Lecture 9

Thursday, November 5, 2020 8:17 AM

Recall Lax equations for

"ordinary linear diff operators with matrix coefficients"

$$L = \sum_{i=0}^{n} u_i(x)\partial_x^i$$

By gauge transformation $L \to g(x) L g^{-1}(x)$

$$u_n^{\alpha\beta} = u_n^\alpha \delta^{\alpha\beta}, \quad u_n^\alpha - \text{const}$$

$$u_{n-1}^{\alpha\alpha} = 0$$

- The linear space of OD operators A s.t

$$[L, A] = O(\partial_x^{n-1})$$

is span by operators

$$A_{m,v} = v\,\partial_x^m + \sum_{j=0}^{m-1} v_{j,m}(x)\,\partial_x^j \qquad A_{m,v}(L)$$

where v is a diagonal matrix

$v_{j,m}^{\alpha\beta}$ are differential polynomials in $u_i^{\alpha\beta}$

"Steps" of the proof

- $L \longrightarrow \Psi(x,\kappa)$ - formal Baker-Akhiezer solution of

$$L\Psi(x,\kappa) = \kappa^n \Psi(x,\kappa)\, u_n$$

$$\Psi(x,\kappa) = \left(\mathbb{1} + \sum_{s=0}^{\infty} \xi_s(x)\kappa^{-s}\right) e^{\kappa x} \qquad \xi_s^{ii} = 0,\ s>0$$

- $\forall\, \Psi$ of the form (1) there is a unique $A_{m,v}$ s.t

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$$A_{m,v}\Psi = k^m \Psi v_m + O(k^{-1})e^{kx}$$

By definition of $A_{m,v}$ the Lax equation

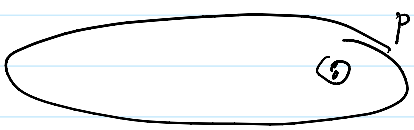
$$\partial_{t_{(m,v)}} L = [A_{m,v}, L]$$ is a well-defined

system of evolutionary equations on the space of operators L.

<u>The BA function</u> $\to$ solutions of the system

<u>Ex</u> <u>Scalar case</u>

Let $\psi(t, \underline{\underline{p}})$ be a <u>"one-point"</u> BA function



$p \in \Gamma$ P_0 $k^{-1}(p) = z(p)$ local coordinate

$z(P_0) = 0$

$\gamma_1, \dots, \gamma_g \in \Gamma$

$\exists!$ (1) $\psi(t,p) = \underline{\left(1 + \sum \xi_s(t) k^{-s}\right) e^{kx + k^2 y + \dots}}$ <u>near P_0</u>

$t = (t_1, \dots\dots, t_n, \dots)$ $\quad \exp\left(\sum t_i k^i(p)\right)$ $\quad$ <u>$x = t_1,\ y = t_2,\ t = t_3$</u>

(2) $\psi(t,p)$ meromorphic on $\Gamma \setminus P_0$ with simple poles at γ_s (if γ_s are distinct)

$\forall\ \underline{\underline{\Psi}}$ $\quad \exists!\ A_m = \partial_x^m + \sum v_j(t)\partial_x^m$ $\quad x = t_1$

s.t. $A_m \psi = \underline{\underline{k}}^m \psi + O(k^{-1})\exp(\quad)$

<u>Th</u> The equation

in the equation

$$\left(\partial_{t_m} - A_m\right)\psi = 0$$

holds

Proof $\left(\partial_m - A_m\right)\psi \doteq \cancel{O(k^{-1})\exp}\ 0$

$\Rightarrow \left[\partial_m - A_m, \partial_{m'} - A_{m'}\right]\psi = 0$ . .

$\Rightarrow \left(-\partial_m A_{m'} + \partial_{m'} A_m + [A_m, A_{m'}]\right) = 0$ . .

