Lecture 8

Thursday, October 29, 2020 9:59 AM

Recall Ex 2

- Lax equations consided as "flow" on a space "operartors" define a hierarchy (family) commuting flows

Phase space "L"

$$L = u_n \partial_x^n + \sum_{i=0}^{n-1} u_i(x)\partial_x^i \qquad u_i(x) \in Mat_{r\times r}(\mathbb{C})$$

$$\underbrace{\begin{matrix} u_n^{\alpha\beta} = u^\alpha \delta^{\alpha\beta} \\ u_{n-1}^{\alpha\alpha} = 0 \end{matrix}} \qquad u^\alpha \neq u^\beta - \text{const}$$

Vector field

$$\dot{L} = [A, L]$$

Lax equations are gauge equivalen gLg^{-1}, gAg^{-1}

$$A = A(L; h) \qquad A\text{ - depends on parameters}$$

For different choices of parameter the flows commute

Commutativity of flows has roots in the following sumple fact:

for a generic matrix L the ring of matrices A s.t $[L, A] = 0$ is commutative

$$A_m = v_m \partial_x^m + \sum_{j=0}^{m-1} v_j(x)\partial_x^j$$

We want to find A_m s.t

$$[A, L] \in O(\partial^{n-1}) \qquad (1)$$

$$[A_m, L] \in O(\partial^{n-1}) \qquad (1)$$

If A_m satisfies (1) then

$$\mathring{L} = [A_m, L] \iff \mathring{u}_i = F_i(u, h)$$

Prop. Coefficients of A_m are differential polynomial in matrix elements of L

The module of formal BA functions

$$\Psi(x,k) = \left(\sum_{s=-N}^{\infty} \xi_s(x) k^{-s}\right) e^{kx}$$

$$\partial_x \Psi = (\quad \dots \quad)' e^{kx} + k (\dots) e^{kx}$$

- $\exists!$ $\psi(x; k; x_0) = \left(\xi_0 + \sum_{s=1}^{\infty} \xi_s(x; x_0) k^{-s}\right) e^{kx}$

- $L\psi(x,k;x_0) = k^n \psi(x,k;x_0) u_n \qquad (2)$

normalized by constraints

$$\xi_0 = \mathbb{1} \qquad \xi_s^{ii}(x_0, x_0) = 0 \quad s > 1$$

- Any solution is of the form

$\Psi(x,k)$ is a solution of (2) then

$$\Psi(x,k) = \psi(x,k;x_0)\, \underline{a(k)}$$

is a diagonal matrix

is a diagonal matrix

Substitution of ψ into (2) gives a reccurent system of eq-ns

$$\underline{[u_n, \xi_0]} = 0 \quad \to \xi_0 - \text{diagonal}$$

$$[u_n, \xi_1] + n u_n \xi_0' + u_{n-1}\xi_0 = 0$$

$F(u_i, \xi_{s'}, s' \leq s)$

$$\underbrace{[u, \xi_s] + n u_n \xi_{s-1}'}\ + u_{n-2}\xi_{s-2} + \cdots$$

$$\xi_0' = 0 \qquad \xi_0 = \text{"const" diagonal}$$

off-diagonal part of ξ_1

Equation " " defines off-diagonal part of ξ_s and diagonal part of ξ_{s-1}

$$L \to \Psi(x, k; x_0)$$

- Claim $\forall$ formal series Ψ of the form (*) $\exists$ $A_{m,v_m} = v_m \partial_x^m + \Sigma \cdots$ s.t.

$$\left(A_{m,v_m}\Psi - k^m \Psi v_m\right) = O(k^{-1})e^{kx}$$

- $[A_{m,v_m}, L] = O(\partial_x^{n-1})$

$$[A, L]\psi = k^n \underline{A \psi u_n} - L\left(k^m \Psi v_m + O(k^{-1})e^{k}\right)$$

$$= \cancel{k^{n+m}\psi v_m u_n} + k^n\left(O(k^{-1})e^{kx}\right) - \cancel{k^{n+m}\psi u_n v_m}$$

$$- L\,O(k^{-1})e^{k} = O(k^{n-1}e^{kx})$$

$$\Rightarrow [A, L] = O(\partial^{n-1})$$

<u>Lemma</u> If $[A_m, L] = O(\partial^{n-1})$ then

$$A\psi = \psi\, a(k) + O(k^{-1})$$

where $a(k)$ is a polynomial in k of degree m with diagonal coefficients

<u>Hint</u> if $[A_m, L] = 0 \Rightarrow$

$$L(A\psi) = k^n (A\psi)\, u_m$$

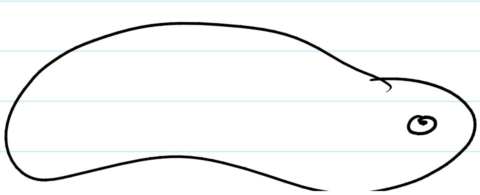
$$\Rightarrow A\psi = \psi\, a(k)$$

<u>Problem</u> Show that

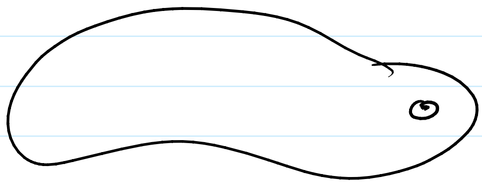
$$\left[\partial_{t_{m,v_m}} - A_{m,v_m},\ \partial_{t_{m',v'_m}} - A_{m',v_m'}\right] = 0$$

Q <u>How to solve the Lax equations</u>

Let Γ be a smooth genus g curve with one marked point and fixed local coordinat at P_0



$$z(P_0) = 0$$



$z(P_0) = 0$

Then · generic divisor $\gamma_1 \dots \gamma_g$ defines the BA function $\psi(t, p)$ that is meromorphic on $\Gamma \setminus P_0$ with $\gamma_1 \dots \gamma_g$

$$\psi = e^{k t_1 + k^2 t_2 \dots} \left(\underline{1} + \sum_{s=1}^{\infty} \xi_s(t) k^{-s} \right)$$

$$t_1 = x$$

$$\forall n \ \exists! \ L_n = \partial_x^n + u_{n-2} \partial_x^{n-2} + \dots + u_0$$

s.t. the equation

$$(\partial_{t_n} - L_n)\psi = 0$$

holds

Define $L_n \psi = k^n \psi + O(k^{-1}) e^k$

$\underline{Ex}$ $n = 2$

$$(\partial^2 + u)\psi = k^2 \psi + O(k^{-1}) e^{kx}$$

$$(\partial^2 + u)\,(1 + \xi_1 k^{-1} \dots) e^{kx} = k^2 (\quad) e^{kx} \qquad x = t_1$$

$$2k(\quad)' + (\quad)'' + u(\quad) = O(k^{-1})$$

$$2\xi_1' + u = 0 \qquad \boxed{u = 2\xi_1'}$$

$2\xi_1' + u = 0 \qquad \boxed{u = 2\xi_1}$

$$\underbrace{(\partial_{t_2} - L)\psi} = \cancel{O(k^{-1})e^{kx\cdots}} \quad 0$$

If ψ is the BA function then

$$(\partial_{t_2} - L)\psi = O(k^{-1})e^{kx+k^2t_2+\cdots}$$