Lecture 7

Thursday, October 22, 2020 9:50 AM

<u>Recall</u> Last time we proved that

$$\psi(t,p) = r(t)\frac{\Theta(A(p)+(t,U)+Z)}{\Theta(A(p)+Z)} \times$$

$$\times \exp(t, \Omega(p))$$

is the BA function corresponding to a generic divisor $\gamma_1 + \dots + \gamma_g$

$$Z = -\sum A(\gamma_s) + \mathcal{K}$$

<u>Ex 1</u> $L(z) = u_0 + \sum \frac{u_i}{z - z_i}$ $(0+1)$

<u>Ex 2</u> $L = u_n \partial_x^n + u_{n-1}(x)\partial_x^{n-1} + \quad + u_0(x)$

$u_j(x) \in Mat_{r\times r}$ $\quad u_n^{\alpha\beta} = u_n^\alpha \delta^{\alpha\beta} = \text{const}$

$u_0, \dots, u_{n-1}$ $\quad u_n^\alpha \neq u_n^\beta$

$$\underline{\dot{L} = [A_m, L]} \qquad A = v_m \partial_x^m + \dots$$

- The equation defines first $A_m = A_m(L)$

<u>Remark</u> <u>Where is the spectral parameter?</u>

For $u_i(x+1) = u_i(x)$ we can define $L(z)$ as L restricted to the space of functions

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$$f(x+1) = z f(x)$$

- ∂_x can define the structure of $\mathcal{D}$-module on the space of formal BA functions

$$\Psi(x,k) = \left(\sum_{s=-N}^{\infty} \xi_s(x) k^{-s} \right) e^{kx}$$

over the field of Laurent series

$$a(k) = \sum_{s=-m}^{\infty} a_s k^{-s}$$

$$\partial_x \Psi(x,k) = k \underline{\left(\sum \xi_s(x) k^{-s} \right)} e^{kx} + \underline{(\quad)}' e^{kx}$$

Consider

$$\mathring{L} = [A_m, L]$$

$$\sum_{i=0}^{n-1} \mathring{u}_i \partial_x^i = ?$$

In order the equs be well defined we need

$$[A_m, L] = O(\partial_x^{n-1})$$

A priory $[A_m, L]$ is of order $n+m-1$

$\Rightarrow$ is equivalent to m equations

We will show that $v_i(x)$ are differential

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$h_i \quad i = 0, \dots, m$

Ex The case of operators with scalar coefficients $u_i(x)$

(Gelfand-Dikii)

$$L \to f L f^{-1}$$

$$L = \partial_x^n + \underbrace{u_{n-2}\partial_x^{n-2} + \dots + u_0}$$

$$\exists!\ \underline{\mathcal{L} = L^{1/n} = \partial + \sum_{i=1}^{\infty} v_i \partial_x^{-i}}$$

pseudo-differential operator

ring structure

$$\partial^{-1} \circ w = w\partial^{-1} - w'\partial^{-2} + w''\partial^{-3} \Big| (-1)^s w^{(s-1)} \partial_x^{-s}$$

$$\partial \circ w = w\partial + w'$$

$$\mathcal{L}^n = L$$

$$L = \partial^2 + u \qquad \left(\partial + v_1\partial^{-1} + v_2\partial^{-2} + \dots\right)^2 = \partial^2 + u$$

$$\partial^2 + 2v_1 + \left(2v_2 + v_1'\right)\partial^{-1} + \dots = \partial^2 + u$$

$$2v_1 = u \qquad 2v_2 + \frac{u'}{2} = 0$$

$$\dots$$

- $$A_m = L^{m/n}_+ \qquad = \left(\partial_x^m + \dots\right)_+$$

$$\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-$$

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nonnegative powers of ∂

$\mathcal{D}_+$ is diff operator such $\mathcal{D} - \mathcal{D}_+ = O(\partial_x^{-1})$

<u>Claim</u> $[L_+^{m/n}, L]$ is of order $\underline{\underline{n-2}}$

$\Rightarrow$ $\mathring{L} = [L_+^{m/n}, L]$ is well-defined

$$0 = [L^{m/n}, L] = [L_+^{m/n}, L] + [L_-^{m/n}, L]$$

$$\underline{[L_+^{m/n}, L]} = -[L_-^{m/n}, L]$$

$$A_m = \sum_{i=0}^{m} c_i L^{i/n}$$

Let us show that

$$\underbrace{[\partial_m - L_+^{m/n}, \partial_k - L_+^{k/n}] = 0}_{\text{commutativity of flows}} \qquad \partial_n := \partial/\partial t_n$$

$$0 = [\partial_m - L_+^{m/n}, L]$$

$$[\partial_m - L_+^{m/n}, \partial_k - L_+^{k/n}] \subset O(\partial^{-1})$$

$$\underline{\partial_m L_+^{k/n}} \approx \partial_m L^{k/n} = [L_+^{m/n}, L^{k/n}] \cong \underline{[L_-^{m/n}, L_+^{k/n}]}$$

$$\underline{\partial_k L_+^{m/n}} \approx [L_+^{k/n}, L^{m/n}] \cong \underline{[L_-^{k/n}, L_+^{m/n}]}$$

$$0 = [L^{m/n}, L^{k/n}] \approx [L_+^{m/n}, L_+^{k/n}] + \underline{[L_-^{m/n}, L_+^{k/n}] + [L_+^{m/n}, L_-^{k/n}]}$$

$$-\partial_m L_+^{k/n} + \partial_k L_+^{m/n} + [L_+^{m/n}, L_+^{k/n}] = O(\partial^{-1})$$

$$\overline{-\partial_m L_+^{k/n} + \partial_k L_+^{m/n} + [L_+^{m/n}, L_+^{k/n}] = O(\partial^{-1})}$$

Ex $\quad L = \partial^2 + u \qquad A = L_+^{3/2} \implies$

$$\dot{u} = \frac{3}{2} u u_x - \frac{1}{4} u_{xxx} \qquad \text{KdV}$$

KP hierarchy $\qquad (2+1)$

$$\boxed{\pm \frac{3}{4} u_{yy} = \left(u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}\right)_x} \qquad u(x, y, t)$$

is defined as commuting flows $v_i(x)$, $i = 1, \ldots, \infty$ on the space of pseudo-diff operators

$$\mathcal{L} = \partial_x + \sum_{i=1}^{\infty} v_i \, \partial_x^{-i}$$

$$\underbrace{\partial_i \mathcal{L} = [\mathcal{L}_+^i, \mathcal{L}]} = -[\mathcal{L}_-^i, \mathcal{L}] \in O(\partial_x^{-1}) \qquad \partial = \partial/\partial t_i$$

$$0 = [\partial_i - \mathcal{L}_+^i, \partial_j - \mathcal{L}_+^j] \qquad i = 2 \quad \mathcal{L}_+^2 = \partial^2 + u$$

$$\mathcal{L}_+^3 = \partial^3 + \frac{3}{2} u \partial + w$$

$$\boxed{u(x, y, t) = -2 \partial_x^2 \ln \Theta(Ux + Vy + Wt + Z)}$$