Zakharov-Shabat form of the KP

Ex $m = 2$ $m' = 3$

$$A_2 = \partial_x^2 + u$$

$$\left(\partial_2 - \partial_x^2 - u\right)\psi = 0 \qquad \partial_2 = \frac{\partial}{\partial t_2} = \partial_y$$

$$\psi = \left(1 + \xi_1 k^{-1} + \dots\right)\exp(\ )$$

$$\left(\partial_2 - \partial_x^2 - u\right)\psi = O(k^{-1})\exp(\ )$$

$$u = -2\xi_1' \qquad (\ )' = \partial_x$$

$$A_3 = \left(\partial_x^3 + \frac{3}{2}u\partial_x + w\right)$$

$$\left[\partial_2 - \partial_x^2 - u, \partial_3 - \partial_x^3 - \frac{3}{2}u\partial_x + w\right] = 0$$

$$\frac{3}{4}u_{yy} = \left(u_t + \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right)_x \qquad \text{(KP equation)}$$

$$\psi(t,p) = \frac{\theta(A(p)+(t,U)+Z)}{\theta(A(p_0)+(t,U)+Z)} \frac{\theta(A(p_0)+Z)}{\theta(A(p)+Z)} \exp\left(\sum_{i=1}^{\infty} t_i \Omega_i(p)\right)$$

θ – Riemann theta function defined by the matrix of b-periods of the normalized holomorphic differentials

$$A(p) = \int_{P_0}^{p} \vec{\omega} \quad \text{– Abel map}$$

$d\Omega_i$ – meromorphic diff. on Γ with pole at P_0 of the form

$$d\Omega_i = d(k^i + O(k^{-1}))$$

$$\oint_{a_i} d\Omega_i = 0$$

$$<t, U> = \Sigma\, t_i U_i \qquad U_i^k = \frac{1}{2\pi\sqrt{-1}} \oint_{b_k} d\Omega_i$$

$$A(p) = A(p_0) - \underline{\underline{U_1}}\, k^{-1} - \tfrac{1}{2} U_2 k^{-2} - \tfrac{1}{i} U_i k^{-i}$$

follows from bilinear relations Riemann

$$\omega_k = dv_k$$

$$\operatorname{res}_{P_0} v_k\, d\Omega_i = \oint_{\partial} v_k\, d\Omega_i \qquad v(p) = A(p)$$

i.e $\left.\frac{\partial}{\partial z}\right|_{z=0} = -\frac{\partial}{\partial x}$, $\quad z = k^{-1}(p)$

$$\rightarrow \quad \xi_1(t) = -\partial_x \ln\left(\theta(A(p_0) + <t, U> + Z)\right) + \ell(t)$$

+const

$$\ell(t) = \sum^{\infty} \ell_i\, t_i \qquad \ldots$$

+const

$$\ell(t) = \sum_{i=1}^{\infty} \ell_i t_i \qquad \text{where}$$

$$\Omega_i(p) = \kappa^i + \ell_i \kappa^{-1} + O(\kappa^{-2}) \qquad \kappa = \kappa(p) \text{ near } P_0$$

$$u = -2\xi_1' = 2\partial_x^2 \ln \theta(A(p_0) + \langle t, U\rangle + Z) - 2\ell_1$$

Denote $\quad A(p_0) + Z + (t_4 U_4 + \dots) = \hat{Z}$

$$\boxed{\underline{\underline{u}}(x,y,t) = 2\partial_x^2 \ln \theta(Ux + Vy + Wt + \hat{Z}) + \text{const}}$$

$y = t_2, \quad t = t_3 \qquad U = U_1, \quad V = U_2, \quad W = U_3$

Q: How to characterize L such that
the formal BA solution is the BA function
on some algebraic curve

Answer $\quad \exists \; A_m$ differential operator of order $(m,n) = 1$

$[L, A_m] = 0$

Lemma (Burchnall - Chaundy, 1921-28)

$$[L, A] = 0 \Longrightarrow R(L, A) = 0$$

$$L^m - A^n + \sum_{ni+jm<nm} \alpha_{ij} L^i A^j = 0$$

Proof (sketch) $\quad$ Consider $\mathcal{L}(E) \ni y(x)$ - the space of solutions of the equation

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$$Ly = Ey \qquad E \text{ - complex}$$

$$A(E) = A\big|_{\mathcal{L}(E)}$$ is a finite dimensional operator

The $R(E, w) = \det(w \cdot 1 - A(E))$

$\Rightarrow$ If $(n, m) = 1 \Rightarrow \exists\ \psi(x, p) \quad p \in \Gamma$

$$L\psi(x,p) = E\psi(x,p)$$
$$A\psi(x,p) = w\psi(x,p)$$

ψ is the BA function