\,\Omega_{2,0} = n\,\Omega_0$$

$$2\pi i\, U = \oint_b d\Omega_0$$

$$e^{\varphi_n} = \frac{\theta(A(p_2) + Un + V\xi + W\eta) + Z)}{\theta(A(p_1) + Un + V\xi + W\eta + Z)} \frac{\theta(A(p_1 + Z))}{\theta(A(p_2) + Z)} e^{\alpha x + \beta\xi + \gamma\eta}$$

Bilinear Rieman identity

$$U = A(p_2) - A(p_1)$$

$$\tilde{Z} \to Z + A(p_1) \qquad Z = \tilde{Z} - A(p_1)$$

$$\boxed{e^{\varphi_n} = \frac{\theta(U(n+1) + V\xi + W\eta + \tilde{Z})}{\theta(Un + V\xi + W\eta + \tilde{Z})} e^{\alpha n + \beta\xi + \gamma\eta}}$$

$\underset{\sim}{N=3}$ $\qquad p_1 \quad k^{n+m}(1 + O(k^{-1}))$

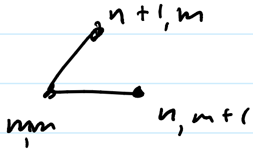
n

$p_2 \quad O(k^{-n}) \qquad p_3 \quad O(k^{-m})$

$\psi_{n,m}(p)$ the Baker-Akhiezer function
with pole of order $n+m$ at P_1
with zeros of order n and m at P_2 and P_3

- Th $\quad \psi_{n+1,m}(p) = \psi_{n,m+1}(p) + v_{n,m}\,\psi_{n,m}(p)$

• Th $\psi_{n+1,m}(p) = \check{\psi}_{n,m+1}(p) + v_{n,m}\,\psi_{n,m}(p)$



Ex Find the formula for $v_{n,m}$

$\psi_{n,m}$

Fay trisecant identity